MAXIMAL SYMMETRIC NORMS ON BANACH SPACES

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Abstract

We study norms and quasi-norms having large groups of isometries (uniquely maximal and almost transitive). It is shown that a uniquely maximal norm on a Banach space is its 'best' equivalent norm with respect to several concepts, such as uniform convexity and smoothness. A similar result for quasi-norms (on non-locally convex) spaces is given. We analyse the connection between uniquely maximal and almost transitive norms: it is proved that both properties coincide for super-reflexive spaces. The existence of an equivalent (quasi-) norm with many symmetries on a given (quasi-) Banach space has a considerable impact on the underlying linear topological structure: for example, an almost transitive Banach space having the Radon–Nikodým property must be super reflexive. A uniquely maximal quasi-normed space having a non-zero bounded linear functional is necessarily a normed space. Also, a uniquely maximal quasi-Banach space has exact type (at least when it is not super reflexive). This gives examples of Banach spaces that do not admit an almost transitive renorming and quasi-Banach spaces (including some Orlicz function spaces) that are not even quotients of any uniquely maximal quasi-Banach space.

0. Introduction

We deal in this paper with norms having a large group of symmetries. There are various ways of measuring the 'size' of the group of symmetries of a given norm on a Banach space [16], [15]. For example, the norm is called maximal if no equivalent norm gives a larger group of isometries. In general we cannot say that a maximal norm is *the* most symmetric norm because other equivalent norms with the same group of isometries could exist. We are concerned with norms that are determined by their groups of symmetries in the following sense.

Definition 0.1. A norm is said to be uniquely maximal if (it is maximal and) there is no equivalent norm with the same group of isometries, apart from its scalar multiples.

Obviously, one can replace 'equivalent' by 'weaker' in the definition. Cowie [4] characterised uniquely maximal norms by means of the action of the group of isometries on the unit sphere; he proved that a norm is uniquely maximal if and only if the convex hull of the orbits under the action of the group of isometries on the unit sphere is dense in the unit ball (that is to say, it is convex transitive). We are also interested in other stronger properties.

Definition 0.2. A norm is said to be almost transitive if its group of isometries acts almost transitively (i.e. with dense orbits) on the unit sphere. A norm is transitive if the group of isometries acts transitively on the unit sphere.

We refer the reader to [15, chapter 9], [16], [2] and the references given therein for examples, basic facts and questions concerning maximal, uniquely maximal, almost transitive and transitive norms on Banach spaces.

We now describe the results of the paper. Firstly we study geometric properties of uniquely maximal norms. It is shown (in a particularly simple manner) that a uniquely maximal norm on a Banach space is the 'best' equivalent norm with respect to several concepts such as uniform convexity and smoothness. As a consequence, every uniquely maximal norm on a super-reflexive Banach space is both uniformly convex and uniformly smooth.

We then apply the previous results to prove that almost transitivity and unique maximality are equivalent in the setting of super-reflexive Banach spaces. This equivalence is no longer true if we omit super reflexivity. We show that all properties previously considered are self-dual in reflexive spaces. This is obvious for uniquely maximal and maximal norms but not for almost transitive or transitive norms.

Finally, we consider uniquely maximal and almost transitive quasi-norms showing that most of the non-locally convex locally bounded Orlicz sequence spaces, and some Orlicz function spaces, are not uniquely maximal for any equivalent quasi-norm.

1. Uniform convexity and smoothness

Let $\|\cdot\|$ be a norm on a (real or complex) vector space X. The modulus of convexity of $\|\cdot\|$ is the function

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\{1-\tfrac{1}{2}\|x+y\| \cdot \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\},$$

defined for $0 \leqslant \varepsilon \leqslant 2$. We shall write $\delta_X(\varepsilon)$ instead of $\delta_{\|\cdot\|}(\varepsilon)$ when there is no risk of confusion. The modulus of convexity of a norm measures, in a certain sense, the degree of strict convexity of its unit ball. In fact, a space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$. It is well known that super reflexivity is the isomorphic property that corresponds to uniform convexity.

Consider a set Γ of norms defined on a given vector space X, k-uniformly equivalent (for some $0 < k \leqslant 1$) in the sense that

$$k\|x\|_1 \leqslant \|x\|_2$$

holds for every $x \in X$ and all norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in Γ. Then Γ has a least upper bound among the norms of X, which is given by

$$\Gamma_{\sup}(x) = \sup\{\|x\| : \|\cdot\| \in \Gamma\}.$$

Clearly, the unit ball of $\Gamma_{\sup}(\cdot)$ coincides with the intersection of the unit balls of the norms of Γ. The following lemma generalises the obvious fact that the intersection of two strictly convex planar figures is again strictly convex.

Lemma 1.1. *Let Γ be uniformly equivalent with constant k. Then the modulus of convexity of $\Gamma_{\sup}(\cdot)$ satisfies the estimate $\delta_{\sup}(\varepsilon) \geqslant \inf\{\delta_{\|\cdot\|}(k\varepsilon) : \|\cdot\| \in \Gamma\}$.*

PROOF. It is a straightforward computation:

$$\begin{aligned}\delta_{\sup}(\varepsilon) &= 1-\tfrac{1}{2}\sup\{\Gamma_{\sup}(x+y)\colon \Gamma_{\sup}(x)\leqslant 1, \Gamma_{\sup}(y)\leqslant 1, \Gamma_{\sup}(x-y)\geqslant \varepsilon\}\\ &= 1-\tfrac{1}{2}\sup\{\|x+y\|: \Gamma_{\sup}(x)\leqslant 1, \Gamma_{\sup}(y)\leqslant 1, \Gamma_{\sup}(x-y)\geqslant \varepsilon, \|\cdot\|\in\Gamma\}\\ &\geqslant 1-\tfrac{1}{2}\sup\{\sup\{\|x+y\|: \|x\|\leqslant 1, \|y\|\leqslant 1, \|x-y\|\geqslant k\varepsilon\}: \|\cdot\|\in\Gamma\}\\ &\geqslant \inf\{\delta_{\|\cdot\|}(k\varepsilon): \|\cdot\|\in\Gamma\}. \quad \blacksquare\end{aligned}$$

Theorem 1.2. *Let $\|\cdot\|$ be a uniquely maximal norm on a vector space X. Then its modulus of convexity is (asymptotically) sharp in the following sense: if $\|\cdot\|_1$ is another equivalent norm on X then there is $0<k\leqslant 1$ such that $\delta_{\|\cdot\|}(\varepsilon)\geqslant \delta_{\|\cdot\|_1}(\mathrm{k}\varepsilon)$ for every $0\leqslant\varepsilon\leqslant 2$.*

PROOF. Let $\|\cdot\|_1$ be equivalent to $\|\cdot\|$. Consider the set of norms given by $\Gamma = \{\|T(\cdot)\|_1: T\in G(\|\cdot\|)\}$, where $G(\|\cdot\|)$ is the group of all isometric automorphisms of $(X,\|\cdot\|)$. It is obvious that the modulus of convexity of every norm in Γ coincides with that of $\|\cdot\|_1$. Moreover, $\Gamma_{\sup}(\cdot)$ is a norm equivalent to $\|\cdot\|$ and clearly $G(\Gamma_{\sup}(\cdot))$ contains $G(\|\cdot\|)$. The hypothesis implies that $\Gamma_{\sup}(\cdot)$ and $\|\cdot\|$ are proportional, and thus their moduli of convexity coincide. Now Lemma 1.1 gives

$$\delta_{\|\cdot\|}(\varepsilon) = \delta_{\sup}(\varepsilon) \geqslant \inf_{T\in G(\|\cdot\|)} \delta_{\|T(\cdot)\|_1}(k\varepsilon) = \delta_{\|\cdot\|_1}(k\varepsilon),$$

where k must be taken so that $k\|T(\cdot)\|_1\leqslant \|L(\cdot)\|_1$ for all T and L in $G(\|\cdot\|)$. Thus the choice

$$k = \inf\{\|T\|_1/\|R\|_1: T, R\in G(\|\cdot\|)\} > 0$$

completes the proof. ■

We have the following (apparent) improvement of [8]. Recall from [13] that the norm of X is said to have modulus of convexity of power type p if there is a constant c such that $\delta_X(\varepsilon)\geqslant c\varepsilon^p$ for ε sufficiently small.

Corollary 1.3. *Let X be a Banach space having an equivalent norm with modulus of convexity of power type p. Then every uniquely maximal renorming of X has modulus of convexity of power type p. Consequently, every uniquely maximal norm on a super-reflexive Banach space is uniformly convex.* ■

We pass to smoothness properties. Consider now the modulus of smoothness

$$\rho_X(\tau) = \sup\{\tfrac{1}{2}[\|x+y\|+\|x-y\|]-1: \|x\|\leqslant 1, \|y\|\leqslant \tau\}.$$

A standard convexity argument shows that the norm of X is uniformly Fréchet differentiable on the unit sphere if and only if $\rho_X(\tau)/\tau\to 0$ as $\tau\to 0$. The modulus of smoothness is in some sense dual to that of convexity. In fact one has

$$\rho_X(\tau) = \sup_{0\leqslant\varepsilon\leqslant 2}\left\{\frac{\tau\varepsilon}{2}-\delta_{X^*}(\varepsilon)\right\}; \rho_{X^*}(\tau) = \sup_{0\leqslant\varepsilon\leqslant 2}\left\{\frac{\tau\varepsilon}{2}-\delta_X(\varepsilon)\right\}. \qquad (*)$$

In this way, the modulus of smoothness of X^* measures the uniform convexity of X and vice versa.

Let Γ be a set of uniformly equivalent norms on a vector space X. Then Γ also has a greatest lower bound (among the norms of X) whose unit ball is the closed convex hull of the unit balls of the norms in Γ. If $\Gamma = \{\|\cdot\|_\gamma\}$, the infimum is given by

$$\Gamma_{\inf}(x) = \inf\{\textstyle\sum_\gamma \|x_\gamma\|_\gamma : x = \Sigma_\gamma x_\gamma\}.$$

Using (*), and taking into account that the dual norm of the greatest lower bound of a set of norms is the least upper bound of the set of dual norms, it is very easy to obtain the following dualisation of Lemma 1.1

Lemma 1.4. *Let Γ be uniformly equivalent with constant k. Then the modulus of smoothness of $\Gamma_{\inf}(\cdot)$ satisfies $\rho_{\inf}(k\tau) \leqslant \sup\{\rho_{\|\cdot\|}(\tau) : \|\cdot\| \in \Gamma\}$.*

Now we have the following.

Theorem 1.5. *Let $\|\cdot\|$ be a uniquely maximal norm on X. Then its modulus of smoothness is (asymptotically) sharp in the following sense: if $\|\cdot\|_1$ is any equivalent norm on X then there is $0 < k \leqslant 1$ such that $\rho_{\|\cdot\|}(k\tau) \leqslant \rho_{\|\cdot\|_1}(\tau)$ for every $0 \leqslant \tau \leqslant 1$.*

As in the above, the norm of X is said to have modulus of smoothness of power type q if there is a constant C such that $\rho_X(\tau) \leqslant C\tau^q$ for τ sufficiently small.

Corollary 1.6. *Let X be a Banach space having an equivalent norm with modulus of smoothness of power type q. Then every uniquely maximal renorming of X has modulus of smoothness of power type q. Consequently, every uniquely maximal norm on a super-reflexive Banach space is uniformly Fréchet differentiable on the unit sphere.*

Remark 1.7. Most of the classical Banach spaces have large groups of isometries. For instance every Banach space with a symmetric basis is either isomorphic to l_2 (which is transitive) or maximal. The L_p spaces are almost transitive in their natural norms for $1 \leqslant p < \infty$ [15]. The space L_∞ is not almost transitive, but it is uniquely maximal since it is convex transitive. Moreover, there are almost transitive L_1-preduals such as the Gurarij space [10]. Spaces of continuous functions are studied in [12], [16], [9] and [3]. Most of them are uniquely maximal under their natural norms and some (such as $C[0, 1]$) admit an almost transitive renorming. Also, there exist transitive renormings of $C(K)$ spaces. Furthermore, Lusky proved in [14] that every Banach space is a complemented subspace of some almost transitive Banach space. So we are interested in the following.

Question 1.8. Does every Banach space admit (*a*) a maximal, (*b*) a uniquely maximal, (*c*) an almost transitive or (*d*) a transitive norm?

2. Uniquely maximal norms on super-reflexive spaces

The following result answers in the negative parts (*c*) and (*d*) of Question 1.8.

Theorem 2.1. *Let X be an almost transitive Banach space. The following statements are equivalent: (a) X is an Asplund space; (b) X has the Radon–Nikodým property; (c) X is reflexive; (d) X is super reflexive.*

PROOF. (For background on Asplund spaces and the Radon–Nikodým property see, respectively, [17] and [7].) Obviously (*d*) implies (*c*), and (*c*) implies (*a*) and (*b*). That (*a*) implies (*d*) is proved in [2]. It remains to prove that (*b*) implies (*d*).

Recall from [5] that a point x belonging to a closed convex bounded set C is called a strongly extreme point if for each $\varepsilon > 0$ there is $\eta > 0$ such that $\|y-z\| \leqslant \varepsilon$ whenever y and z are points in C with $\|y+z-2x\| \leqslant \eta$. It is easy to see that a Banach space has the Radon–Nikodým property if and only if each of its bounded closed convex subsets has strongly extreme points.

Now let x be a point of the unit sphere of a Banach space X. We define its modulus of strong extremality as the function (of $0 \leqslant \varepsilon \leqslant 1$)

$$\lambda(x, \varepsilon) = \inf\{1-\rho\colon \text{there is } y \in X \text{ such that } \|y\| \geqslant \varepsilon \text{ and } \|\rho x \pm y\| \leqslant 1\}.$$

This function characterises the strongly extreme points of the unit ball and measures its degree of strong extremality in the following sense: a point $x \in S(X)$ (the unit sphere of X) is a strongly extreme point of $B(X)$ (the unit ball of X) precisely when $\lambda(x, \varepsilon) > 0$ for every $\varepsilon > 0$. The behaviour of the modulus of strong extremality on the unit sphere of a Banach space is related to the uniform convexity of the norm. In fact, the following identity (see [1]) holds in every Banach space:

$$\delta_X(2\varepsilon) = \inf_{\|x\|=1} \lambda_X(x, \varepsilon).$$

We are ready to show that an almost transitive norm on a Banach space having the Radon–Nikodým property is uniformly convex. Since in a Banach space with the Radon–Nikodým property there is some ξ in the unit sphere such that $\lambda(\xi, \varepsilon) > 0$ for every $\varepsilon > 0$, one only has to show that, in an almost transitive Banach space, the value of $\lambda(x, \varepsilon)$ does not depend on x.

To this end we need some computations. Let $x \in S(X)$. Fix $\varepsilon > 0$. Obviously, for each $\rho > 0$ with $1-\rho > \lambda(x, \varepsilon)$ there is $\|y\| \geqslant \varepsilon$ such that $\|\rho x \pm y\| \leqslant 1$. If $u \in S(X)$ it is clear that $\|\rho u \pm y\| \leqslant 1+\rho\,\|u-x\|$, hence $\|\rho u \pm y\|/(1+\rho\,\|u-x\|) \leqslant 1$. It follows that

$$\lambda\left(u, \frac{\varepsilon}{1+\rho\,\|u-x\|}\right) \leqslant 1 - \frac{\rho}{1+\rho\,\|u-x\|}.$$

Finally, if X is assumed to be almost transitive and z is another point in $S(X)$ then there exists a sequence (z_n) converging to x in the orbit of z. Since $\lambda(\cdot,\cdot)$ is invariant under isometries of the first variable, one has $\lambda(z_n, \cdot) = \lambda(z, \cdot)$ for all n, hence

$$\lambda\left(z, \frac{\varepsilon}{1+\rho\,\|z_n-x\|}\right) = \lambda\left(z_n, \frac{\varepsilon}{1+\rho\,\|z_n-x\|}\right) \leqslant 1 - \frac{\rho}{1+\rho\,\|z_n-x\|}.$$

Since $\lambda(\cdot,\cdot)$ is non-decreasing and left-continuous in the second variable this implies that

$$\lambda(z, \varepsilon) = \lambda(z, \varepsilon^-) \leqslant 1-\rho,$$

and, taking the greatest lower bound for the right-hand side, we obtain the inequality $\lambda(z, \varepsilon) \leqslant \lambda(x, \varepsilon)$; and thus, by symmetry, $\lambda(x, \varepsilon) = \lambda(z, \varepsilon)$, as desired. ■

Examples 2.2. The following spaces do not have an almost transitive renorming: c_0, l_1, $l_p \hat{\otimes} l_p (1 < p < \infty)$, $l_2(l_1^n)$, $C(K)$ and all its (infinite dimensional) subspaces for K dispersed, quasi-reflexive spaces, the Tsirelson's space, its dual and their (infinite dimensional) subspaces.

Surprisingly, we have the following.

Theorem 2.3. *Every uniquely maximal norm on a super-reflexive Banach space is almost transitive.*

Proof. Let X be a uniquely maximal super-reflexive Banach space. By 1.3, X is uniformly convex and every point of $S(X)$ is an extreme point of $B(X)$. Fix $x \in S(X)$. Since the norm is convex transitive the (norm) closed convex hull of the orbit $G(X)x$ contains $B(X)$ and the same happens with respect to the weak topology. Now, reflexivity implies that $B(X)$ is weakly compact and the Krein–Milman theorem gives that the weak closure of $G(X)x$ contains the set of all extreme points of $B(X)$, i.e. the whole of $S(X)$. Finally, $G(X)x$ is also norm dense in $S(X)$ because norm and weak topologies coincide on the unit sphere of a super-reflexive space. ■

Corollary 2.4. *A reflexive space X is maximal (respectively uniquely maximal, almost transitive or transitive) if and only if X* is also maximal.*

Proof. Clearly it suffices to prove the 'only if' parts. If we observe that groups of isometries are nothing but bounded subgroups of the group of all linear automorphisms, then all is clear for maximal and uniquely maximal reflexive spaces, because the bounded subgroups of $\mathrm{Aut}(X)$ are in exact correspondence (by transposition) with those of $\mathrm{Aut}(X^*)$ and the norms on X are also in exact correspondence with those of X^*. The almost transitive case follows immediately from Theorems 2.1 and 2.3 and the uniquely maximal case.

The assertion concerning transitivity is less clear, since transitivity is defined in terms of the action of the group of isometries on the unit sphere and not by properties of the group of isometries itself. Fortunately, the results of the first section provide us with a suitable connection between $S(X)$ and $S(X^*)$. (The following proof also works in the almost transitive case without 2.3.) First of all, observe that a reflexive (almost) transitive Banach space is necessarily super reflexive. So let X be an (almost) transitive super-reflexive Banach space. Then both X and X^* are uniformly convex and uniformly smooth and thus the duality mapping $J\colon S(X) \to S(X^*)$ given by

$$J(x) = \{x^* \in S(X^*) : x^*(x) = 1\}$$

is well defined, single-valued and, in fact, a uniform homeomorphism. It is easy to see that $T^*JT = J$ holds for every $T \in G(X)$, from which it follows that J carries the orbits of $G(X)$ into exactly that of $G(X^*)$. Thus $S(X)$ is an orbit of $G(X)$ precisely when $S(X^*)$ is an orbit of $G(X^*)$ (in the almost transitive case, every orbit of $G(X)$ is norm dense in $S(X)$ if and only if every orbit of $G(X^*)$ is norm dense in $S(X^*)$) and the proof is complete. ■

Remark 2.5. Corollary 2.4 is false for arbitrary Banach spaces. Let X be a transitive Lindenstrauss space (say an ultrapower of the Gurarij space [10]). Then X^* cannot be uniquely maximal, since X^* is isometrically representable as $l_1(\Gamma) \oplus_1 L_1(\mu)$, where Γ is a non-void set and μ is a non-atomic measure. Writing the elements of X^* as pairs (f, g) with $f \in l_1(\Gamma)$ and $g \in L_1(\mu)$ one can define a new norm, putting

$$\|(f, g)\|_{new} = \|f\|_{l_1(\Gamma)} + \|f\|_{l_2(\Gamma)} + \{\|g\|_{L_1(\mu)}.$$

This norm is equivalent to the original dual norm of X^* and has the same group of symmetries.

Remark 2.6. From Theorem 2.1 we know that there exist Banach spaces that do not admit an almost transitive renorming. Unfortunately, none of them is super reflexive. So we have another problem.

Problem 2.7. Does every super-reflexive Banach space admit an almost transitive (transitive) norm?

This problem is implicit in [6], where it is remarked that a positive answer to it would provide a positive answer to the following.

Problem 2.8. [6, p. 176] If a Banach space admits a norm with modulus convexity of power type p and admits a norm with modulus of smoothness of power type q, does it admit a norm that simultaneously shares both these properties?

Clearly, an answer in the affirmative to part (*b*) of Question 1.8 would solve these problems.

3. Almost transitive and uniquely maximal quasi-norms

There is no clear intrinsic reason to restrict attention to the locally convex setting in a study of symmetries. In fact, some of the early examples of almost transitive spaces (the L_p spaces for $0 < p < 1$) are quasi-normed non-locally convex spaces. Recall from [11] that a quasi-norm on a (real or complex) vector space X is a non-negative real-valued function on X satisfying:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$;
(iii) $\|x + y\| \leqslant K[\|x\| + \|y\|]$ for some fixed $K \geqslant 1$ and all $x, y \in X$.

A quasi-normed space is a vector space X together with a specified quasi-norm. On such a space one has a (linear) topology defined as the smallest linear topology for which the set

$$B = \{x \in X: \|x\| \leqslant 1\}$$

(the unit ball of X) is a neighbourhood of 0. In this way, X becomes a locally bounded space (i.e. it has a bounded neighbourhood of 0); and, conversely, every locally bounded topology on a vector space comes from a quasi-norm. A quasi-Banach space is a complete quasi-normed space.

The main problem with quasi-norms is that a quasi-norm can be discontinuous with respect to the topology it itself induces. This has some unpleasant consequences. Consider the standard quasi-norm on the space $L_p(0,1)$ defined by

$$\|f\|_p = \left[\int_0^1 |f(t)|^p \, dt\right]^{1/p}.$$

Rolewicz and Pelczynski proved that this quasi-norm is almost transitive [15]. Consider also the new quasi-norm defined as $\|f\|_{new} = \|f\|_p$ if f vanishes on a set of positive measure and $\|f\|_{new} = 2\,\|f\|_p$ otherwise. Then $\|\cdot\|_{new}$ and $\|\cdot\|_p$ have the same group of symmetries for every $p \neq 2$. Thus an almost transitive quasi-norm need not be uniquely maximal with respect to all equivalent quasi-norms. In fact a quasi-norm is uniquely maximal with respect to all its equivalent quasi-norms if and only if it is transitive. In the sequel, by a uniquely maximal quasi-norm we mean a quasi-norm that is uniquely maximal with respect to all equivalent continuous quasi-norms.

An interesting class of continuous quasi-norms are the so-called p-norms ($0 < p \leqslant 1$). These are quasi-norms satisfying the inequality $\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$ for all x, y. For a p-norm one has $|\,\|x\|^p - \|y\|^p| \leqslant \|x-y\|^p$, from which continuity follows immediately. A well-known result of Aoki and Rolewicz [15, p. 95] guarantees that every quasi-norm is equivalent to a p-norm for some $0 < p \leqslant 1$.

Proposition 3.2. *Every almost transitive quasi-norm is continuous. Consequently, almost transitive quasi-norms are uniquely maximal* (*in the sense of the previous remark*).

PROOF. Let $\|\cdot\|$ be an almost transitive quasi-norm on X. Take an equivalent p-norm $[\![\cdot]\!]$ and define a quasi-norm on X putting $\|x\|_p = \sup\{[\![Tx]\!] : T \in G(\|\cdot\|)\}$. Clearly $\|\cdot\|_p$ is also a p-norm equivalent to $\|\cdot\|$. We shall show that $\|\cdot\|$ is proportional to $\|\cdot\|_p$. Fix $x \in X$, $\|x\| = 1$. For every y in the unit sphere of $\|\cdot\|$ and $\varepsilon > 0$, there is $T_\varepsilon \in G(\|\cdot\|)$ such that $\|y - T_\varepsilon x\| < \varepsilon$. So, $T_\varepsilon(x) \to y$ as $\varepsilon \to 0$ and the continuity of $\|\cdot\|_p$ implies that $\|T_\varepsilon x\|_p \to \|y\|_p$ as $\varepsilon \to 0$. Since $\|\cdot\|_p$ is invariant under the symmetries of $\|\cdot\|$, we have $\|T_\varepsilon x\|_p = \|x\|_p$ and thus $\|y\|_p = \|x\|_p$ whenever $\|x\| = \|y\| = 1$, from which it follows that $\|\cdot\|$ is a scalar multiple of $\|\cdot\|_p$. Since $\|\cdot\|_p$ is a continuous p-norm so is $\|\cdot\|$.

The second part is obvious. ■

The preceding proof also shows that every uniquely maximal quasi-norm is a p-norm for some $0 < p \leqslant 1$ and thus continuous. We shall prove the following stronger result, which can be understood as 1.2 without local convexity.

Theorem 3.3. Let $\|\cdot\|$ be a uniquely maximal quasi-norm on a vector space X. Then $\|\cdot\|$ is a p_0-norm, where $p_0 = \sup\{p$: there is a non-zero real-valued homogeneous p-convex map on X that is bounded on the unit ball$\}$.

PROOF. Let $f: X \to \mathbb{R}$ be non-zero bounded homogeneous and p-convex with $0 < p \leqslant 1$. Then $F(x) = \sup\{|f(Tx)| : T \in G(\|\cdot\|)\}$ is again a non-zero bounded homogeneous (absolutely) p-convex map and $\|x\|_1 = \|x\| + F(x)$ is a continuous quasi-norm equivalent to $\|\cdot\|$. It is evident that $\|\cdot\|_1$ is invariant under the symmetries of $\|\cdot\|$. The hypothesis concerning $\|\cdot\|$ implies that $\|\cdot\|_1$, and thus $F(\cdot)$, is proportional to $\|\cdot\|$. Since $F(\cdot)$ is p-convex, so is $\|\cdot\|$.

Now, let $x, y \in X$. For every $p < p_0$ we have $\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$. Letting $p \to p_0$ we obtain that $\|\cdot\|$ is also a p_0-norm. This completes the proof. ■

Taking $p_0 = 1$ in the theorem, we obtain the following.

Corollary 3.4. *Let X be a uniquely maximal quasi-normed space. If there is a non-zero continuous linear functional on X, then X is locally convex and its quasi-norm is actually a norm.*

Example 3.5. No equivalent uniquely maximal quasi-norm can be given on a non-normable quasi-normed space X with $X^* \neq 0$. This includes l_p spaces and Hardy spaces H_p for $0 < p < 1$, all non-locally convex (locally bounded) Orlicz sequence spaces and for all non-locally convex spaces having a locally convex quotient.

The examples provided by 3.4 (and those of 2.2) are in a sense 'small' spaces. There are also 'big' non-locally convex spaces without any equivalent almost transitive quasi-norm.

Example 3.6. There exists an Orlic function space with trivial dual in which no equivalent quasi-norm is uniquely maximal with respect to its equivalent continuous quasi-norms.

PROOF. (See [15] or [11] for information about Orlicz spaces.) Fix $0 < p_0 \leqslant 1$. Let $h(u)$ be a positive decreasing continuous convex function defined on $[0, +\infty)$ such that $h(0) < p_0$ and $h(u) \to 0$ as $u \to +\infty$. Consider the Orlicz function $\phi(u) = u^{p_0 - h(u)}$ and the Orlicz space $L_\phi = L_\phi(0,1)$ associated with ϕ, that is, the space of measurable functions on (0, 1) for which the integral

$$\mathrm{I}_\phi(af) = \int_0^1 \phi(a\,|f(t)|)\,dt$$

is finite for some $a > 0$. The topology of L_ϕ is given by the quasi-norm $\|f\| = \inf\{\varepsilon > 0 : \mathrm{I}_\phi(f/\varepsilon) \leqslant 1\}$. By results of Rolewicz [15, p. 115], for every $p < p_0$ there is an equivalent p-norm on L_ϕ, although the topology of L_ϕ cannot be induced by any p_0-norm. Thus L_ϕ has trivial dual and no equivalent quasi-norm on L_ϕ is uniquely maximal among all continuous equivalent quasi-norms. Of course, no equivalent quasi-norm on L_ϕ is almost transitive. ■

Remark 3.7. Let X be a uniquely maximal quasi-Banach space. Then X is a p-Banach space for some $0 < p \leqslant 1$ (which is given by Theorem 3.3). Moreover, every quotient (in particular every complemented subspace) of X is also a p-Banach space. Thus, for example, the Orlicz spaces of 3.6 or $L_p \oplus L_q$ ($p \neq q$, $p < 1$) are not isomorphic to a quotient of a uniquely maximal quasi-Banach space. This implies the failure of the result of Lusky, quoted in 1.7, in a very strong way for general quasi-normed spaces.

We close the paper noting that from Theorem 3.3 it follows that uniquely maximal quasi-Banach spaces have exact type at least if they are not uniformly convex. We do not know if a uniformly convex almost transitive Banach space must have exact type.

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