Invariant Einstein metrics on compact simple Lie groups and Stiefel manifolds

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A Riemannian manifold (M, g) is called Einstein if $\mathrm{Ric}(g) = \lambda g$ for some $\lambda \in \mathbb{R}$. For compact Lie groups, G. Jensen 1979 proved the existence of left-invariant Einstein metrics. Later J. E. D'Arti and W. Ziller 1979 in the work *Naturally reductive metrics and Einstein metrics on compact Lie groups*, they obtained a large number of left-invariant Einstein metrics that are *naturally reductive.* They also asked whether G admits a non-naturally reductive Einstein metric. The problem of finding *non-naturally reductive* left-invariant Einstein metrics on compact simple Lie groups seems to be harder, and in fact this is stressed by the above authors. In some previous works of K. Mori 1994, A. Arvanitoyeorgos, K. Mori, Y. Sakane 2012, Z. Chen, K. Liang 2014, and I. Chrysikos, Y. Sakane 2015 the authors found new non-naturally reductive Einstein metrics on several Lie groups. In this talk we will present the existence of invariant Einstein metrics first on compact Lie groups and second on Stiefel manifolds.

For the first problem we will work for the compact Lie groups $G \in \{\mathrm{SO}(n), \mathrm{Sp}(m)\}$, $n \geq 7$, $m \geq 3$ and $\mathrm{SU}(\ell + 3)$, $\ell \geq 2$, and will prove that these groups admit left invariant Einstein metrics that are not naturally reductive. The space of metrics has been studied by using the generalized Wallach spaces $G/(G_1 \times G_2 \times G_3)$, $G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$, $i = 1, 2, 3$, the generalized flag manifold $\mathrm{SU}(1+2+\ell)/\,\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(\ell))$, and the homogeneous space $\mathrm{SU}(\ell+3)/(\mathrm{U}(1)\times\mathrm{SO}(3)\times\mathrm{SU}(\ell))$. In the last two cases the Einstein metrics obtained, are different from those obtained by Mori.

For the second problem, the difficulty is that the isotropy representation of Stiefel manifolds $G/K = V_k\mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}, \mathbb{C}\}$ contains equivalent summands, hence the Ricci tensor for an invariant metric is not easy to describe. Invariant Einstein metrics on Stiefel manifolds have been originally studied by S. Kobayashi, A. Sagle, G. Jensen. Later, A. Arvanitoyeorgos, D.D. Dzhepko and Yu. G. Nikonorov obtained new invariant Einstein metrics on the Stiefel manifolds $V_{sk}\mathbb{F}^{sk+l} \cong G(sk + l)/G(l)$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$ and $G(\ell) \in \{\mathrm{SO}(\ell), \mathrm{Sp}(\ell)\}$ respectively, by making some extra symmetry assumptions. In our work we take some suitable subset of the set of all invariant inner products of $\mathfrak{m} \cong T_o(G/K)$ for which the Ricci tensor becomes diagonal.

Homogeneous geodesics in certain homogeneous manifolds

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Let $(M = G/H, g)$ be a homogeneous Riemannian manifold. A geodesic $\gamma(t)$ through $o = eK$ is called *homogeneous* if it is an orbit of a 1-parameter subgroup G, i.e. $\gamma(t) = \exp tX \cdot o$ for some $0 \neq X \in \mathfrak{g}$ the Lie algebra of G. Then $M = G/K$ is called *g.o. space* (or space with homogeneous geodesics) if any geodesic γ of M is homogeneous. A Riemannian manifold (M, g) is called *g.o. manifold* (or a manifold with homogeneous geodesics) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g). Some examples of g.o. manifolds are the symmetric spaces, naturally reductive spaces, normal homogeneous spaces and weakly symmetric spaces. Besides their mathematical significance, homogeneous geodesics appear in physics and they have applications in optimization and deep learning.

In the present talk I will present joint works with Y. Wang, G. Zhao and H. Qin, concerning the classification of g.o. metrics in generalized Wallach spaces, M-spaces, and generalized C-spaces.