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Diameter preserving linear maps and isometries

By

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Abstract. We study linear bijections of $C(X)$ which preserve the diameter of the range, that is, the seminorm $\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in X\}$.

1. Introduction and statement of the results. In a recent paper [2], Győry and Molnár studied linear bijections of $C(X)$ (the space of real or complex continuous functions on the compact Hausdorff space X) which preserve the diameter of the range, that is, the seminorm

$$\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in X\}.$$

They proved the following nice

Theorem 1. *Let X be a first countable compact Hausdorff space. A linear bijection $T : C(X) \to C(X)$ is diameter preserving if and only if there is a homeomorphism $\varphi : X \to X$, a linear functional $\mu : C(X) \to \mathbb{K}$ and a number τ with $|\tau| = 1$ and $\mu(1_X) + \tau \neq 0$ such that $Tf = \tau f \circ \varphi + \mu(f) 1_X$ for every $f \in C(X)$.*

In this note, we prove that Theorem 1 holds without first countability. In particular, we obtain the form of all invertible (linear) maps on l_∞ preserving the supremum of the distances between the coordinates (the diameter of the spectrum, if we think l_∞ as diagonal operators acting on a separable Hilbert space; see the final remark in [2]). Our approach is quite different from that of [2] and depends on the analysis of the isometry group of certain Banach spaces which appear naturally in connection with diameter preserving operators.

Let $C_\varrho(X)$ denote the quotient space $C(X)/\ker\varrho$. Clearly, it is a Banach space under the norm

$$\|\pi(f)\|_{C_\varrho(X)} = \varrho(f),$$

where $\pi : C(X) \to C(X)/\ker\varrho$ is the natural quotient map.

Now, suppose that $T : C(X) \to C(X)$ is a diameter preserving linear bijection. Then there exists a (unique) isometry T_ϱ of $C_\varrho(X)$ making the diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{T} & C(X) \\ \pi\downarrow & & \downarrow\pi \\ C_\varrho(X) & \xrightarrow{T_\varrho} & C_\varrho(X) \end{array}$$

commute.

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Our main result is the following characterization of the isometries of $C_\varrho(X)$.

Theorem 2. *A linear map $T_\varrho : C_\varrho(X) \to C_\varrho(X)$ is a surjective isometry if and only if there is a homeomorphism φ of X and $\tau \in \mathbb{K}$, with $|\tau| = 1$, such that $T_\varrho(\pi(f)) = \pi(\tau f \circ \varphi)$, for all $f \in C(X)$.*

After the proof of Theorem 2 it will be clear that a linear bijection $T : C(X) \to C(Y)$ (resp. $T_\varrho : C_\varrho(X) \to C_\varrho(Y)$) is diameter preserving (resp. an isometry) if and only if there is a homeomorphism $\varphi : Y \to X$, a linear functional $\mu : C(X) \to \mathbb{K}$ and $\tau \in \mathbb{K}$ with $|\tau| = 1$ and $\mu(1_X) + \tau \neq 0$ such that $Tf = \tau f \circ \varphi + \mu(f) 1_Y$ (resp. $T_\varrho \pi f = \pi(\tau f \circ \varphi)$) for all $f \in C(X)$. So, we have the following Banach-Stone type theorem.

Theorem 3. *Let X and Y be compact Hausdorff spaces. The following statements are equivalent:*

(a) *X and Y are homeomorphic.*
(b) *$C(X)$ and $C(Y)$ are isometric.*
(c) *$C_\varrho(X)$ and $C_\varrho(Y)$ are isometric.*
(d) *There is a (not necessarily continuous) diameter preserving linear bijection $C(X) \to C(Y)$.*

2. Proofs. Before going into the proof of Theorem 2, we derive Theorem 1 (without first countability) from Theorem 2.

Proof of Theorem 1. Let $T : C(X) \to C(X)$ be a diameter preserving bijection and let $T_\varrho : C_\varrho(X) \to C_\varrho(X)$ be the corresponding isometry. According to Theorem 2, one has $T_\varrho(\pi(f)) = \pi(\tau f \circ \varphi)$, for suitable φ and τ. Since $T_\varrho \circ \pi = \pi \circ T$ one has

$$\pi T f = \pi(\tau f \circ \varphi),$$

so that $f \to Tf - \tau f \circ \varphi$ takes values in the subspace of constant functions of $C(X)$ (which is the kernel of π). This obviously implies that there is $\mu : C(X) \to \mathbb{K}$ such that

$$Tf = \tau f \circ \varphi + \mu(f) 1_X$$

for every $f \in C(X)$. □

Remark 1. Observe that T need not be continuous. In fact, T is continuous if and only if μ is.

For the proof of Theorem 2, we need a description of the extreme points of the unit ball of $C_\varrho(X)^*$. Recall that if $C(X)$ is endowed with the usual supremum norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\},$$

then $C(X)^*$ equals the space $M(X)$ of all regular Borel measures on X with values in the ground field. The duality is defined by $\mu(f) = \int_X f d\mu$. Moreover the norm of the measure μ acting as a functional on $C(X)$ equals its total variation:

$$\|\mu\|_{C(X)^*} = \|\mu\|_1 = |\mu|(X).$$

Since $C_\varrho(X)$ is isomorphic to a quotient of $C(X)$, its dual space is isomorphic (although not generally isometric) to a subspace of $M(X)$. In fact, $C_\varrho(X)^*$ can be viewed as $\{\mu \in M(X) : \mu(X) = 0\}$ equipped with the following equivalent norm:

$$\|\mu\|_{C_\varrho(X)^*} = \sup\{|\mu(f)| : \varrho(f) \leqq 1, f \in C(X)\}.$$

As usual, for $z \in X$, we denote by δ_z the evaluation functional $f \in C(X) \to f(z) \in \mathbb{K}$.

Lemma 1. *Let $\mu \in M(X)$. Then μ is an extreme point of the unit ball of $C_\varrho(X)^*$ if and only if $\mu = \sigma(\delta_x - \delta_y)$, where x and y are distinct points of X and $|\sigma| = 1$.*

Proof of Lemma 1. *Necessity.* Consider the linear operator $L : C_\varrho(X) \to C(X^2)$ given by $L\pi(f)(x, y) = f(x) - f(y)$. Obviously,

$$\|\pi(f)\|_{C_\varrho(X)} = \|Lf\|_\infty,$$

so that L is an isometric embedding.

Let $L^* : C(X^2)^* \to C_\varrho(X)^*$ be the adjoint map. Clearly, L^* is *weak to *weak continuous and carries the unit ball of $C(X^2)^*$ (which is a *weakly compact set) exactly into the unit ball of $C_\varrho(X)$. Thus the Krein-Milman theorem implies that each extreme point of the unit ball of $C_\varrho(X)^*$ is the image under L^* of some extreme point of the unit ball of $C(X^2)^*$.

Hence, if μ is an extreme point of the unit ball of $C_\varrho(X)^*$, then

$$\mu = L^*(\sigma\delta_{(x,y)}) = \sigma L^*\delta_{(x,y)} = \sigma(\delta_x - \delta_y),$$

for some $x, y \in X$ with $x \neq y$ and $|\sigma| = 1$. This proves the 'only if' part.

Sufficiency. Let us assume for a moment that $\mathbb{K} = \mathbb{R}$. One then has

$$\varrho(f) = 2\inf\{\|f - \lambda 1_X\|_\infty : \lambda \in \mathbb{R}\}$$

for all $f \in C(X)$. This means that $C_\varrho(X)$ is, up to a constant factor 2, isometric to the quotient of $(C(X), \|\cdot\|_\infty)$ by the subspace of constant functions (which is not true if $\mathbb{K} = \mathbb{C}$). Therefore, the space $C_\varrho(X)^*$ is, up to a factor 1/2, isometric (and not only isomorphic) to a subspace of $C(X)^*$. In fact, for every $\mu \in M(X)$ with $\mu(X) = 0$, one has $2\|\mu\|_{C_\varrho(X)^*} = \|\mu\|_1$.

So, we can work with $\|\cdot\|_1$ instead of the original norm of $C_\varrho(X)^*$. Let λ^+ and λ^- denote respectively the positive and negative part of the measure λ. Clearly, $\|\lambda\|_1 = \|\lambda^+\|_1 + \|\lambda^-\|_1$. Moreover, it is easily seen that if $\lambda = \lambda_1 - \lambda_2$ is a decomposition of λ with λ_1 and λ_2 positive measures and $\|\lambda\|_1 = \|\lambda_1\|_1 + \|\lambda_2\|_1$, then $\lambda^+ = \lambda_1$ and $\lambda^- = \lambda_2$.

After this preparation, let $x, y \in X$. Suppose that $\mu, \nu \in C_\varrho(X)^*$ are such that $\delta_x - \delta_y = \mu + \nu$, with $\|\delta_x - \delta_y\|_{C_\varrho(X)^*} = \|\mu\|_{C_\varrho(X)^*} + \|\nu\|_{C_\varrho(X)^*}$. Writing $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$, and taking into account that $\|\cdot\|_1$ is additive on the positive cone of $M(X)$, it is clear that

$$\begin{aligned}\|\delta_x - \delta_y\|_1 &= \|\mu\|_1 + \|\nu\|_1\\ &= \|\mu^+\|_1 + \|\mu^-\|_1 + \|\nu^+\|_1 + \|\nu^-\|_1\\ &= \|\mu^+ + \nu^+\|_1 + \|\mu^- + \nu^-\|_1.\end{aligned}$$

Since $\delta_x - \delta_y = \mu^+ - \mu^- + \nu^+ - \nu^-$, it follows that $\delta_x = (\delta_x - \delta_y)^+ = \mu^+ + \nu^+$ and $\delta_y = (\delta_x - \delta_y)^- = \mu^- + \nu^-$. Hence $\|\delta_x\|_1 = \|\mu^+\|_1 + \|\nu^+\|_1$ and $\|\delta_y\|_1 = \|\mu^-\|_1 + \|\nu^-\|_1$ and since δ_x and δ_y are extreme points in the unit ball of $M(X)$, one obtains

$$\mu^+ = \mu^+(X)\delta_x, \quad \nu^+ = \nu^+(X)\delta_x \quad \mu^- = \mu^-(X)\delta_y, \quad \nu^- = \nu^-(X)\delta_y.$$

But μ and ν belong to $C_\varrho(X)^*$ so we have $\mu^+(X) = \mu^-(X)$ and $\nu^+(X) = \nu^-(X)$ and therefore

$$\mu = \mu^+(X)(\delta_x - \delta_y), \quad \nu = \nu^+(X)(\delta_x - \delta_y).$$

This shows that $\delta_x - \delta_y$ is an extreme point of the unit ball of $C_\varrho(X)^*$ in the real case.

To end with the proof of the Lemma, let $\mathbb{K} = \mathbb{C}$. It obviously suffices to see that $\delta_x - \delta_y$ is an extreme point of the unit ball of the complex $C_\varrho(X)^*$. Suppose that

$$\delta_x - \delta_y = \mu + \nu \quad \text{and} \quad \|\delta_x - \delta_y\|_{C_\varrho(X)^*} = \|\mu\|_{C_\varrho(X)^*} + \|\nu\|_{C_\varrho(X)^*}.$$

By the Hahn-Banach theorem there exist $\tilde{\mu}, \tilde{\nu} \in M(X^2)$ such that

$$L^*\tilde{\mu} = \mu \quad \text{with} \quad \|\tilde{\mu}\|_1 = \|\mu\|_{C_\varrho(X)^*}$$
$$L^*\tilde{\nu} = \nu \quad \text{with} \quad \|\tilde{\nu}\|_1 = \|\nu\|_{C_\varrho(X)^*}.$$

Now, observe that $\|\Re(\eta)\|_1 \leqq \|\eta\|_1$ for every $\eta \in M(X^2)$, with equality only if η is real, which implies that $\tilde{\mu}$ and $\tilde{\nu}$ are real measures. Hence $\mu(\pi(f))$ and $\nu(\pi(f))$ are real for every real-valued $f \in C(X)$ and

$$(\delta_x - \delta_y)|_{C_\varrho(X,\mathbb{R})} = \mu|_{C_\varrho(X,\mathbb{R})} + \nu|_{C_\varrho(X,\mathbb{R})}$$

with

$$\|(\delta_x - \delta_y)|_{C_\varrho(X,\mathbb{R})}\|_{C_\varrho(X,\mathbb{R})^*} = \|\mu|_{C_\varrho(X,\mathbb{R})}\|_{C_\varrho(X,\mathbb{R})^*} + \|\nu|_{C_\varrho(X,\mathbb{R})}\|_{C_\varrho(X,\mathbb{R})^*}$$

since obviously $\|(\delta_x - \delta_y)|_{C_\varrho(X,\mathbb{R})}\|_{C_\varrho(X,\mathbb{R})^*} = \|\delta_x - \delta_y\|_{C_\varrho(X)^*}$. On the other hand $(\delta_x - \delta_y)|_{C_\varrho(X,\mathbb{R})}$ is an extreme point of the unit ball of $C_\varrho(X,\mathbb{R})^*$ and, therefore, μ and ν are proportional to $\delta_x - \delta_y$ when restricted to real functions. By complex linearity one obtains that μ and ν also are proportional to $\delta_x - \delta_y$ as complex functionals. This completes the proof of Lemma 1. □

Beginning of the proof of Theorem 2. Let T be a surjective isometry of $C_\varrho(X)$. Then the adjoint map $T^* : C_\varrho(X)^* \to C_\varrho(X)^*$ is an isometry as well and, therefore, it carries extreme points into extreme points. Taking Lemma 1 into account, it is clear that, given $x, y \in X$ with $x \neq y$, there are $u, v \in X$, $u \neq v$ and $\sigma \in \mathbb{K}$ with $|\sigma| = 1$ such that

$$T^*(\delta_x - \delta_y) = \sigma(\delta_u - \delta_v).$$

Let X_2 stand for the collection of all subsets of X having exactly two elements. Plainly, T induces a bijection $\Phi : X_2 \to X_2$ by

$$\Phi\{x, y\} = \operatorname{supp}(T^*(\delta_x - \delta_y)).$$

Let $|S|$ denote the cardinality of the set S.

Lemma 2. *For all* $\{x, y\}, \{u, v\} \in X_2$, *one has* $|\{x, y\} \cap \{u, v\}| = |\Phi\{x, y\} \cap \Phi\{u, v\}|$.

Proof of Lemma 2. Simply observe that if $\{x, y\} \neq \{u, v\}$, then $\{x, y\} \cap \{u, v\}$ is non-empty if and only if there is a nontrivial linear combination of $\delta_x - \delta_y$ and $\delta_u - \delta_v$ that is an extreme point of the unit ball of $C_\varrho(X)^*$. □

Lemma 3. *There is a bijection* $\varphi : X \to X$ *such that* $\Phi\{x, y\} = \{\varphi(x), \varphi(y)\}$ *for every* $x, y \in X$.

Proof of Lemma 3. (We follow [2], step 7). Suppose that $|X| > 4$. Fix $x \in X$ and take $y_1, y_2 \in X$ with $y_1 \neq y_2, y_1 \neq x, y_2 \neq x$. Let $\varphi(x)$ be the unique point in

$$\Phi\{x, y_1\} \cap \Phi\{x, y_2\}.$$

The proof will be complete if we see that $\varphi(x) \in \Phi\{x, y\}$ for every $y \neq x$. Suppose there is $y \notin \{x, y_1, y_2\}$ such that $\varphi(x) \notin \Phi\{x, y\}$. Write

$$\Phi\{x, y_1\} = \{\varphi(x), a_1\}, \Phi\{x, y_2\} = \{\varphi(x), a_2\}.$$

Then $\Phi\{x, y\} = \{a_1, a_2\}$ and the injectivity of Φ implies that y is the only point in X for which $\varphi(x) \notin \Phi\{x, y\}$. Pick $y_3 \in K, y_3 \notin \{x, y, y_1, y_2\}$. Then $\varphi(x) \in \Phi\{x, y_3\}$ and there is $a_3 \notin \{a_1, a_2\}$ such that

$$\Phi\{x, y_3\} = \{\varphi(x), a_3\}.$$

Replacing a_2 by a_3 we obtain again

$$\Phi\{x, y\} = \{a_1, a_3\} \neq \{a_1, a_2\},$$

a contradiction.

This proves the Lemma for $|X| > 4$. If $|X| < 4$ the result is trivial. For $|X| = 4$ there is a little problem: there are bijections $\Phi : X_2 \to X_2$ satisfying the statement of Lemma 2 that cannot be induced by a map $\varphi : X \to X$. This is clear thinking X and X_2 respectively as the set of vertices and edges of a tetraedron. (Indeed, let $X = \{1, 2, 3, 4\}$. Define a bijective mapping $\Phi : X_2 \to X_2$ by $\Phi(\{1, 2\}) = \{3, 4\}, \Phi(\{3, 4\}) = \{1, 2\}$ and leaving fixed the remaining edges. Then $|\{x, y\} \cap \{u, v\}| = |\Phi\{x, y\} \cap \Phi\{u, v\}|$ for every $x, y, u, v \in X$ but there is no map $\varphi : X \to X$ for which $\Phi(\{x, y\}) = \{\varphi(x), \varphi(y)\}$.) Nevertheless, it is easily checked that such maps cannot be induced by isometries of $C_\varrho(X)$. □

End of the proof of Theorem 2. Let $\varphi : X \to X$ be the (obviously bijective) map of the preceding Lemma. Clearly

$$T^*(\delta_x - \delta_y) = \sigma(x, y)(\delta_{\varphi(x)} - \delta_{\varphi(y)}),$$

where $|\sigma(x, y)| = 1$. We want to see that $\sigma(x, y)$ does not depend on x, y. Let $z \notin \{x, y\}$. Then

$$\begin{aligned} \sigma(x, y)(\delta_{\varphi(x)} - \delta_{\varphi(y)}) &= T^*(\delta_x - \delta_y) \\ &= T^*(\delta_x - \delta_z + \delta_z - \delta_y) \\ &= T^*(\delta_x - \delta_z) + T^*(\delta_z - \delta_y) \\ &= \sigma(x, z)(\delta_{\varphi(x)} - \delta_{\varphi(z)}) + \sigma(z, y)(\delta_{\varphi(z)} - \delta_\varphi(y)), \end{aligned}$$

so that

$$\sigma(x, y) = \sigma(x, z) = \sigma(z, y).$$

Since x, y and z are arbitrary, the equality $\sigma(x, y) = \sigma(z, y)$ means that $\sigma(\cdot, \cdot)$ does not depend on the first variable, while $\sigma(x, y) = \sigma(x, z)$ implies that the same occurs with the second one. Hence $\sigma(x, y) = \tau$ for some unimodular τ.

We now prove that $\varphi : X \to X$ is continuous. Without loss of generality, assume that $\tau = 1$. For y arbitrary, but fixed in X, consider the map $\Psi_y : X \backslash \{y\} \to C_\varrho(X)^*$ given by $\Psi_y(x) = \delta_x - \delta_y$. Clearly Ψ_y is an into homeomorphism when $C_\varrho(X)^*$ is endowed with the *weak topology. Moreover T^* (as an adjoint mapping) is *weak to *weak continuous. Obviously

$$\varphi(x) = \Psi^{-1}_{\varphi(y)} T^* \Psi_y(x)$$

for all $x \neq y$ and φ is continuous at every $x \neq y$. Since y was arbitrary, φ is continuous on the whole of X and, in fact, it is a homeomorphism.

Finally, define $T_{(\tau,\varphi)} : C_\varrho(X) \to C_\varrho(X)$ as $T_{(\tau,\varphi)}(\pi f) = \pi(\tau f \circ \varphi)$. Since

$$T^*(\delta_x - \delta_y) = T^*_{(\tau,\varphi)}(\delta_x - \delta_y)$$

for all $x, y \in X$, the Krein-Milman theorem implies that $T = T_{(\tau,\varphi)}$. This completes the proof of Theorem 2. $\square$

3. Locally compact spaces. We close the paper with some remarks about diameter preserving bijections on $C_0(X)$ (the space of real or complex continuous functions on the locally compact space X vanishing at infinity) for noncompact X. In this case there are no constants in $C_0(X)$ and

$$\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in X\}$$

is a norm on $C_0(X)$. In fact,

$$\|f\|_\infty \leqq \varrho(f) \leqq 2\|f\|_\infty$$

for every $f \in C_0(X)$.

Let $\alpha X = X \cup \{\infty\}$ denote the one-point compactification of X. Then $C_0(X)$ can be regarded as a subspace of $C(\alpha X)$ in the obvious way. Moreover, if $\pi : C(\alpha X) \to C_\varrho(\alpha X)$ is the natural map, it is clear that the restriction of π to $C_0(X)$ yields a surjective isometry between $(C_0(X), \varrho(\cdot))$ and $C_\varrho(\alpha X)$.

In this way, T is a diameter preserving linear bijection of $C_0(X)$ for noncompact X if and only if there is a surjective isometry T_ϱ of $C_\varrho(\alpha X)$ making commute the following diagram

$$\begin{array}{ccc} C_0(X) & \xrightarrow{T} & C_0(X) \\ \pi\downarrow & & \downarrow\pi \\ C_\varrho(\alpha X) & \xrightarrow{T_\varrho} & C_\varrho(\alpha X) \end{array}$$

Hence, in view of Theorem 2, we have:

Theorem 4. *Let X be a locally compact, noncompact space. A linear bijection T of $C_0(X)$ is diameter preserving if and only if there is a homeomorphism φ of αX and a number τ with $|\tau| = 1$ such that $Tf = \tau(f \circ \varphi - f(\varphi(\infty))1_{\alpha X})$ for every $f \in C_0(X)$.* $\square$

Observe that φ need not leave fixed the infinity point of αX and, therefore, T need not preverve the usual supremum norm. Thus, contrarily to what happens in the compact case, the group of diameter preserving automorphisms of the real space $C_0(X)$ may be strictly larger than the isometry group of $(C_0(X), \|\cdot\|_\infty)$ (take, for instance, $X = \mathbb{R}$).

Also, it is clear that, given locally compact noncompact spaces X and Y, there is a diameter preserving bijection between $C_0(X)$ and $C_0(Y)$ if and only if αX and αY are homeomorphic. Hence considering $X = [0,1)$ and $Y = [0,1/2) \cup (1/2,1]$, we see that the existence of a diameter preserving bijection between $C_0(X)$ and $C_0(Y)$ does not imply that X and Y are homeomorphic.

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