We now have all the tools needed for the proof of Theorem 1. As observed in §1, if $\mathbf{X}$ and $\mathbf{Y}$ are not disjoint then $\mathbf{Y}$ has some $(\mathbf{X}/K)^{n\odot}$ as a factor. By Theorem 4.1 it follows that $(\mathbf{X}/K)^{n\odot}$ has a countable ergodic self-joining $\mathbf{Z}$ which is a classical factor of $\mathbf{Y}$. A classical factor of $\mathbf{Y}$ is a compact factor of some Gaussian factor of $\mathbf{Y}$, so it is virtually divisible by Theorem 3.2.

On the other hand, since $(\mathbf{X}/K)^{n\odot}$ is a factor of $\mathbf{X}^n$, a countable ergodic self-joining of $(\mathbf{X}/K)^{n\odot}$ lifts to a countable ergodic self-joining of $\mathbf{X}^n$, which must be isomorphic to $\mathbf{X}^{\mathbb{N}}$, by simplicity of $\mathbf{X}$. This means that $\mathbf{Z}$ is isomorphic to a factor of $\mathbf{X}^{\mathbb{N}}$ so Theorem 3.3 tells us that $\mathbf{Z}$ cannot be virtually divisible. This contradiction completes the proof of Theorem 1. ∎

References

[C,F,S] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer, Berlin, 1982.

[G,H,R] E. Glasner, B. Host and D. Rudolph, *Simple systems and their higher order self-joinings*, Israel J. Math. 78 (1992), 131–142.

[I,L,R,S] A. Iwanik, M. Lemańczyk, T. de la Rue et J. de Sam Lazaro, *Quelques remarques sur les facteurs des systèmes dynamiques gaussiens*, Studia Math. 125 (1997), 247–254.

[J,L,M] A. del Junco, M. Lemańczyk and M. K. Mentzen, *Semisimplicity, joinings and group extensions*, ibid. 112 (1995), 141–164.

[J,R] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems 7 (1987), 531–557.

[L,P,Th] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, preprint.

[Th1] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, in: Ergodic Theory and its Connections with Harmonic Analysis (Proc. Alexandria Conference), K. E. Petersen and I. Salama (eds.), London Math. Soc. Lecture Note Ser. 205, Cambridge Univ. Press, Cambridge, 1995, 207–235.

[Th2] —, *Utilisation des processus gaussiens en théorie ergodique*, preprint.

[Th3] —, *The metrical structure of some Gaussian processes*, in: Proc. Conf. on Ergodic Theory and Related Topics II (Georgenthal, 1986), Teubner Texte Math. 94, Teubner, Leipzig, 1987, 195–198.

[V] W. A. Veech, *A criterion for a process to be prime*, Monatsh. Math. 94 (1982), 335–341.

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A theorem on isotropic spaces

by

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Abstract. Let X be a normed space and $G_F(X)$ the group of all linear surjective isometries of X that are finite-dimensional perturbations of the identity. We prove that if $G_F(X)$ acts transitively on the unit sphere then X must be an inner product space.

1. Introduction and statement of the result. During the thirties some people studied *isotropic* spaces. These are normed spaces in which the group of linear surjective isometries acts transitively on the unit sphere. Clearly, inner product spaces are isotropic. That the converse is also true for finite-dimensional spaces was proved by S. Mazur [7] in 1938 (see also [2]):

THEOREM 1. *Isotropic finite-dimensional normed spaces are euclidean (in the sense that the norm comes induced by an inner product).*

There are, however, isotropic normed spaces that are not isomorphic to inner product spaces (this was discovered in the sixties by A. Pełczyński and S. Rolewicz [8]): for instance, if μ is a homogeneous non-σ-finite measure, the space $L_p(\mu)$ is isotropic for every finite p (see also [6]). These examples are necessarily non-separable. Also, isotropic separable normed (not complete) non-euclidean spaces are known: for example, the subspace of all functions in $L_p(-\infty,\infty)$ having bounded support. In spite of these examples the Mazur problem on the existence of a separable isotropic Banach space which is not a Hilbert space remains open [3]. (A recent survey on this problem and related topics is [4], which contains an extensive bibliography.) In this note, we generalize Mazur's result replacing the hypothesis on the dimension of the space by a weaker one concerning the structure of the isometry group.

So, let X be a (real or complex) normed space with unit sphere $S(X)$. We denote by $G(X)$ the group of all isometric automorphisms of X. An operator $T : X \to X$ is said to be a *finite-dimensional perturbation* of the identity if the difference $T - \mathrm{Id}$ is a finite rank operator. If we write

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$F(X)$ for the ideal of finite rank operators on X, then the set of all finite-dimensional perturbations of the identity is $\mathrm{Id}+F(X)$. Clearly, $\mathrm{Id}+F(X)$ is closed under composition and under taking inverses, so that the set $G_F(X)$ of all isometries which are finite-dimensional perturbations of the identity is a subgroup of $G(X)$. Our result is the following.

THEOREM 2. *Let X be a normed space. Suppose that given $x, y \in S(X)$ there is $T \in G_F(X)$ such that $y = Tx$. Then X is an inner product space.*

2. Proof of Theorem 2. Having in mind the classical proofs of Theorem 1 via invariant inner products, we shall construct an invariant inner product for $G_F(X)$ which *a priori* is defined only on a suitable subspace of X. The crucial step is Lemma 2. Lemma 1 collects some obvious facts needed for the proof.

LEMMA 1. *Let Γ be a set of finite-dimensional perturbations of the identity on a vector space X. For each finite $\gamma \subset \Gamma$, put*

$$E(\gamma) = \mathrm{span}\Big\{\bigcup_{T\in\gamma}(T - \mathrm{Id})X\Big\}.$$

Then:

(a) *$E(\gamma)$ is finite-dimensional for every finite γ;*
(b) *$E(\gamma) \neq 0$ if $\gamma \neq \{\mathrm{Id}\}$;*
(c) *$E(\gamma) \subset E(\eta)$ for $\gamma \subset \eta$;*
(d) *$TE(\gamma) \subset E(\gamma)$ for each $T \in \gamma$ and all finite $\gamma \subset \Gamma$.*

LEMMA 2. *For every normed space X there exist a linear subspace $H \subset X$ and a (possibly degenerate) inner product $(\cdot,\cdot)$ on H such that*:

(a) *$H \neq 0$;*
(b) *$TH = H$ for every $T \in G_F(X)$;*
(c) *$(\cdot,\cdot)$ is invariant under $G_F(X)$ in the sense that $(Tx, Ty) = (x, y)$ for every $T \in G_F(X)$ and all $x, y \in H$;*
(d) *$(x_0, x_0) = 1$ for some $x_0 \in H \cap S(X)$.*

REMARK. We claim neither that H is closed in X (even if X is a Banach space) nor that the norm induced on H by $(\cdot,\cdot)$ is weaker (or stronger) than the original norm of X restricted to H.

Proof of Lemma 2. Let $\wp$ be the net of all finite subsets of $G_F(X)$ ordered by inclusion and for each $\gamma \in \wp$, let $E(\gamma)$ be as in Lemma 1. Put

$$Y = \bigcup_{\gamma\in\wp} E(\gamma).$$

Clearly, Y is a linear subspace of X which is invariant under $G_F(X)$. Observe that the net $(E(\gamma))_{\gamma\in\wp}$ contains every $x \in Y$ eventually, that is, for every $x \in Y$ there is $\eta \in \wp$ such that $x \in E(\gamma)$ for all $\gamma \geq \eta$. Now, by an old result of Auerbach [1], for each $\gamma \in \wp$ there exists some inner product $(\cdot,\cdot)_\gamma$ on $E(\gamma)$ such that every isometry of $(E(\gamma), \|\cdot\|_X)$ preserves $(\cdot,\cdot)_\gamma$ and, consequently, one has $(Tx, Ty)_\gamma = (x, y)_\gamma$ for each $T \in \gamma$ and all $x, y \in E(\gamma)$. Fixing x_0 in Y such that $\|x_0\|_X = 1$, one can renormalize the inner products introduced above by the condition $(x_0, x_0)_\gamma = 1$ for $x_0 \in E(\gamma)$ and $(x_0, x_0)_\gamma = 0$ for $x_0 \notin E(\gamma)$.

Let

$$c_{00}(\wp) = \Big\{(t_\gamma)_\gamma \in \prod_\wp \mathbb{K} : \text{there is } \eta \in \wp \text{ such that } t_\gamma = 0 \text{ for every } \gamma \geq \eta\Big\}.$$

Given $x, y \in Y$, one sees that $(x, y)_\gamma$ is eventually defined, so that the mapping

$$(x, y) \in Y \times Y \to B(x, y) = ((x, y)_\gamma) \in \prod_\wp \mathbb{K}/c_{00}(\wp)$$

defines an "inner product" on Y which has values in the quotient algebra $\prod_\wp \mathbb{K}/c_{00}(\wp)$. Moreover, B is invariant under $G_F(X)$: let T be in $G_F(X)$ and let x, y be in Y. Pick $\eta \in \wp$ such that $x, y \in E(\eta)$ and $T \in \eta$. Then, for $\gamma \geq \eta$, one has $T(E(\gamma)) = E(\gamma)$ and $(Tx, Ty)_\gamma = (x, y)_\gamma$ and thus $B(Tx, Ty) = B(x, y)$.

Consider $l_\infty(\wp)/(c_{00}(\wp) \cap l_\infty(\wp))$, the subspace of families in $\prod_\wp \mathbb{K}/c_{00}(\wp)$ having a bounded representative, and define

$$H = \{x \in Y : B(x, x) \in l_\infty(\wp)/(c_{00}(\wp) \cap l_\infty(\wp))\}.$$

Then H is a linear subspace of Y: that H is closed under multiplication by scalars easily follows from 2-homogeneity of B; that H is closed under sums is a consequence of the "parallelogram law"

$$B(x + y, x + y) + B(x - y, x - y) = 2B(x, x) + 2B(y, y)$$

and the fact that $B(z, z) \geq 0$ in the obvious order of $\prod_\wp \mathbb{K}/c_{00}(\wp)$. Moreover, Schwarz' inequality implies that $B(x, y)$ belongs to $l_\infty(\wp)/(c_{00}(\wp) \cap l_\infty(\wp))$ for all $x, y \in H$. On the other hand, it is clear that H is $G_F(X)$-invariant and contains at least x_0 since $B(x_0, x_0) = 1 + c_{00}(\wp)$.

Finally, observe that the system $V = \{\gamma \in \wp : \gamma \geq \eta\}_{\eta\in\wp}$ is a filter base on $\wp$. Let U be a proper ultrafilter on $\wp$ extending V and define the desired (scalar) inner product on H as

$$(x, y) = \lim_U B(x, y)$$

(this definition is correct because the functional $\lim_U : l_\infty(\wp) \to \mathbb{K}$ vanishes on $c_{00}(\wp)$). It is clear that $(\cdot,\cdot)$ is invariant under the maps of $G_F(X)$ since $B(\cdot,\cdot)$ is, and, obviously $(x_0, x_0) = 1$. ∎

Proof of Theorem 2. If $G_F(X)$ acts transitively on $S(X)$, then the only $G_F(X)$-invariant subspaces of X are the trivial ones, hence $H = X$ in Lemma 2 and $(\cdot,\cdot)$ is defined on the whole X. Moreover, $(\cdot,\cdot)$ is $G_F(X)$-invariant and $(x_0, x_0) = \|x_0\|^2 = 1$. Transitivity of $G_F(X)$ now implies that $(x, x) = \|x\|^2 =$ for every $x \in X$, which proves the theorem. ∎

3. Concluding remarks and questions. In a sense, the proof of Theorem 2 is algebra. Thus, it is not surprising that Theorem 2 holds if X is assumed to be a quasi-normed space. But, actually, an almost isotropic quasi-normed space having a non-trivial finite-dimensional perturbation of the identity must be locally convex, its quasi-norm being, in fact, a norm [5]. Recall that an *almost isotropic* quasi-normed space is one in which the isometry group acts with dense orbits on the unit sphere. Theorem 2 suggests the following questions:

QUESTION 1. Let X be a normed space for which, given $x, y \in S(X)$ and $\varepsilon > 0$, there exists $T \in G_F(X)$ such that $\|y - Tx\| \le \varepsilon$. Must X be an inner product space?

QUESTION 2. Find operator ideals J (containing F) for which Theorem 2 remains true if $T - \mathrm{Id} \in F$ is replaced by $T - \mathrm{Id} \in J$.

References

[1] H. Auerbach, *Sur les groupes linéaires bornés I*, Studia Math. 4 (1934), 113–127.

[2] —, *Sur une propriété caractéristique de l'ellipsoïde*, ibid. 9 (1940), 17–22.

[3] S. Banach, *Théorie des opérations linéaires*, Monograf. Mat. 1, Warszawa–Lwów, 1932.

[4] F. Cabello Sánchez, *Regards sur le problème des rotations de Mazur*, Extracta Math. 12 (1997), 97–116.

[5] —, *Maximal symmetric norms on Banach spaces*, Proc. Roy. Irish Acad., to appear.

[6] P. Greim, J. E. Jamison and A. Kamińska, *Almost transitivity of some function spaces*, Math. Proc. Cambridge Philos. Soc. 116 (1994), 475–488.

[7] S. Mazur, *Quelques propriétés caractéristiques des espaces euclidiens*, C. R. Acad. Sci. Paris 207 (1938), 761–764.

[8] S. Rolewicz, *Metric Linear Spaces*, Monograf. Mat. 56, PWN and D. Reidel, 1984.

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Weighted inequalities and the shape of approach regions

by

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Abstract. We characterize geometric properties of a family of approach regions by means of analytic properties of the class of weights related to the boundedness of the maximal operator associated with this family.

1. Introduction. In [NS], Nagel and Stein studied under which conditions on a general domain $\Omega \subset \mathbb{R}_+^{n+1}$, the associated maximal operator M_Ω is of weak type $(1,1)$. J. Sueiro [Su] gave an extension of this result for spaces of homogeneous type. Following the ideas of [Su], Pan [Pa] studied weak type weighted norm estimates for M_Ω also in spaces of homogeneous type. Later, Sánchez-Colomer and Soria [SS1] gave strong-type weighted norm estimates for M_Ω in the Euclidean space, and they also studied the relationship between weighted inequalities for this operator and the geometry of Ω (see [SS2]).

In this paper we find another (easier) characterization of the weak-type inequalities for M_Ω, in terms of the classical A_p condition plus an extra property related to being a Carleson measure (see Theorem 2.12). For this, we use some of the techniques given in [AC].

This result allows us to prove that the equivalence of weighted inequalities for M_Ω and the classical Hardy–Littlewood maximal function M completely determines the geometry of the family of approach regions Ω (see Theorem 3.4). We work in the setting of spaces of homogeneous type, extending previous results in $\mathbb{R}^n$ for the special case of regions obtained by translation of a fixed one (see [SS2]).

To this end, we observe that there exists a class of "power" weights (see Corollary 3.2) which are the key to establishing the correspondence between analytic properties (boundedness of maximal operators) and geometric properties of the domains Ω. The main idea behind this technique

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