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Uniform Boundedness Theorems for Nearly Additive Mappings

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Abstract. We deal with mappings from a (not necessarily commutative) group G into a Banach space Y which are nearly additive in the sense of satisfying that for some constant $K \geq 0$,

$$\Big\|\sum_{i=1}^{n} F(x_i) - \sum_{j=1}^{m} F(y_j)\Big\| \leq K\Big\{\sum_{i=1}^{n} \rho(x_i) + \sum_{j=1}^{m} \rho(y_j)\Big\},$$

whenever x_i and $y_i \in G$ are such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$, where ρ is a fixed (non-negative) "control" functional on G. Such maps, called zero-additive, appear in various contexts. The smallest constant K for which the inequality holds shall be noted by $Z(F)$.

For mappings $G \to Y$ we consider the (possibly infinite) distance

$$\operatorname{dist}(F, A) = \inf\{C \geq 0 \mid \|F(x) - A(x)\| \leq C\rho(x) \text{ for all } x \in G\}.$$

Then one may ask whether or not a zero-additive map F must be near to a true additive map $A\colon G \to Y$ in the sense of $\operatorname{dist}(F, A) < \infty$ and how $Z(F)$ and $\operatorname{dist}(F, A)$ are related (a question which goes back to ULAM). We prove the following "uniform boundedness" result, thus solving a problem stated by CASTILLO and the present author.

Theorem. *Let (G, ρ) be a controlled group and Y a Banach space. Suppose that every zero-additive map $G \to Y$ is at finite distance from some additive map. Then there exists a constant $K \geq 0$ such that, for each zero-additive map $F\colon G \to Y$, there is an additive map $A\colon G \to Y$ for which* $\operatorname{dist}(F, A) \leq K Z(F)$.

This result generalizes a theorem of FORTI. A similar statement holds for mappings satisfying estimates

$$\|F(x + y) - F(x) - F(y)\| \leq K\rho(x);$$
$$\|F(x + y) - F(x) - F(y)\| \leq K\{\rho(x) + \rho(y) - \rho(x + y)\},$$

for some constant K and all $x, y \in G$.

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1 Introduction and statement of the main results

In two earlier papers [2, 3], we have studied mappings $F\colon G \to Y$ from a (not necessarily commutative) group $(G, +)$ into a Banach space which are "nearly additive" in the sense of satisfying that for some constant $K \geq 0$,

$$\left\|\sum_{i=1}^{n} F(x_i) - \sum_{j=1}^{m} F(y_j)\right\| \leq K\left\{\sum_{i=1}^{n} \rho(x_i) + \sum_{j=1}^{m} \rho(y_j)\right\},$$

whenever x_i and $y_i \in G$ are such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$, where ρ is a fixed (non-negative) "control" functional on G. Such maps, appearing in various contexts [4, 7, 8, 9], are called zero-additive in [2, 3] and the smallest constant K for which the inequality holds shall be referred to as the zero-additivity constant of F and denoted by $Z(F)$

For mappings $G \to Y$ consider the (possibly infinite) distance

$$\operatorname{dist}(F, A) = \inf\{C \geq 0 \mid \|F(x) - A(x)\| \leq C\rho(x) \text{ for all } x \in G\}.$$

(Maps at finite distance from the zero map shall be called bounded.) Then one may ask whether or not a zero-additive map F must be asymptotically additive, that is, near to a true additive map $A\colon G \to Y$ in the sense of $\operatorname{dist}(F, A) < \infty$ and how $Z(F)$ and $\operatorname{dist}(F, A)$ are related (a question which goes back to ULAM[10]). Here, $\operatorname{Hom}(G, Y)$ is the group of all additive maps from G to Y. Obviously a mapping which is at finite distance from an additive one must be zero-additive. In this setting, the pair $\{(G, \rho); Y\}$ is said to have the property $K\mathbf{Z}$ [3] if to each zero-additive map $F\colon G \to Y$ there corresponds an additive map A such that $\operatorname{dist}(F, A) \leq KZ(F)$.

In [2] it is proved that if G is a weakly commutative group, then the pair $\{(G, \rho); \mathbb{R}\}$ has the property $1\mathbf{Z}$ for every ρ, while if G is an amenable group, then $\{(G, \rho); \mathbb{R}\}$ has the property $2\mathbf{Z}$ provided ρ is symmetric (i.e. $\rho(x) = \rho(-x)$ for all $x \in G$).

In this note we prove the following "uniform boundedness" result, thus solving a problem stated in [2] and [3].

Theorem 1. *Let (G, ρ) be a controlled group and Y a Banach space. Suppose that every zero-additive map $G \to Y$ is asymptotically additive. Then the pair $\{(G, \rho); Y\}$ has the property $K\mathbf{Z}$ for some $K \geq 0$.*

It should be noted that there are non-trivial pairs $(G; Y)$ for which every zero-additive map $G \to Y$ is asymptotic additive:

a) $G = l_1$ (equipped with its natural norm) and Y any Banach space;
b) $G = L_1[0, 1]$ and Y a Banach space complemented in its second dual;
c) Let $G = l_1/B$, where B ist Bourgain's uncomplemented copy of l_1 inside l_1. Then all zero-additive maps $G \to l_2$ are at finite distance from additive ones, although there are zero-additive maps $G \to l_1$ which are not asymptotically additive;
d) G any controlled group such that $\{G; \mathbb{R}\}$ have some property $K\mathbf{Z}$ and $Y = l_\infty$;
e) G a countable generated group such that $\{G; \mathbb{R}\}$ have some property $K\mathbf{Z}$ and $Y = c_0(\Gamma)$ for any index set Γ;

The proof of these facts, as well as other examples, can be found in [3], where a theory of nearly additive maps is developed.

Since zero-additive maps with respect to $\rho = 1$ coincide with approximately additive maps in Hyers sense (that is, maps with bounded Cauchy difference) our result generalizes Theorem 2 in [6].

We remark that the scalar case $Y = \mathbb{R}$ is not easier than the general one. The somewhat involved proof combines ideas of GIUDICI (see [7], pp. 149-150) with a decomposition result for arbitrary groups and a stronger version of a previous result in [3] (roughly speaking, that Theorem 1 holds for G a vector space over $\mathbb{Q}$).

The main step in the proof of Theorem 1 is the following result which has its own intrinsic interest.

Theorem 2. *Let (G, ρ) be a controlled group. Then there exists a function $\rho^* : G \to \mathbb{R}$ (depending only on ρ) with the following property: for every Banach space Y and every zero-additive map $F\colon G \to Y$ there is an additive map $A\colon G \to Y$ fulfilling $\|F(x) - A(x)\| \leq Z(F)\rho^*(x)$ for all $x \in G$. Moreover, when Y is fixed, A depends linearly on F.*

Theorem 2 seems to be new even for $\rho = 1$ and shows that the space of all mappings with bounded Cauchy difference (modulo homomorphisms) is always a Banach space under the norm $\|F\| = \sup\{\|F(x+y) - F(x) - F(y)\| : x, y \in G\}$. (see the recent paper [5] for a description of some related spaces.)

2 Decomposition of Groups

It is well known that every group G contains a normal subgroup G_1 (the so-called commutator subgroup) with the following universal property: every homomorphism from G to a commutative group factorizes through the quotient $G \to G/G_1$. Moreover G/G_1 is itself a commutative group, so that one has an exact sequence $0 \to G_1 \to G \to G/G_1 \to 0$ which decomposes G into G_1 and the "abelian part" G/G_1.

Let T be the torsion subgroup of G/G_1. One has another exact sequence $0 \to T \to G/G_1 \to (G/G_1)/T \to 0$, where $(G/G_1)/T$ is torsion-free (although not generally free). Nevertheless $(G/G_1)/T$ (as every torsion-free abelian group) can be seen as a subgroup of a vector space over $\mathbb{Q}$ by means of $x \in (G/G_1)/T \to x \otimes 1 \in (G/G_1)/T \otimes_{\mathbb{Z}} \mathbb{Q}$ (see for instance [1], Ch. 3, ex. 12). Now, let $G_0 = q^{-1}\big(T(G/G_1)\big)$, where $q\colon G \to G/G_1$ is the natural map. Clearly, G/G_0 is isomorphic to $(G/G_1)/T$. So we have a decomposition

$$0 \to G_0 \to G \to G/G_0 \to 0,$$

where G_0 contains both the commutator subgroup G_1 and the "torsion elements" of G and G/G_0 is a subgroup of a vector space over $\mathbb{Q}$.

Remark 1. It is not hard to see that $G_0 = \cap_A \ker A$, where the intersection is taken over all $A \in \mathrm{Hom}(G, \mathbb{R})$. To identify G/G_0, define $\Phi\colon G \to \prod_A \mathbb{R}$ by $\big(\Phi(x)\big)_A = A(x)$. Then Φ is a group homomorphism whose kernel is G_0, so that G/G_0 is isomorphic to $\Phi(G)$. Thus G_0 comes defined by the universal property that every real-valued homomorphism on G factorizes through the quotient $G \to G/G_0$.

Remark 2. Actually, the following generalization of [7], Lemma 1, p. 149 holds. Let G be a group endowed with a symmetric, subadditive functional. For a real-valued map F on G the following statements are equivalent:

a) F is bounded on G_0 and zero-additive on G;
b) F is bounded on G_1 and zero-additive on G;
c) F is at finite distance from some additive $A\colon G \to \mathbb{R}$.

3 Proof of Theorem 2

In this section we prove Theorem 2. The crucial property of G_0 appears in Lemma 4. Lemma 3 will simplify the proof.

Lemma 3. *For every zero-additive mapping F from (G, ρ) into a Banach space one has the estimate $\|F(x) + F(-x)\| \leq Z(F)\big(\rho(x) + \rho(-x)\big)$ for all $x \in G$.*

Proof. For every n one has

$$\left\| \sum_{i=1}^{n} F(x) + \sum_{i=1}^{n} F(-x) - F(0) \right\| \leq Z(F)\big(n\rho(x) + n\rho(-x) + \rho(0)\big),$$

that is, $\|F(x)+F(-x)-F(0)/n\| \leq Z(F)\big(\rho(x)+\rho(-x)+\rho(0)/n\big)$, and the result follows. □

Lemma 4. *Let (G, ρ) be a controlled group and let G_0 be as before. There is a functional $\eta\colon G_0 \to \mathbb{R}$ such that, for every zero-additive map F from G into a Banach space, one has $\|F(x)\| \leq Z(F)\eta(x)$ for all $x \in G_0$.*

Proof. Let $x \in G_0$. Then there is $n(x) \geq 0$ such that $n(x)x \in G_1$. Since G_1 is generated by the set $\{z + y - z - y : z, y \in G\}$, there are $m(x)$ and $z_i(x), y_i(x)$, $1 \leq i \leq m(x)$, such that

$$\sum_{i=1}^{n(x)} x = \sum_{i=1}^{m(x)} \{z_i(x) + y_i(x) - z_i(x) - y_i(x)\}.$$

Hence,

$$\begin{aligned}
\|n(x)F(x)\| &\leq \left\| n(x)F(x) - \sum_{i=1}^{m(x)} \{Fz_i(x) + Fy_i(x) + F(-z_i(x)) + F(-y_i(x))\} \right\| \\
&\quad + \sum_{i=1}^{m(x)} \|Fz_i(x) + Fy_i(x) + F(-z_i(x)) + F(-y_i(x))\| \\
&\leq Z(F)\Big\{ n(x)\rho(x) \\
&\quad + 2\sum_{i=1}^{m(x)} \{\rho(z_i(x)) + \rho(y_i(x)) + \rho(-z_i(x)) + \rho(-y_i(x))\} \Big\}.
\end{aligned}$$

Thus, the choice

$$\eta(x) = \rho(x) + \frac{2}{n(x)} \sum_{i=1}^{m(x)} \{\rho(z_i(x)) + \rho(y_i(x)) + \rho(-z_i(x)) + \rho(-y_i(x))\}$$

ends the proof. □

End of the proof of Theorem 2. Let G_0 be as in Section 2 and let $\{e_i\}$ be a fixed basis for $G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Without loss of generality we may assume that e_i belongs to G/G_0 for all i. For each i, choose $g_i \in G$ such that $\pi g_i = e_i$. Now, let Y be a Banach space and $F\colon (G, \rho) \to Y$ a zero-additive map. We define a $\mathbb{Q}$-linear map $L : G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q} \to Y$ by putting $L(e_i) = F(g_i)$. Finally, let $A\colon G \to Y$ be given by

$$A(x) = L(\pi x).$$

Clearly, A is additive and depends linearly on F. We shall estimate $\|F(x) - A(x)\|$ as a function of $Z(F)$ and x. Indeed, let $x \in G$ be fixed. Since $\{e_i\}$ is a basis of $G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ there are rational numbers q_i such that $\pi x = \sum_i q_i e_i$. Write $q_i = m_i/n_i$ as an irreducible fraction with $n_i > 0$. Let moreover $N = \prod_i n_i$ and $M_i = \prod_{j \neq i} n_j$. Clearly,

$$\pi N x = N \pi x = \sum_i M_i e_i.$$

Let $y = \sum_i M_i g_i$. Obviously $Nx - y$ lies in G_0, so that $ANx = Ay$. One has

$$\begin{aligned}
\Big\| \sum_{j=1}^{N} F(x) - \sum_{j=1}^{N} A(x) \Big\| &= \Big\| \sum_{j=1}^{N} F(x) - \sum_{j=1}^{N} A(y) \Big\| \\
&= \Big\| \sum_{j=1}^{N} F(x) - \sum_i M_i F(g_i) \Big\| \\
&\leq Z(F)\eta\big(Nx - \sum_i M_i g_i\big) \\
&\quad + \Big\| \sum_{j=1}^{N} F(x) - \sum_i M_i F(g_i) - F\big(\sum_{j=1}^{N} x - \sum_i M_i g_i\big) \Big\| \\
&\leq Z(F)\Big\{\eta\big(Nx - \sum_i M_i g_i\big) \\
&\quad + \rho\big(Nx - \sum_i M_i g_i\big) + N\rho(x) + \sum_i M_i \rho(g_i)\Big\},
\end{aligned}$$

where η is given by Lemma 4. Therefore, the choice

$$\rho^* = \frac{1}{N}\Big\{\eta\big(Nx - \sum_i M_i g_i\big) + \rho\big(Nx - \sum_i M_i g_i\big) + N\rho(x) + \sum_i M_i \rho(g_i)\Big\}$$

completes the proof. □

4 Completeness of the space of all zero-additive maps

For the proof of Theorem 1 we need first to develop some more ideas. Let (G, ρ) be a controlled group and Y a Banach space. Let $Z(G, Y)$ be the (real) vector space of all zero-additive maps from G to Y. Consider the functional $F \in Z(G, Y) \to Z(F) \in \mathbb{R}$. Clearly, $Z(\cdot)$ is a seminorm on $Z(G, Y)$ whose kernel is $\mathrm{Hom}(G, Y)$, so that $Z(\cdot)$ defines a norm on the quotient space $Z(G, Y)/\mathrm{Hom}(G, Y)$.

Theorem 5. *Let (G, ρ) be a controlled group and Y a Banach space. Then the space $Z(G, Y)/\mathrm{Hom}(G, Y)$ endowed with $Z(\cdot)$ is a Banach space.*

Proof. Let $[G_n]$ be a Cauchy sequence in $Z(G, Y)/\mathrm{Hom}(G, Y)$ for $Z(\cdot)$. We want to see that there are representatives F_n of $[G_n]$ such that F_n is pointwise convergent on G. Let ρ^* be as in Theorem 2 and put $F_n = G_n - A_n$, where A_n is the additive map associated with G_n in the proof of Theorem 2. Clearly, one has

$$\|F_n(x) - F_m(x)\| \leq Z(G_n - G_m)\rho^*(x),$$

so that $F_n(x)$ converges in Y for each $x \in G$.

Finally, let $F(x) = \lim_n F_n(x)$. It is easily seen that F is zero-additive on G, and also that $[G_n] = [F_n]$ converges to $[F]$ with respect to $Z(\cdot)$ which proves Theorem 5. □

Lemma 6. *Let (G, ρ) be a controlled group, Y a Banach space and $B(G, Y)$ the linear space of all bounded maps $G \to Y$. Then the quotient space $\{B(G, Y) + \mathrm{Hom}(G, Y)\}/\mathrm{Hom}(G, Y)$ endowed with the norm $[F] \to \mathrm{dist}\big(F, \mathrm{Hom}(G, Y)\big)$ is a Banach space.*

Proof. Simply observe that $\{B(G, Y) + \mathrm{Hom}(G, Y)\}/\mathrm{Hom}(G, Y)$ endowed with $\mathrm{dist}\big([\cdot], \mathrm{Hom}(G, Y)\big)$ is naturally isomorphic to $B(G, Y)/\{B(G, Y) \cap \mathrm{Hom}(G, Y)\}$, where $B(G, Y)$ is equipped with $F \to \mathrm{dist}(F, 0)$, which is easily proved to be a Banach space. □

With all this machinery, we are ready to end the

Proof of Theorem 1. First we observe that since for any controlled group G the space $Z(G, Y)$ contains $B(G, Y) + \mathrm{Hom}(G, Y)$, the hypothesis 'every zero-additive map $G \to Y$ is at finite distance from additive ones' means that both spaces coincide. Thus both $Z(\cdot)$ and $\mathrm{dist}\big([\cdot], \mathrm{Hom}(G, Y)\big)$ are defined on $Z(G, Y)/\mathrm{Hom}(G, Y)$ making it complete by Theorem 5 and Lemma 6. Since it is quite clear that $Z(\cdot)$ is dominated by $\mathrm{dist}\big([\cdot], \mathrm{Hom}(G, Y)\big)$ the open mapping theorem implies that $Z(\cdot)$ and $\mathrm{dist}\big([\cdot], \mathrm{Hom}(G, Y)\big)$ are K-equivalent on $Z(G, Y)/\mathrm{Hom}(G, Y)$ for some $K \geq 0$ which is nothing but a restatement of the property $K\mathbf{Z}$. □

5 Concluding remarks

The proofs of Theorems 1, 2 and 5 can be adapted for other types of nearly additive maps. Recall that a mapping $F\colon G \to Y$ is said to be Ger-additive (respectively pseudoadditive) [3] if there is a constant K such that $\|F(x + y) - F(x) - F(y)\| \leq K\rho(x)$ (resp. $\|F(x+y) - F(x) - F(y)\| \leq K\{\rho(x) + \rho(y) - \rho(x+y)\}$) holds for all

$x, y \in G$. The meaning of the constant $G(F)$ (resp. $P(F)$ and the space $G(G, Y)$ (resp. $P(G, Y)$ should be clear. As in the zero-additive case, let us say that the pair $\{(G, \rho); Y\}$ has the property $K\mathbf{G}$ (resp. $K\mathbf{Z}$) if for every Ger-additive (resp. pseudo-additive) map $F: G \to Y$ there is an additive map A such that $\operatorname{dist}(F, A) \leq KG(F)$ (resp. $\operatorname{dist}(F, A) \leq KP(F)$). Observe that since $Z(F) \leq \min\{G(F), P(F)\}$ the property $K\mathbf{Z}$ implies both $K\mathbf{G}$ and $K\mathbf{P}$ (the converse is false: If G is a commutative group and Y a Banach space complemented in its second dual then $\{G; Y\}$ has both $1\mathbf{G}$ and $2\mathbf{Z}$, while every super-reflexive (infinite-dimensional) Banach space admits zero-additive maps into itself which are not asymptotically additive [3]). We have

Theorem 7. *Let (G, ρ) be a controlled group and Y a Banach space. Suppose that every Ger-additive (resp. pseudo-additive) map $G \to Y$ is asymptotically additive. Then the pair $\{(G, \rho); Y\}$ has the property $K\mathbf{G}$ (resp. $K\mathbf{P}$) for some $K \geq 0$.*

Theorem 8. *Let (G, ρ) be a controlled group and Y a Banach space. Then one has*

a) *The space $G(G, Y)/\operatorname{Hom}(G, Y)$ endowed with $G(\cdot)$ is a Banach space;*
b) *The space $P(G, Y)/\operatorname{Hom}(G, Y)$ endowed with $P(\cdot)$ is a Banach space.*

References

[1] M. F. ATIYAH and I. G. MACDONALD, *Introducción al álgebra commutativa.* Reverté, Barcelona, 1980.

[2] F. CABELLO SÁNCHEZ, Some remarks stemming from Ulam's problem about nearly additive mappings. *Aequat. Math.* **56** (1998), 233–242.

[3] F. CABELLO SÁNCHEZ and J. M. F. CASTILLO, Banach space techniques underpinning a theory of nearly additive mappings. *Universidad de Extremadura* Preprint 1998.

[4] J. M. F. CASTILLO and M. GONZÁLEZ, *Three-space problems in Banach space theory.* Lecture Notes in Math **1667**, Springer 1997.

[5] V. A. FAIZIEV, Pseudocharacters on free semigroups invariant with respect to their automorphism groups. *Proc. Amer. Math. Soc.* **126** (1998), 333-342.

[6] G. L. FORTI, The stability of homomorphisms and amenability, with applications to functional equations. *Abh. Math. Sem. Univ. Hamburg* **57** (1987), 215–226.

[7] ———, Hyers-Ulam stability of functional equations in several variables. *Aequat. Math.* **50** (1995), 143–190.

[8] B. E. JOHNSON, Approximately multiplicative maps between Banach algebras. *J. London Math. Soc.* **37** (1988), 294–316.

[9] P. SEMRL, The stability of approximately additive functions. In: *Stability of mappings of Hyers-Ulam type* (ed. by Th. M. Rassias and J. Tabor) Hadronic Press, Florida (1994), 135–140.

[10] S. M. ULAM, *An Anecdotal History of the Scottish Book.* In: The Scottish Book, ed. by R. D. Mauldin, Birkhäuser, 1981.

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