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Report on Twisted Sums of Banach Spaces

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1. Introduction

This note is to report some of the advances obtained as a follow-up of the book [2] on the topic of twisted sums of Banach spaces. Since this announcement is not longer enough to contain the theory being developed, we submit the interested reader to [2] and to [1], where full details and proofs shall appear.

2. Basics on twisted sums

A twisted sum of Banach spaces Y, Z is a short exact sequence

$$0 \to Y \to X \to Z \to 0.$$

The open mapping theorem implies that Y is then a closed subspace of X and $X/Y = Z$.

The classical theory of Kalton-Peck [5] describes twisted sums in terms of homogeneous maps $F : Z \to Y$ satisfying

$$\|F(x+y) - F(x) - F(y)\| \leq K\left(\|x\| + \|y\|\right),$$

which are called quasi-linear maps.

Given a twisted sum, the quasi-linear map that defines it can be obtained as the difference $B - L$ between a bounded homogeneous and a linear selection for the quotient map. Conversely, if $F : Z \to Y$ is a quasi-linear map, the product space $Y \times Z$ endowed with the quasi-norm

$$\|(y,z)\| = \|y - F(z)\| + \|z\|$$

is denoted $Y \oplus_F Z$ and provides a twisted sum

$$0 \to Y \to Y \oplus_F Z \to Z \to 0.$$

For instance, the direct sum $Y \oplus Z$ is the twisted sum space provided by any linear map $Z \to Y$. When a twisted sum is equivalent to the direct sum (i.e. when Y is complemented in X) we also say that the twisted sum splits. Kalton and Peck [5] showed that two twisted sums $Y \oplus_F Z$ and $Y \oplus_G Z$ are equivalent if and only if for some linear map $L : Z \to Y$

$$\operatorname{dist}(F - G, L) < +\infty.$$

In particular, the twisted sum defined by F splits if and only if $\operatorname{dist}(F, L) < +\infty$ for some linear map $L : Z \to Y$.

Observe that the twisted sum space $Y \oplus_F Z$ is not necessarily locally convex: in fact, the expression $\|(y, z)\| = \|y - F(z)\| + \|z\|$ is just a quasi-norm. Kalton [4] proved that when Y and Z are B-convex Banach spaces then $Y \oplus_F Z$ is a Banach space too.

3. News on twisted sums

The problem of when $Y \oplus_F Z$ is a Banach space can be completely solved (see [2]). Let us call a homogeneous map $F : Z \to Y$ 0-linear if whenever $\sum_{i=1}^{n} x_i = 0$ then

$$\left\| \sum_{i=1}^{n} F(x_i) \right\| \;\leq\; K \sum_{i=1}^{n} \|x_i\|$$

for some constant $K > 0$ independent of the points x_i.

THEOREM. *The expression $\|(y, z)\| = \|y - F(z)\| + \|z\|$ is equivalent to a norm if and only if F is 0-linear.*

(To this, one should add the following result of D. Yost: the expression $\|(y, z)\| = \|y - F(z)\| + \|z\|$ is itself a norm if and only if F is pseudo-linear in the sense that $\|F(x + y) - F(x) - F(y)\| \leq \|x\| + \|y\| - \|x + y\|$).

4. The nonlinear Hahn-Banach theorem

Therefore, by the Hahn-Banach theorem, if $F : Z \to \mathbb{R}$ is a 0-linear map, the twisted sum it defines splits and then, by the criterium of Kalton and

Peck, for some linear map $L : Z \to \mathbb{R}$

$$\operatorname{dist}(F, L) < +\infty.$$

Theorem. *An explicit construction of a linear map at finite distance of a 0-linear given map $F : Z \to \mathbb{R}$.*

5. Sobczyk's theorem Kalton's way

Sobczyk's theorem asserts that c_0 is complemented in any separable Banach space containing it. In our language, this means that every 0-linear map $F : Z \to c_0$ with Z separable admits a linear map $L : Z \to c_0$ at finite distance.

Theorem. *An explicit construction of a linear map $L : Z \to c_0(I)$ at finite distance of a 0-linear given map $F : Z \to c_0(I)$, when Z is separable.*

6. Nonlinear duality

It is well-known that if $0 \to Y \to X \to Z \to 0$ is an exact sequence then $0 \to Z^* \to X^* \to Y^* \to 0$ is also exact. This means that if $F : Z \to Y$ is a 0-linear map there should be

Theorem. *An explicit method to construct the adjoint 0-linear map $F^* : Y^* \to Z^*$.*

Knowing the form of F^* one can prove that the space Z_2 of Kalton-Peck [5] is isomorphic to its dual.

Derived from this we consider two related

7. Three-space problems on duality

It is an open problem to know if "being a dual space" is a three-space property, in the sense that given an exact sequence $0 \to Y^* \to X \to Z^* \to 0$ the space X must be a dual space. We construct

Theorem. *An exact sequence $0 \to Y^* \to X \to Z^* \to 0$ that is not a dual sequence,*

which solves a question that goes back to Vogt [7]. Moreover, THEOREM. *If* $0 \to Y^* \to X \to R \to 0$ *is an exact sequence, where* R *is reflexive, then it is a dual sequence.*

Related to this is the question: is the property of being complemented in its bidual a three-space property? This question is also open. A simplification of the preceding argument shows:

THEOREM. *If* $0 \to Y \to X \to R \to 0$ *is an exact sequence where* Y *is complemented in its bidual and* R *is reflexive then* X *is complemented in its bidual (see also [3]).*

In some cases, there is a positive answer:

THEOREM. *If* $0 \to Y \to X \to Z \to 0$ *is an exact sequence where* Y *is complemented in some dual space and* Z *is an* $\mathcal{L}_\infty$*-space then the sequence splits.*

This contains an old result of Lindenstrauss [6].

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