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STABILITY OF ADDITIVE MAPPINGS ON LARGE SUBSETS

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Abstract. We study mappings from a group into a Banach space which are "nearly additive" on large subsets.

1. Introduction and statement of the results

This note is concerned with the stability of additive maps on restricted domain. The basic problem (which goes back to Ulam [16]) can be stated in a vague manner as follows: let G be a group (written additively in what follows), B a suitable subset of G, Y a Banach space and $F\colon B \to Y$ a mapping which is, in some sense to be made precise, "nearly additive". Must F be near to a mapping $A\colon B \to Y$ additive on B? If so, can A be extended as an additive map from B to G? We refer the reader to the survey papers [4, 9] for general information on the subject.

A partial affirmative answer has been recently given by Hyers, Isac and Rassias [10]: given a real normed space Z and a real Banach space Y, let numbers $k > 0$, $\varepsilon > 0$ and $0 \le p < 1$ be chosen. Suppose that the mapping $F\colon Z \to Y$ satisfies the inequality $\|F(x+y) - F(x) - F(y)\| \le \varepsilon\{\|x\|^p + \|y\|^p\}$ for all $x, y \in Z$ such that $\|x\|, \|y\|, \|x+y\| > k$. Then there is an additive mapping $A\colon Z \to Y$ satisfying $\|F(x) - A(x)\| \le 2\varepsilon(2-2^p)^{-1}\|x\|^p$ for all $x \in Z$ such that $\|x\| > k$. Moreover, A is given by $A(x) = \lim_n 2^{-n}F(2^n x)$.

Well-known examples [5, 11, 13, 14] show that this result cannot be extended to the case $p = 1$, even if $F\colon \mathbb{R} \to \mathbb{R}$ is a mapping satisfying $|F(x+y) - f(x) - F(y)| \le \varepsilon\{|x| + |y|\}$ for all x, y. This leads Johnson [11], Ger [7], Semrl [15], Forti [4] and others [1, 2] to deal with other types of "nearly additive" mappings.

Definition 1. Let G be a group on which a nonnegative "control functional" $\rho\colon G \to \mathbb{R}$ is given, B a subset of G, Y a Banach space and $F\colon B \to Y$ a mapping.

(a) F is called pseudo-additive (with constant K) on B if $\|F(x+y) - F(x) - F(y)\| \le K(\rho(x) + \rho(y) - \rho(x+y))$ holds for every $x, y \in G$ such that x, y and $x+y$ belong to B.

(b) F is said to be Ger-additive (with constant K) on B if $\|F(x+y) - F(x) - F(y)\| \le K\rho(x)$ holds for every $x, y \in G$ such that x, y and $x+y$ belong to B.

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Finally, we consider relations between an arbitrary (but finite) number of variables.

(c) A mapping $F\colon B \to Y$ is zero-additive (with constant K) on B if, for all $n \in \mathbb{N}$, one has $\|F(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} F(x_i)\| \leq K(\sum_{i=1}^{n} \rho(x_i))$ whenever $x_i, \sum_i x_i \in B$.

(See [1, 2] for background.) Our approach is quite different from the direct methods of [10] and strongly depends on the existence of invariant means for the group on which the maps are defined.

Recall from [8] that a (not necessarily commutative) group G is said to be (left) amenable if there is a (left) invariant mean for G; that is, a bounded linear functional m on $B(G)$ (the Banach space of all bounded maps $G \to \mathbb{R}$ with the sup norm) such that $m\{f\} \geq 0$ for all $f \geq 0$, $m\{1\} = 1$ and with the following invariance property: $m\{f_x\} = m\{f\}$ for all $f \in B(G)$ and all $x \in G$, where $f_x(y) = f(x + y)$. Right amenability is defined in a similar way. Commutative groups are always amenable.

Definition 2. Let G be a group and let m be an invariant mean for G. A subset B of G shall be called big for m if $m(1_B) = 1$, where 1_B denotes the characteristic function of B. A set is called big provided it is big for some (left or right) invariant mean on G.

Abundant examples of big sets (including the complements of bounded sets and linear manifolds in normed spaces) are given in Section 2. Our main result is the following.

Theorem 3. *Let G be a commutative group endowed with a control functional ρ and let B a "big" subset of G. Suppose that $F\colon B \to \mathbb{R}$ is a zero-additive (resp. Ger-additive or pseudo-additive) map with constant K on B. Then there exists an additive map $A\colon G \to \mathbb{R}$ such that $|F(x) - A(x)| \leq K\rho(x)$ for every $x \in B$.*

In particular, additive maps can be extended from a given big subset to the whole group. (See Theorem 8 for a stronger result.) For vector valued maps, we have:

Theorem 4. *Let G be an amenable group endowed with ρ, B a big subset of G and Y a Banach space complemented in its second dual by a projection π. Suppose that $F\colon B \to Y$ is Ger-additive on B with constant K. Then there exists an additive mapping $A\colon G \to Y$ such that $\|F(x) - A(x)\| \leq K\|\pi\|\rho(x)$ for every $x \in B$.*

Corollary 5. *Let G be an amenable group endowed with a symmetric control functional ρ (i.e., $\rho(-x) = \rho(x)$ for all $x \in G$), B a symmetric big subset of G and Y a Banach space complemented in its second dual by a projection π. Suppose that $F\colon B \to Y$ is pseudo-additive on B with constant K. Then there exists an additive mapping $A\colon G \to \mathbb{R}$ such that $\|F(x) - A(x)\| \leq 2K\|\pi\|\rho(x)$ for every $x \in B$.*

2. Big subsets of amenable groups

In this section, we give simple examples of big sets. Let m be an invariant mean for G. Obviously, B is a big set for m if and only if its complement is a residual set for m, that is, $m(1_{G\setminus B}) = 0$. Clearly, the intersection of finitely many big sets for m is a big set for m too. The following result yields more examples of big sets.

Lemma 6. *Let C be a subset of a group G. Suppose that for each n there exist points $s_1, \ldots, s_n$ in G such that $s_k + C$ are pairwise disjoint. Then C is a residual set for any (left) invariant mean on G.*

Proof. Simply note that, for every left invariant mean m and for all n, one has

$$1 = m\{1\} \geq m\left\{\sum_{k=1}^{n} 1_{s_k+C}\right\} = \sum_{k=1}^{n} m\{1_{s_k+C}\} = n(m\{1_C\}).$$

Corollary 7. (a) *Let G be a commutative group and H a subgroup of G. If G/H is infinite, then H is a residual set for any invariant mean on G. In particular, proper subspaces and manifolds of vector spaces are residual sets.*

(b) *Let d be an unbounded invariant metric on a group G. Then bounded sets are residual sets for any invariant mean on G. In particular, bounded sets in normed spaces are residual.*

(c) *Let X be a vector space and $f\colon X \to \mathbb{K}$ a nonzero linear functional. Let K be a bounded subset of $\mathbb{K}$. Then the "infinite strip" $\{x \in X\colon f(x) \in K\}$ is a residual set for every invariant mean on X.*

Remark. Despite the previous examples it should be noted that a big set for a given invariant mean need not be big for all invariant means. In fact, for every $0 \leq c \leq 1$ there is a (two-sided) invariant mean m on $\mathbb{Z}$ such that $m(1_{\mathbb{N}}) = c$.

3. Extending additive maps from big sets

Let us start with the following result.

Theorem 8. *Let G be an amenable group, B a subset of G, V a real vector space and $a\colon B \to V$ a mapping additive on B (that is, such that $a(x+y) = a(x) + a(y)$ whenever x, y and $x+y$ belong to B). If B is a big set for some invariant mean on G, then a admits a unique additive extension $A\colon G \to V$.*

For the proof we need to develop some ideas. Let $m\{\cdot\}$ be a left invariant mean for G. Consider the following subspace of $B(G)$:

$$N_m = \{f \in B(G)\colon m\{1_{\mathrm{sop}(f)}\} = 0\},$$

where $\mathrm{sop}(f) = \{y\colon f(y) \neq 0\}$. Clearly, $m\{f\} = 0$ for all $f \in N_m$, so that $m\{\cdot\}$ is well-defined on the quotient space $B(G)/N_m$ by $m\{[f]\} = m\{f\}$. Observe that $[g]$ can be regarded as an element of $B(G)/N_m$ even if g is defined only on a subset of G and bounded on some m-big subset of G. For such a g one can define $m\{g\} = m\{[g]\}$. Moreover, if g is defined (resp. bounded) on D, then g_x is defined (resp. bounded) on $-x+D$ (which is a big set for m if D is) and one has $m\{g_x\} = m\{g\}$.

Proof of Theorem 8. To fix ideas, assume that B is a big set for some left invariant mean m on G. We first prove the theorem for $V = \mathbb{R}$.

Observe that, for every $x \in G$, there exist $x_1, x_2 \in B$ such that $x = x_1 - x_2$. (It obviously suffices to see that the set $\{x_2 \in B\colon x+x_2 \in B\} = B\cap(-x+B)$ is nonvoid, which is clear since, actually, $m\{1_{B\cap(-x+B)}\} = 1$.) Now, put $A(x) = a(x_1) - a(x_2)$. We want to see that $A(x)$ does not depend on x_1 or on x_2 but only on x:

$$\begin{aligned}
a(x_1) - a(x_2) &= m_y\{a(x_1+y) - a(y)\} - m_y\{a(x_2+y) - a(y)\}\\
&= m_y\{a(x_1+y) - a(x_2+y)\}\\
&= m_y\{a(x_1-x_2+y) - a(y)\}\\
&= m_y\{a(x+y) - a(y)\}.
\end{aligned}$$

(The subscript y indicates that the mean is applied to a function of the variable y.) This also shows that $A(x)$ can be defined as $A(x) = m_y\{a(x+y) - a(y)\}$ on G. That A is an extension of a is clear since for $x \in B$ one has $a(x+y) - a(y) = a(x)$ for every y in the big set $B \cap (-x + B)$.

Finally, let $x, z \in G$. Then

$$\begin{aligned} A(x+z) &= m_y\{a(x+z+y) - a(y)\} \\ &= m_y\{a(x+z+y) - a(z+y) + a(z+y) - a(y)\} \\ &= m_y\{a(x+z+y) - a(z+y)\} + m_y\{a(z+y) - a(y)\} \\ &= m_y\{a(x+y) - a(y)\} + m_y\{a(z+y) - a(y)\} \\ &= A(x) + A(z), \end{aligned}$$

so that A is additive.

We pass to the vector-valued case. Let V' denote the algebraic dual of V over $\mathbb{R}$. For $x \in G$, pick $x_1, x_2 \in B$ such that $x = x_1 - x_2$ and define, as before, $A(x) = a(x_1) - a(x_2) \in V$. Observe that for every $f \in V'$ one has $f(a(x_1) - a(x_2)) = m_y\{f(a(x+y) - a(y))\}$; hence $A(x)$ depends only on x. That A extends a is obvious. Finally, the additivity of A is a consequence of the fact that for every $f \in V'$ the map fA is additive.

4. Proof of Theorems 3 and 4

Proof of Theorem 4. Assume that B is a big set for some left invariant mean m on G. For each $x \in B$, define $a(x)$ on Y^* as

$$a(x)y* = m_y\{y^*(F(x+y) - F(y))\}.$$

The definition of $a(x)$ makes sense since $y^*(F(x+y) - F(y))$ is bounded by

$$\|y^*\|(\|Fx\| + K\rho(x))$$

on the big set $B \cap (-x + B)$. Clearly, $a(x)\colon Y^* \to \mathbb{R}$ is a linear map. The boundedness of $a(x)$ follows from the estimate

$$|a(x)y^* - y^*F(x)| = m_y\{y^*(F(x+y) - F(y) - F(x))\} \le \|y^*\|K\rho(x),$$

which also shows that $\|a(x) - F(x)\|_{Y^{**}} \le K\rho(x)$. Let us prove that a acts additively on B. Let $x, z \in G$. Then

$$\begin{aligned} a(x+z)y^* &= m_y\{y^*(F(x+z+y) - F(y))\} \\ &= m_y\{y^*(F(x+z+y) - F(z+y))\} + m_y\{y^*(F(z+y) - F(y))\} \\ &= (a(x) + a(x))y^*. \end{aligned}$$

Finally, let $\pi\colon Y^{**} \to Y$ be a bounded projection. Then πa is an additive map from B to Y with $\|\pi a(x) - F(x)\|_Y \le \|\pi\|K\rho(x)$ for every $x \in B$. Now apply Theorem 8.

Proof of Corollary 5. Observe that the hypotheses imply that $\|F(x+y) - F(x) - F(y)\| \le 2K\rho(x)$ and apply Theorem 4.

The proof of Theorem 3 is based on the following variation of [6, Theorem 3] which can be understood as a "Sandwich theorem" on a restricted domain.

Lemma 9. *Let G and B be as in Theorem 3. Suppose that $\alpha, \beta\colon B \to \mathbb{R}$ are such that α is superadditive on B (i.e., $\alpha(x+y) \ge \alpha(x) + \alpha(y)$ whenever x, y and $x+y$ belong to B), β is subadditive on B and $\alpha(x) \le \beta(x)$ for all $x \in B$. Then there* *exists an additive mapping $A\colon G \to \mathbb{R}$ separating α from β on B, that is, satisfying $\alpha(x) \le A(x) \le \beta(x)$ for every $x \in B$.*

Proof of Lemma 9. Assume that B is a big subset for some left (hence two-sided, by commutativity) invariant mean m on G. For a real-valued map f defined on an m-big subset of G, put

$$\operatorname{ess.inf}_y\{f\} = \inf\{t \in \mathbb{R}\colon m(\{y \in G\colon f(y) \le t\}) \ne 0\}.$$

Now the proof closely follows [6]. Note that if x and y are such that $x, y, x+y \in B$, one has

$$\beta(x+y) - \alpha(x) \ge \alpha(x+y) - \alpha(x) \ge \alpha(y).$$

Hence, for $x \in B$, one can define

$$h(x) = \operatorname{ess.inf}_y\{\beta(x+y) - \alpha(y)\} \ge \alpha(x).$$

Suppose that x, y and $x+y$ are in B. Then

$$\begin{aligned} h(x+y) &= \operatorname{ess.inf}_z\{\beta(x+y+z) - \alpha(z)\} \\ &\le \operatorname{ess.inf}_z\{\beta(x) + \beta(y+z) - \alpha(z)\} \\ &= B(x) + h(y). \end{aligned}$$

Moreover,

$$\begin{aligned} h(x+y) &= \operatorname{ess.inf}_z\{\beta(x+y+z) - \alpha(z)\} \\ &\ge \operatorname{ess.inf}_z\{\beta(x+y+z) + \alpha(x) - \alpha(z+x)\} \\ &= \alpha(x) + \operatorname{ess.inf}_z\{\beta(x+y+z) - \alpha(z+x)\} \\ &= \alpha(x) + \operatorname{ess.inf}_w\{\beta(y+w) - \alpha(w)\} \\ &= \alpha(x) + h(y). \end{aligned}$$

(This is the only point where the commutativity is needed.) Therefore one has

$$\alpha(x) \le h(x+y) - h(y) \le \beta(x)$$

whenever x, y and $x+y$ belong to B. Finally, define a map $a\colon B \to \mathbb{R}$ by

$$a(x) = m_y\{h(x+y) - h(y)\}.$$

The argument used in the proof of Theorem 4 shows that a is additive on B and an appeal to Theorem 8 completes the proof of the lemma.

Proof of Theorem 3. Notice that the "Ger-additive" part has been already proved. We now prove the statement about pseudo-additive maps. Let $F\colon B \to \mathbb{R}$ be such that $|F(x+y) - F(x) - F(y)| \le K(\rho(x) + \rho(y) - \rho(x+y))$ for $x, y, x+y \in B$. Then $F + K\rho$ is subadditive on $B, F - K\rho$ is superadditive on B and $(F - K\rho)(x) \le (F + K\rho)(x)$ for every $x \in B$. Lemma 9 yields an additive map $A\colon G \to \mathbb{R}$ such that $F(x) - K\rho(x) \le A(x) \le F(x) + K\rho(x)$ for every $x \in B$, which obviously implies that $|F(x) - A(x)| \le K\rho(x)$ for every $x \in B$, as desired.

Finally, suppose that F is zero-additive on B with constant K. For $x \in B$, define

$$\begin{aligned} \alpha(x) &= \inf\left\{\sum_{i=1}^n F(x_i) + K\sum_{i=1}^n \rho(x_i)\colon x = \sum_i x_i, x_i \in B\right\}, \\ \beta(x) &= \sup\left\{\sum_{i=1}^n F(x_i) - K\sum_{i=1}^n \rho(x_i)\colon x = \sum_i x_i, x_i \in B\right\}. \end{aligned}$$

Clearly, α is superadditive on B and β is subadditive on B. We claim that $\alpha(x) \leq \beta(x)$ for $x \in B$ (which implies that both functions take only finite values on B). Indeed, let $x \in B$. One has to verify that if x_i and y_j are points in B such that $x = \sum_i x_i = \sum_j y_j$, then

$$\sum_{i=1}^{n} F(x_i) - K \sum_{i=1}^{n} \rho(x_i) \leq \sum_{j=1}^{m} F(y_j) + K \sum_{j=1}^{m} \rho(y_j),$$

or, in other words, that

$$\sum_{i=1}^{n} F(x_i) - \sum_{u=1}^{m} F(y_j) \leq K \left[\sum_{i=1}^{n} \rho(x_i) + \sum_{j=1}^{m} \rho(y_j) \right],$$

which immediately follows from zero-additivity. Lemma 9 yields an additive map A fulfilling

$$F(x) - K\rho(x) \leq \alpha(x) \leq A(x) \leq \beta(x) \leq F(x) + K\rho(x),$$

hence $|F(x) - A(x)| \leq K\rho(x)$ for every $x \in B$, and the proof is complete.

5. Concluding remarks and questions

One may ask about the rôle of the hypotheses about G and Y in Theorems 3 and 4 and Corollary 5. We know from [3] that there exists a real-valued mapping F on $\mathbb{F}_2$ (the free group with two generators) such that $F(x+y) - F(x) - F(y) \in \{-1, 0, 1\}$ for all x, $y \in \mathbb{F}_2$ (thus F is zero-additive, Ger-additive and even pseudo-additive with respect to $\rho(x) = 1$ on $\mathbb{F}_2$) but such that $F(x) - A(x)$ is unbounded on $\mathbb{F}_2$ for any additive A. Hence some condition on G is necessary to get stability in 3, 4 and 5 (even in the scalar case) and also to obtain the separating map in Lemma 9.

On the other hand, we do not know if the hypothesis about Y can be removed in 4 and 5. If Theorem 4 were true for any Banach space Y (not necessarily complemented in its bidual) and $B = G$ a Banach space endowed with its norm, then the long-standing problem of whether or not subspaces of a Banach space whose metric projection admits a uniformly continuous selection are complemented would have an affirmative answer. (See [2] for details.) Also, if Corollary 5 remains true for all Banach spaces Y (and $B = G$ a Banach space endowed with its norm), then absolutely Chebyshev subspaces are always complemented subspaces. (Absolutely, Chebyshev subspaces are important in approximation theory, see [12].)

Finally, the statement of Theorem 3 concerning zero-additive maps is false if $\mathbb{R}$ is replaced by an arbitrary Banach space Y. In fact, let Y be a closed subspace of a Banach space X and let $G = X/Y$ endowed with the quotient norm. Then a zero-additive map $F\colon G \to Y$ can be obtained as follows: choose a bounded (not necessarily continuous nor linear) homogeneous selection $B\colon G \to X$ for the quotient map $\pi\colon X \to G$ (i.e., such that $\|B(x)\| \leq K\|x\|$ for some K and all $x \in G$). Let $L\colon G \to X$ be a linear (not necessarily bounded) selection for π. Then the difference $F = B - L$ takes values in Y (instead of X) and is zero-additive since, for $x_i \in B$, one has

$$\left\| F\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} F(x_i) \right\| = \left\| B\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} B(x_i) \right\| \leq 2K \sum_{i=1}^{n} \|x_i\|.$$

Moreover, an additive map $A\colon G \to Y$ fulfilling $\|F(x) - A(x)\| \leq M\|x\|$ for some M and every $x \in G$ exists if and only if Y is complemented in X; see [2] for details.

Hence vector-valued zero-additive maps need not be close to additive maps, even if they act on a Banach space.

Added in proof

The unrestricted domain version of Theorem 4 was essentially proved by R. Ger in *The singular case in the stability behaviour of linear mappings* (Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian-Polish Seminar, Graz, 1991), Grazer Math. Ber. **316** (1992), 59–70.

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