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Isometries of Finite-Dimensional Normed Spaces†

Félix Cabello Sánchez and Jesús M.F. Castillo

Dpto. de Matemáticas, Univ. de Extremadura, 06071-*Badajoz, Spain*

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A fundamental result in Functional Analysis establishes that no matter which norm is defined on a finite-dimensional space, the underlying topological space is the same. A different question is the equality of the underlying *metric* spaces. The basic examples of norms in $\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, are the p-norms, $1 \leq p \leq \infty$, which generalize the euclidean modulus (p=2):

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p < \infty,$$
$$\|x\|_\infty = \max |x_i|, \quad p = \infty.$$

We denote by ℓ_p^n the space $\mathbb{K}^n$ endowed with the $\| \ \|_p$ norm. Here we present several proofs that the spaces ℓ_p^n and ℓ_q^n are not isometric when p is different from q. This result is certainly well-known to specialists and it appears mentioned in several books on real analysis and/or functional analysis. Nevertheless, it is not easy to find an explicit proof. For instance, in [3], it appears as an exercise to prove the cases $p = 1, 2, \infty$; and in [13, p. 280, Prop. 37.6] it is established that the Banach-Mazur distance between ℓ_p^n and $\ell_{p^*}^n$ (the only case that matters, as we shall see) is proportional to $n^{1/p-1/2}$; the proof there presented, using Khintchine's and Kahane's inequalities, has little overlap with ours. Besides this, Pelczynski [2] attributes to Gurarii, Kadec and Macaev [5, 6] the exact calculus of the Banach-Mazur distance between ℓ_p^n spaces:

If either $1 \leq p < q \leq 2$ *or* $2 \leq p < q \leq \infty$, *then* $\mathrm{d}(\ell_p^n, \ell_q^n) = n^{1/p-1/q}$.

If $1 \leq p < 2 < q \leq \infty$, *then* $(\sqrt{2}-1)\,\mathrm{d}(\ell_p^n, \ell_q^n) \leq \max(n^{1/p-1/2}, n^{1/2-1/q}) \leq \sqrt{2}\,\mathrm{d}(\ell_p^n, \ell_q^n)$.

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It is enough to consider the case of linear isometries since, by an old theorem of Ulam and Mazur [10], an isometry of a real normed space that carries 0 to 0 must be linear (cf. [1, p. 166]). In fact, if f is an isometry between normed spaces then for some linear isometry T one has that $f(x) = T(x)+f(0)$ (see [4, p. 107, Ex. 3(b)]).

The set $\{x \in E : \|x\| = 1\}$ will be termed the unit *sphere* of $\| \,\|$. From now on, the unit sphere of the scalar field shall be denoted $\mathbf{D}$. An isometry between the normed spaces $(E, \| \,\|_1)$ and $(F, \| \,\|_2)$ is a linear application $T\colon E \to F$ such that, for all $x \in E$, $\|Tx\|_2 = \|x\|_1$. It is clear that an isometry transforms the unit sphere of one space into exactly the unit sphere of the other. Let S_p be the unit sphere of $\| \,\|_p$.

Our first proof is based on the idea: how many "peaks" has S_p?

THEOREM. *If p is different from q, the spaces ℓ_p^n and ℓ_q^n are not linearly isometric except in the case: $\mathbb{K} = \mathbb{R}$, $p, q \in \{1, \infty\}$, and $n = 2$.*

Let us start with:

An obvious case: $\mathbb{K} = \mathbb{R}$, $p, q \in \{1, \infty\}$, and $n = 2$. The isometry is an easy consequence of the equality $2\max\{|a|, |b|\} = |a + b| + |a - b|$.

An impossible case: $p \in \{1, \infty\}$ and $q \notin \{1, \infty\}$ (or viceversa). In this case, S_p contains segments, which are preserved by linear applications, while S_q does not.

We now calculate the points where S_p intersects the smallest sphere μS_2 that containis it.

An intermission: comparison with the $\| \,\|_2$ norm. It is a direct consequence of Hölder's inequality that $\| \,\|_2 \le \| \,\|_p \le n^{1/p-1/2}\| \,\|_2$, if $1 \le p < 2$, and that $\| \,\|_q \le \| \,\|_2 \le n^{1/2-1/q}$, if $2 < q < \infty$. Besides, one easily verifies:

$(1 < p < 2)$ The norms $\| \,\|_p$ and $\| \,\|_2$ coincide exactly on the points $x = \sigma e_i$, $\sigma \in \mathbf{D}$. Moreover, $\|x\|_p = n^{1/p-1/2}\|x\|_2$ if and only if $x = \sum \sigma_i e_i$, $\sigma_i \in \mathbf{D}$.

$(2 < q < \infty)$ The norms $\| \,\|_q$ and $\| \,\|_2$ coincide exactly on the points $x = \sigma e_i$, $\sigma \in \mathbf{D}$. Moreover $\|x\|_2 = n^{1/2-1/q}\|x\|_q$ if and only if $x = \sum \sigma_i e_i$, $\sigma_i \in \mathbf{D}$.

For the proof of the second parts of these assertions just verify that if $\alpha < \beta$ then the minimum of $\|x\|_\beta$ over the unit sphere S_α is attained if and only if all coordinates are equal in modulus.

The Proof

The first maybe not-entirely-trivial step is to show that

Claim 1. An isometry between ℓ_p^n and ℓ_q^n necessarily implies $q = p^*$.

Proof. To see this, let $1 < p \neq q < \infty$. Without loss of generality we can assume that $p < q$. Let $T\colon \ell_p^n \to \ell_q^n$ be an isometry represented by a matrix (a_{ij}) with respect to the natural basis (e_i) and (e_j), $1 \leq i, j \leq n$. It is clear that the transposed application $T^*\colon \ell_{q^*}^n \to \ell_{p^*}^n$ with $1/r + 1/r^* = 1$, $r = p, q$, must also be an isometry. Since $1 = \|e_i\|_p = \|Te_i\|_q$, and $1 = \|e_i\|_{q^*} = \|Te_i\|_{p^*}$, one obtains the equalities

$$1 = \sum_{j=1}^{n} |a_{ij}|^q, \ (1 \leq i \leq n) \quad \text{ and } 1 = \sum_{i=1}^{n} |a_{ji}|^{p^*}, \ (1 \leq j \leq n).$$

Summing all equations one obtains

$$n = \sum_{i,j} |a_{ij}|^q = \sum_{i,j} |a_{ij}|^{p^*}.$$

It is clear that $|a_{ij}| \leq 1$. The case $|a_{ij}| \in \{0, 1\}$ directly leads to an application T having the form $Tx = (r_i x_{\pi(i)})$, where $|r_i| = 1$ and π is a permutation of $\{1, \ldots, n\}$, and this yields $p = q$. Otherwise, the last equation is only consistent when $q = p^*$ (and, therefore, $p < 2$). ∎

To complete the proof, the idea is quite simple: why, in the real case, S_1 and S_∞ cannot be (except in the case $n = 2$) linearly isometric?: Because S_1 has 2^n "peaks", and a linear application must transform "peaks" into "peaks". Put it otherwise, let $\cup_n \mathbf{D}$ be the disjoint union of n copies of $\mathbf{D}$ and let $\mathbf{D}^n$ be the product of n copies of $\mathbf{D}$. The vertices of S_1 form the set $\cup_n \mathbf{D}$ and the vertices of S_∞ form the set $\mathbf{D}^n$. In the real case, $\cup_n \mathbf{D}$ has $2n$ elements and $\mathbf{D}^n$ has 2^n. In the complex case, $\cup_n \mathbf{D}$ is a one-dimensional (real) manifold with n connected components (n circumferences) and $\mathbf{D}^n$ is a connected n-dimensional manifold (an n-torus). Exception made of the obvious case $n = 1$ (and, perhaps, $n = 2$ real) they cannot be continuously transformed one into the other.

Thus, what we want to make is to mimic this proof and make it work with other p. To carry that program through we consider as "peaks" of the norm $\| \, \|_p$ the points where its unit sphere intersects the smallest ellipsoid μS_2 that contains it. These points have been calculated in the preceding section. Now, the core of our argumentation appears:

Claim 2. Let $1 < p < 2$. If $T\colon \ell_p^n \to \ell_{p^*}^n$ is an isometry, then $n^{1/p-1/2}T\colon \ell_n^2 \to \ell_n^2$ is also an isometry.

Proof. Let T be such an isometry. If $x_1, \dots, x_n$ are points in S_{p^*}, $q > 2$, then

$$\min_{\pm} \| \sum \pm x_i \|_{p^*} \le n^{1/p},$$

(consequence of the parallelogram law plus Holder's inequality) with strict inequality except if $x_i = n^{-1/p^*} \sum_{ij} \sigma_{ij} e_i$ for all i (with a similar argument to that of the second parts of the assertions in the intermission). Since

$$\sqrt[p]{n} = \| \sum \sigma_i e_i \|_p = \| \sum \sigma_i T e_i \|_{p^*}$$

it follows that

$$T e_i = n^{-1/p^*} \sum \sigma_{ij} e_j,$$

which, taking into account the form of T^{-1} and that $(e_i, e_j) = (T^{-1}Te_i, e_j) = (Te_i, (T^{-1})^* e_j) = \delta_{ij}$, yields $(Te_i, Te_j) = n^{1/p^*-1/2}\delta_{ij}$.

Hence $\|Tx\|_2 = \| \sum x_i T e_i \|_2 = \sqrt{\sum |x_i|^2 \|Te_i\|_2^2} = n^{1/p-1/2}\|x\|_2$. ∎

The immediate effect all this has is that:

$$T(S_2 \cup S_p) = TS_2 \cup S_p = n^{1/p-1/2} S_2 \cup S_{p^*}.$$

And the lasting surprise: $S_2 \cup S_p = \cap_n \mathbf{D}$, while $n^{1/p-1/2} S_2 \cup S_{p^*} = \mathbf{D}^n$.

Epilogue. There is one case overlooked: $\mathbb{K} = \mathbb{R}$ and $n = 2$; here, the only possibility for an operator to be isometry is to be $T(x, y) = 2^{-1/p^*}(x+y, x-y)$. That it is not can be seen as follows: let $p < 2 < p^*$ and put $d = p^*/p$; this makes $p^* = d + 1$ and $p = (1 + d)/d$. Consider points $(1, r)$ with $r > 1$. The equality $\|(1, r)\|_p = \|T(1, r)\|_{p^*}$ implies the equality

$$2\left(1 + r^{d+1/d}\right)^d = (r + 1)^{d+1} + (r - 1)^{d+1}.$$

If $f(r)$ denotes the function on the left and $g(r)$ denotes the function on the right it is an elementary matter of calculus that $\lim_{r\to\infty} g'(r)/f'(r) = 0$.

A second proof after claim 1. Our second proof starts once it has been shown that an isometry between ℓ_p^n and ℓ_q^n implies $q = p^*$. If T is an isometry between ℓ_p^n and ℓ_q^n then since $\|Te_i\|_p = 1$ and $\|T(e_i + e_j)\|_p = 2^{1/p}$ it should be possible to find three points a, b and c such that $\|c - a\|_q = 2^{1/p}$, $\|b - a\|_q =$

$1 = \|b-c\|_q$. Ther is no problem identifying a and c as $(\alpha, 0)$ and $(0,0)$, where α denotes a number of modulus $2^{1/p}$. The third point $b = (x, y)$ should satisfy simultaneously the equations

$$\begin{cases} |x|^q + |y|^q = 1 \\ |\alpha - x|^q + |y|^q = 1 \end{cases}$$

which is impossible since x should have modulus $2^{1/q}$ and this leaves no room for y.

A second proof for claim 2. There is a unique ellipsoid of maximal volume inscribed in the unit ball of a finite dimensional norm called John's ellipsoid (see [11, 12]). Therefore, an isometry must send John's ellipsoid inscribed in B_p into John's ellipsoid inscribed in B_q. This, and the comparison with the $\| \ \|_2$ norm, prove Claim 2.

Concluding remarks. i) During the preparation of the manuscript, the authors learnt of a relevant result that was obtained by Lyubich and Vasertein [9]: If one has an isometric embedding $\ell_p^n \to \ell_q^m$ with $1 < p, q < \infty$, then $p = 2$, $q$is an even integer and m satisfies the inequality

$$\binom{n + q/2 - 1}{n - 1} \leq m \leq \binom{n + q - 1}{n - 1}$$

ii) A different line of proof has been suggested to us by Prof. R. Payá, verifying that the modulus of convexity of $\| \ \|_p$ and $\| \ \|_q$ are different. If the calculus of Banach-Mazur distances is "rather difficult" (cf. [2, p. 231]) the exact calculus of the modulus of convexity of a given space is still harder. For L_p and ℓ_p spaces it was calculated by Hanner [7] and Kadec [8] (see [2, p. 238]) obtaining for $1 < p < \infty$ the formula $\delta(t) = a_p t^k + o(t^k)$, where $k = \max\{2, p\}$ and a_p are suitable positive constants depending only on p. The result of the paper would also follow from this.

iii) In an infinite-dimensional Banach space, the numbers

$$b_n = \sup_{\|x_i\|=1} \inf_{\pm} \|\textstyle\sum^n \pm x_i\|$$

are used to define B-convexity ($\lim n^{-1} b_n = 0$). Here we used them to find the "peaks" of the finite dimensional norms $\| \cdot \|_p$.

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