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Some remarks stemming from Ulam's problem about nearly additive mappings

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Summary. We consider the following problems: Let $(S,+)$ be a (not necessarily commutative) semigroup and let $\rho\colon S \to \mathbb{R}$ be a given "control" functional (which is *not* assumed to be subadditive). Assume that $F\colon S \to \mathbb{R}$ is a "nearly additive" mapping in either of the following ways:

(1) $|F(x+y) - F(x) - F(y)| \leq \rho(x) + \rho(y) - \rho(x+y)$ holds for all $x, y \in S$.

(2) $\left|\sum_{i=1}^{n} F(x_i) - \sum_{j=1}^{m} F(y_j)\right| \leq \sum_{i=1}^{n} \rho(x_i) + \sum_{j=1}^{m} \rho(y_j)$ holds for all n and m whenever x_i and y_j are elements of S such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$.

Must F be "near" to an additive mapping $A\colon S \to \mathbb{R}$ in the sense that

(3) $|F(x) - A(x)| \leq K\rho(x)$ for some K and all $x \in S$?

We prove

Theorem. *Let $(S,+)$ be a semigroup and let K be a fixed number. The following statements are equivalent:*

(a) *For every ρ and every F satisfying* (1) *there is an additive A fulfilling* (3).

(b) *For every ρ and every F satisfying* (2) *there is an additive A fulfilling* (3).

Moreover, for $K = 1$ both (a) *and* (b) *are equivalent to*

(c) *For every $\alpha, \beta\colon S \to \mathbb{R}$ such that α is superadditive (i.e., $\alpha(x+y) \geq \alpha(x) + \alpha(y)$), β is subadditive and $\alpha \leq \beta$, there exists an additive A separating α from β (that is, satisfying $a(x) \leq A(x) \leq \beta(x)$).*

Combining this theorem with previous results of Ger and Gajda and Kominek we obtain that every weakly commutative group satisfies (b) for $K = 1$ and every amenable group satisfies (b) for $K = 2$, which gives a partial answer to a problem suggested by Forti, improving results by Šemrl and Castillo and the present author.

We close the paper with some remarks about vector-valued mappings and the connections between "nearly additive" mappings and the theory of extensions of Banach spaces.

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1. Introduction

This note is largely motivated by an old problem of S. Ulam (see [20] and [11]) which can be stated in a vague manner as follows: under which conditions a "nearly

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additive" mapping must be near to an additive mapping?

The first partial answer to the problem was given by D. H. Hyers [11]. He proved that, for any mapping $F\colon Z \to Y$ acting between Banach spaces which satisfies

$$\|F(x+y) - F(x) - F(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and every $x, y \in Z$, there exists an additive $A\colon Z \to Y$ such that

$$\|F(x) - A(x)\| \leq \varepsilon$$

for every $x \in Z$. We refer the reader to [7] (and also to [12]) for a nice survey on the topic.

Although Ulam's original problem referred to topological groups, in this paper we shall mainly think about scalar-valued and vector-valued mappings defined on semigroups on which a "control" functional is given.

So, let $(S, +)$ be a (not necessarily commutative) semigroup and let Y be a Banach space. In view of Hyers's paper, we consider ε-additive mappings, that is, mappings $F\colon S \to Y$ satisfying an estimate

$$\|F(x+y) - F(x) - F(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and every $x, y \in S$.

There are other ways of considering "near additivity". The following one is inspired by Å. Lima and D. Yost [15] (see also R. Ger [10]). Let ρ be a fixed real-valued non-negative "control" function on S. A mapping $F\colon S \to Y$ is said to be pseudo-additive with respect to ρ if, for all $x, y \in S$, one has

$$\|F(x+y) - F(x) - F(y)\| \leq \rho(x) + \rho(y) - \rho(x+y). \tag{1}$$

(This implies the subadditivity of ρ, i.e., $\rho(x+y) \leq \rho(x) + \rho(y)$.)

Finally, following [4], let us say that F is *zero-additive* with respect to ρ (which is *not* assumed to be necessarily subadditive) if, for every $n, m \in \mathbb{N}$, it satisfies

$$\left\|\sum_{i=1}^{n} F(x_i) - \sum_{j=1}^{m} F(y_j)\right\| \leq \sum_{i=1}^{n} \rho(x_i) + \sum_{j=1}^{m} \rho(y_j), \tag{2}$$

whenever x_i and y_j are such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$.

The basic problem is whether or not a mapping $F\colon S \to Y$ which is approximately additive in some sense (such as (1), (2) or that of Hyers) must be asymptotically additive (with respect to ρ) in the sense that there is an additive mapping $A\colon S \to Y$ such that

$$\|F(x) - A(x)\| \leq K\rho(x) \tag{3}$$

holds for some K and all $x \in S$.

2. Scalar-valued mappings

There are some connections between the various kinds of "near additivity" considered in the introduction. Clearly, Hyers's condition reduces to pseudo-additivity with respect to the subadditive functional defined by $\rho(x) = \varepsilon$ for all $x \in S$. Less obvious is the following.

Lemma 1. *Pseudo-additivity implies zero-additivity.*

Proof. Let us show that a pseudo-additive mapping satisfies the inequality

$$\Big\| F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big\| \le \sum_{i=1}^{n} \rho(x_i) - \rho\Big(\sum_{i=1}^{n} x_i\Big), \tag{4}$$

for each n and all $x_i \in S$. The proof is by induction on n. The statement (4) is trivial for $n = 1$. The induction step is as follows: assume that (4) holds for $n = k$. Let x_i be in S for $1 \le i \le k+1$. Then

$$\begin{aligned}
&\Big\| F\Big(\sum_{i=1}^{k+1} x_i\Big) - \sum_{i=1}^{k+1} F(x_i)\Big\| = \Big\| F\Big(\sum_{i=1}^{k+1} x_i\Big) - F\Big(\sum_{i=1}^{k} x_i\Big) + \Big(\sum_{i=1}^{k} x_i\Big) - \sum_{i=1}^{k+1} F(x_i)\Big\| \\
&\le \Big\| F\Big(\sum_{i=1}^{k+1} x_i\Big) - F\Big(\sum_{i=1}^{k} x_i\Big) - F(x_{k+1})\Big\| + \Big\| F\Big(\sum_{i=1}^{k} x_i\Big) - \sum_{i=1}^{k} F(x_i)\Big\| \\
&\le \rho\Big(\sum_{i=1}^{k} x_i\Big) + \rho(x_{k+1}) - \rho\Big(\sum_{i=1}^{k+1} x_i\Big) + \sum_{i=1}^{k} \rho(x_i) - \rho\Big(\sum_{i=1}^{k} x_i\Big) \\
&= \sum_{i=1}^{k+1} \rho(x_i) - \rho\Big(\sum_{i=1}^{k+1} x_i\Big),
\end{aligned}$$

as desired.

It seems to be interesting to know for which semigroups zero-additivity (or pseudo-additivity) implies asymptotic additivity. For real-valued mappings, we have the following theorem which is the main result of the paper.

Theorem 2. *Let* $(S, +)$ *be a semigroup and let* $K \ge 1$ *be a fixed number. The following statements are equivalent:*

(a) *For each* ρ *and every real-valued* F *which is pseudo-additive with respect to* ρ *there is an additive* $A\colon S \to \mathbb{R}$ *fulfilling* $|F(x) - A(x)| \le K\rho(x)$ *for all* $x \in S$.

(b) *For each* ρ *and every real-valued* F *which is zero-additive with respect to* ρ *there is an additive* $A\colon S \to \mathbb{R}$ *fulfilling* $|F(x) - A(x)| \le K\rho(x)$ *for all* $x \in S$.

Moreover, for $K = 1$ (a) *and* (b) *are both equivalent to*

(c) *For every* $\alpha, \beta\colon S \to \mathbb{R}$ *such that* α *is superadditive,* β *is subadditive and* $\alpha \le \beta$, *there exists an additive* A *separating* α *from* β (*that is, satisfying* $\alpha(x) \le A(x) \le \beta(x)$).

Recall that a mapping $\alpha\colon S \to \mathbb{R}$ is said to be superadditive if $\alpha(x+y) \geq \alpha(x) + \alpha(y)$ for every $x, y \in S$.

Proof. That (b) implies (a) for any K is clear after Lemma 1. We prove now that (c) implies (b) for $K = 1$. We need some notation. For a real-valued mapping a on a semigroup S, define

$$a^*(x) = \sup\left\{\sum_i a(x_i) : x = \sum_{i=1}^n x_i\right\},$$
$$a_*(x) = \inf\left\{\sum_i a(x_i) : x = \sum_{i=1}^n x_i\right\}.$$

Obviously, $a \leq a^*$ and $a_* \leq a$. Moreover, a^* is superadditive and a_* is subadditive.

Suppose that S has property (c) and let $F\colon S \to \mathbb{R}$ be zero-additive with respect to ρ. We claim that $(F-\rho)^*(x) \leq (F+\rho)_*(x)$ for all $x \in S$ (which implies that both functions take only finite values). Indeed, let $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$. One has to verify that

$$\sum_{i=1}^n F(x_i) - \sum_{i=1}^n \rho(x_i) \leq \sum_{j=1}^m \rho(y_j) + \sum_{j=1}^m F(y_j)$$

or, in other words, that

$$\sum_{i=1}^n F(x_i) - \sum_{j=1}^m F(y_j) \leq \sum_{j=1}^m \rho(y_j) + \sum_{i=1}^n \rho(x_i),$$

which immediately follows from zero-additivity. The hypothesis on S implies the existence of an additive A separating $(F-\rho)^*$ from $(F+\rho)_*$. Hence $F(x)-\rho(x) \leq A(x) \leq F(x) + \rho(x)$ for all $x \in S$, i.e.,

$$|F(x) - A(x)| \leq \rho(x),$$

and, thus, S has property (b) for $K = 1$.

It remains to show that the property (a) for $K = 1$ implies the separation property (c). We need a way to translate a problem about separation into an equivalent problem about approximation of pseudo-additive mappings by additive mappings. So, let α and β be respectively superadditive and subadditive mappings such that $\alpha \leq \beta$. Define

$$F = \frac{\alpha+\beta}{2},$$
$$\rho = \frac{\beta-\alpha}{2}.$$

Obviously ρ is non-negative and subadditive. The pseudo-additivity of F with respect to ρ follows from the hypotheses on α and β by routine computations. Finally, observe that $|F - A| \leq \rho$ implies that A separates α from β. Hence property (a) for $K = 1$ implies (c).

To complete the proof we show that (a) implies (b) for any $K \geq 1$. Assume that S has property (a) for some $K \geq 1$ and let F be zero-additive with respect to a given ρ. Put

$$\alpha = (F - \rho)^*, \qquad \beta = (F + \rho)_*.$$

As above, α is superadditive, β is subadditive and $\alpha \leq \beta$. Hence taking

$$G = \frac{\alpha + \beta}{2} = \frac{1}{2}\{(F - \rho)^* + (F + \rho)_*\},$$
$$\sigma = \frac{\beta - \alpha}{2} = \frac{1}{2}\{(F + \rho)_* - (F - \rho)^*\}$$

one obtains that G is pseudo-additive with respect to σ. The hypothesis gives an additive A such that $|G - A] \leq K\sigma$. This can be written as

$$\frac{1}{2}\{(1-K)(F+\rho)_* + (1+K)(F-\rho)^*\} \leq A \leq \frac{1}{2}\{(1+K)(F+\rho)_* + (1-K)(F-\rho)^*\}.$$

Since $K \geq 1$, taking into account that $(F + \rho)_* \leq F + \rho$ and $(F - \rho)^* \geq F - \rho$ one obtains that

$$\frac{1}{2}\{(1 - K)(F + \rho) + (1 + K)(F - \rho)\} \leq A \leq \frac{1}{2}\{(1 + K)(F + \rho) + (1 - K)(F - \rho)\}$$

also holds. Rearranging the parentheses, one has

$$F - K\rho \leq A \leq F + K\rho$$

which completes the proof. □

Remarks. 1. No semigroup has property (a) (nor (b)) for $K < 1$: given a semigroup S, consider the constant mapping $F = 1$ which is pseudo-additive (hence zero-additive) with respect to $\rho = 1$. It is easily seen that there is no additive $A\colon S \to \mathbb{R}$ such that $|F(x) - A(x)| \leq K$ for all $x \in S$ for any $K < 1$.

2. The following example, due to G. L. Forti, shows that not every ε-additive mapping (let alone every pseudo-additive or zero-additive mapping) is asymptotically additive, even if we assume $Y = \mathbb{R}$.

Let $\mathbb{F}(a, b)$ be the free semigroup with two generators a and b, the operation being superposition. For $x \in \mathbb{F}(a, b)$, let $F(x)$ be the number of pairs of the form ab in x. It is easily seen that F is 1-additive, but obviously $F - A$ is not bounded on $\mathbb{F}(a, b)$ for any additive $A\colon \mathbb{F}(a, b) \to \mathbb{R}$. See [6] or [12] for details.

Applying Theorem 2, some results about separation of subadditive and superadditive functionals can be read as well as results about approximation of pseudo-additive or zero-additive real-valued mappings by additive functionals.

Recall that S is said to be weakly commutative if for every $x, y \in S$ there is $n \in \mathbb{N}$ such that

$$2^n(x + y) = 2^n x + 2^n y.$$

Gajda and Kominek proved in [9] that every weakly commutative *group* has the separation property (c) of Theorem 2, generalizing previous results of Mazur and Orlicz [16], Sikorski [19], Ptak [17], Kaufman [13] and Kranz [14]. Thus, we obtain:

Corollary 3. *Weakly commutative groups have the approximation properties* (a) *and* (b) *of Theorem 2 for* $K = 1$.

We do not know if this remains true for semigroups.

Corollary 3 is in a sense a best result: it asserts that, given a "control" functional ρ on a weakly commutative group, for every zero-additive F (with respect to ρ) there exists an additive mapping A such that $|F(x) - A(x)| \leq \rho(x)$. The converse holds in every semigroup, since, given a real-valued mapping F, the existence of an additive A such that $|F(x) - A(x)| \leq \rho(x)$ clearly implies that F is zero-additive with respect to ρ.

Remark. Observe that, for a mapping F on an arbitrary semigroup S, the statement

$$\left\| F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i) \right\| \leq \sum_{i=1}^{n} \rho(x_i) \tag{*}$$

is weaker than pseudo-additivity (see Lemma 1) and stronger than zero-additivity. Hence, Corollary 3 shows that for weakly commutative groups the condition (*) implies the existence of some additive A such that $|F(x) - A(x)| \leq \rho(x)$. We know from [7] that P. Šemrl proved in [18] that for any *continuous* mapping $F\colon \mathbb{R} \to \mathbb{R}$ satisfying $\big|F\big(\sum_{i=1}^n t_i\big) - \sum_{i=1}^n F(t_i)\big| \leq \sum_{i=1}^n |t_i|$ there is an additive A such that $|F(t) - A(t)| \leq |t|$ for all t. Corollary 3 and Corollary 5 below improve this result (as well as Theorem 1 of [2]) and give a partial answer to a problem suggested by Forti in [7, p. 11].

Another generalization of commutativity is amenability (which involves analytic properties of the dual space of the Banach space $l_\infty(S, \mathbb{R})$, see [5]). In [8] it is proved that, if S is an amenable semigroup and α and β are respectively superadditive and subadditive functionals such that $\alpha \leq \beta$, then there is an additive A such that $\alpha \leq A \leq \beta$ provided either

$$\sup\{|\alpha(x + y) - \alpha(y + x)| : x, y \in S\}$$

or

$$\sup\{|\beta(x + y) - \beta(y + x)| : x, y \in S\}$$

is finite.

Since the meaning of these assumptions in the setting of approximation properties is not clear, let us examine approximation properties for amenable semigroups.

We need the following result presented in [2]. Since [2] was written we have discovered from [7] that a similar result was early proved by Ger.

Theorem 4. (Ger). *Let $F\colon S \to Y$ be a pseudo-additive mapping with respect to some subadditive $\rho\colon S \to \mathbb{R}$. If S is an amenable group and Y is complemented in its second dual, then there is an additive $A\colon S \to Y$ such that*

$$\|F(x) - A(x)\| \leq K\rho(x)$$

holds for some K and every $x \in S$. Moreover K can be taken as $2\|P\|$, where P is any projection from Y^{**} onto Y.

Combining Theorem 2 with Theorem 4, one has

Corollary 5. *Amenable groups have the approximation properties* (a) *and* (b) *of Theorem 2 for $K = 2$.*

Remark. We do not know whether or not the constant 2 can be replaced by 1 in Theorem 4 (or in its Corollary 5). The answer is affirmative if only bounded "control" functionals are considered. In the scalar case, this problem is equivalent to know whether or not amenable groups have property (c) of Theorem 2 (see [9]).

3. Vector-valued mappings, examples and counter-examples

In this section we deal with nearly additive mappings taking values in a (real) Banach space.

Important examples of pseudo-additive and zero-additive mappings appear in connection with extensions of Banach spaces. Recall from [4] that, given Banach spaces Y and Z, an extension of Y by Z is a Banach space X having a subspace linearly isomorphic to Y whose corresponding quotient is linearly isomorphic to Z. An extension X of Y by Z can be viewed as a short exact sequence of Banach spaces and (linear continuous) operators

$$0 \to Y \to X \to Z \to 0.$$

When such a sequence is given, a zero-additive mapping from Z to Y can be obtained as follows. Let $B\colon Z \to X$ be a homogeneous bounded selection for the quotient map $X \to Z$ (i.e., satisfying $\|B(z)\| \leq K\|z\|$ for some K and all $z \in Z$). We do not assume either continuity or linearity for B. Let $L\colon Z \to X$ be a linear (not necessarily continuous) selection for the quotient map $X \to Z$. Clearly, the difference $F = B - L$ takes values in Y instead of X.

Now, let x_i and y_j be points of Z such that $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$. One has

$$\left\| \sum_{i=1}^n F(x_i) - \sum_{j=1}^m F(y_j) \right\| = \left\| \sum_{i=1}^n B(x_i) - \sum_{j=1}^m B(y_j) \right\| \le K \left[\sum_i \|x_i\| + \sum_j \|y_j\| \right],$$

so that $F\colon Z \to Y$ is zero-additive with respect to $K\|\cdot\|$ and homogeneous.

This mapping $F\colon Z \to Y$ characterizes X as an extension of Y by Z (actually X as a linear topological space can be obtained by means of F [1], see below). For instance, the space X is the trivial extension (i.e., $X = Y \oplus Z$) if and only if F is at finite distance from additive mappings in the sense that there exists an additive $A\colon Z \to Y$ (which must be necessarily homogeneous but not generally bounded) such that

$$\|F(x) - A(x)\| \le M\|x\|$$

for some M and all $x \in Z$ (see [1] and references therein). In this way, every uncomplemented subspace Y of a Banach space X gives a Y-valued zero-additive mapping on X/Y (with respect to some scalar multiple of the norm of X/Y) which cannot be near to any additive mapping from X/Y to Y. Thus, all results of Section 2 (exception made of Lemma 1) are false for vector valued mappings even if we assume that S is a commutative group.

Suppose now that a homogeneous zero-additive $F\colon Z \to Y$ (with respect to some scalar multiple of the norm of Z) is given. Then an extension (termed $Y \oplus_F Z$) of Y by Z can be obtained considering the product space $Y \times Z$ equipped with the quasi-norm

$$\|(y,z)\|_F = \|y - Fz\| + \|z\|.$$

(Observe that $\{(y,0)\colon y \in Y\}$ is a subspace of $Y \oplus_F Z$ isometric to Y whose corresponding quotient is isometric to Z.) The functional $\|(\cdot,\cdot)\|_F$ may not be a norm: it is only a quasi-norm, but it is always equivalent to a norm provided F is zero-additive [1]. Sometimes $\|(\cdot,\cdot)\|_F$ is a norm: we know from [15] that $\|(\cdot,\cdot)\|_F$ is a norm if and only if F is pseudo-additive with respect to the norm of Z. In this case Y becomes a semi-L-summand in $Y \oplus_F Z$ and, in fact, every semi-L-summand can be obtained in this way.

The following problem is open.

Problem. [15] Let $F\colon Z \to Y$ be homogeneous and pseudo-additive with respect to the norm of Z. Must F be at finite distance from some additive mapping from Z to Y?

A negative answer to this problem is equivalent to the existence of an uncomplemented semi-L-summand (or an absolutely Chebyshev subspace) of a Banach space.

In connection with this problem we know some "uniform boundedness principles" for nearly additive mappings. The following one can be derived easily from [3]: let Y be a Banach space and let S be a vector space over $\mathbb{Q}$. Suppose that every zero-additive $\mathbb{Q}$-homogeneous mapping $F\colon S \to Y$ (with respect to a fixed $\rho : S \to \mathbb{R}$) is at finite distance from some $\mathbb{Q}$-linear mapping $S \to Y$. Then there is a constant C such that every zero-additive $\mathbb{Q}$-homogeneous mapping $S \to Y$ (with respect to ρ) is at distance at most C from $\mathbb{Q}$-linear mappings $S \to Y$.

We do not know if a similar result is true for general semigroups equipped with a fixed "control" function ρ (for the case $\rho =$ constant, the main theorem of [8] shows that the answer is affirmative).

We close the paper with a remark about the structure of real-valued homogeneous pseudo-additive mappings with respect to a symmetric subadditive functional in the spirit of [15]. In some sense, the only semi-L-summand we know is $\mathbb{R}$ viewed as constants in $C(K)$. The corresponding pseudo-additive homogeneous mapping $F\colon C(K)/\mathbb{R} \to \mathbb{R}$ is given by

$$F(f) = \frac{1}{2}\{\max f(K) + \min f(K)\}$$

(this mapping is well-defined from $C(K)/\mathbb{R}$ and not just from $C(K)$). It is easily seen that F is pseudo-additive with respect to

$$\rho(f) = \frac{1}{2}\{\max f(K) - \min f(K)\}$$

which is the (quotient) norm of $C(K)/\mathbb{R}$. This gives ρ and F respectively as the "even" and "odd" parts of the mapping $\beta(f) = \max f(K)$. We have the following result asserting that essentially this is the only way in which such a pseudo-additive mapping can be given.

Proposition. *Let S be a group. Assume that $F\colon S \to \mathbb{R}$ is pseudo-additive with respect to ρ. Moreover, assume that F is homogeneous in the sense that $F(-x) = -F(x)$ and that ρ is symmetric (i.e., $\rho(-x) = \rho(x)$). Then there is a unique subadditive real-valued β for which*

$$F(x) = \frac{1}{2}\{\beta(x) - \beta(-x)\}, \qquad \rho(x) = \frac{1}{2}\{\beta(x) + \beta(-x)\}$$

holds for every $x \in S$.

Proof. Simply define $\beta(x) = F(x) + \rho(x)$.

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