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Polynomials on dual-isomorphic spaces

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In this note we study isomorphisms between spaces of polynomials on Banach spaces. Precisely, we are interested in the following question raised in [5]: If X and Y are Banach spaces such that their topological duals X' and Y' are isomorphic, does this imply that the corresponding spaces of homogeneous polynomials $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for every $n\geq 1$?

Díaz and Dineen gave the following partial positive answer [5, Proposition 4]: Let X and Y be dual-isomorphic spaces; if X' has the Schur property and the approximation property, then $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for every n. Observe that the Schur property of X' makes all bounded operators from X to X' (and also from Y to Y') compact. That hypothesis can be considerably relaxed. Following [6], [7], let us say that X is regular if every bounded operator $X\to X'$ is weakly compact. We prove the following result.

Theorem 1. *Let X and Y be dual-isomorphic spaces. If X is regular then $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for every $n\geq 1$.*

In fact, it is even true that the corresponding spaces of holomorphic maps of bounded type $\mathcal{H}_b(X)$ and $\mathcal{H}_b(Y)$ are isomorphic Fréchet algebras. Observe that the approximation property plays no rôle in Theorem 1. This is relevant since, for instance, the space of all bounded operators on a Hilbert space is a regular space (as every C^*-algebra [7]) but lacks the approximation property.

Our techniques are quite different from those of [5] and depend on certain properties of the extension operators introduced by Nicodemi in [10]. For stable spaces (that is, for spaces isomorphic to its square) one has the following stronger result.

Theorem 2. *If X and Y are dual-isomorphic stable spaces, then $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for every $n\geq 1$.*

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At the end of the paper we present examples of Banach spaces X, Y with $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ isomorphic for every $n\geq 1$ despite the following facts.

Example 1. All polynomials on X are weakly sequentially continuous, while Y contains a complemented subspace isomorphic to l_2 (thus there are plenty of polynomials which are not weakly sequentially continuous).

Example 2. The space X is separable and Y is not.

Example 3. Every infinite-dimensional subspace of X contains a copy of l_2, X has the Radon–Nikodym property and Y is isomorphic to c_0.

1. Multilinear maps and Nicodemi operators

Our notation is standard and follows [5]. Let $Z_1, \dots, Z_n$ be Banach spaces. Then, for each $1\leq i\leq n$, there is an isomorphism

$$(\,\cdot\,)_i\colon \mathcal{L}(Z_1, \dots, Z_n) \longrightarrow \mathcal{L}(Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n; Z_i')$$

given by

$$\langle A_i(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n), z_i\rangle = A(z_1, \dots, z_n).$$

The inverse isomorphism will be denoted $(\,\cdot\,)^i$. Thus, for any vector-valued multilinear map $B\in\mathcal{L}(Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n; Z_i')$, we have

$$B^i(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_n) = \langle B(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n), z_i\rangle.$$

Our main tool are the extension operators introduced by Nicodemi in [10] whose construction we briefly sketch (see also [6]). Let X and Y be Banach spaces. Given an operator $\Phi\colon X'\to Y'$, one can construct a sequence of bounded operators $\Phi^{(n)}\colon \mathcal{L}(^nX)\to\mathcal{L}(^nY)$ between the spaces of multilinear forms as follows. For $1\leq i\leq n$, define

$$\Phi_i^{(n)}\colon \mathcal{L}(X, \overset{(i)}{\dots}, X, Y, \overset{(n-i)}{\dots}, Y) \longrightarrow \mathcal{L}(X, \overset{(i-1)}{\dots}, X, Y, \overset{(n-i+1)}{\dots}, Y)$$

as

$$\Phi_i^{(n)}(A) = (\Phi\circ A_i)^i.$$

Finally, define $\Phi^{(n)}$ by

$$\Phi^{(n)} = \Phi_1^{(n)}\circ\Phi_2^{(n)}\circ\dots\circ\Phi_{n-1}^{(n)}\circ\Phi_n^{(n)}.$$

Clearly, if $\Phi\colon X'\to Y'$ is an isomorphism, so is every $\Phi_i^{(n)}$. Hence we have the following lemma.

Lemma 1. *Let* $\Phi: X' \to Y'$ *be an isomorphism. Then* $\Phi^{(n)}$ *is an isomorphism for every* $n \geq 1$*.*

Corollary 1. *If* X *and* Y *are dual-isomorphic spaces, then* $\mathcal{L}(^n X)$ *and* $\mathcal{L}(^n Y)$ *are isomorphic for every* $n \geq 1$*.*

We are ready to prove Theorem 2.

Proof of Theorem 2. The hypothesis on X and Y together with [5, Theorem 2(ii)] and Corollary 1 above yields $\mathcal{P}(^n X) \approx \mathcal{L}(^n X) \approx \mathcal{L}(^n Y) \approx \mathcal{P}(^n Y)$, as desired. □

Identifying $\mathcal{P}(^n X)$ with the space of symmetric forms $\mathcal{L}_s(^n X)$ (and also $\mathcal{P}(^n Y)$ with $\mathcal{L}_s(^n Y)$) one might think that, given an isomorphism $\Phi: X' \to Y'$, the restriction of $\Phi^{(n)}$ to $\mathcal{L}_s(^n X)$ could give an isomorphism between the spaces of polynomials. Unfortunately, we are unable to prove that $\Phi^{(n)}(A)$ is symmetric when A is (we believe that not all isomorphisms Φ achieve this). Fortunately, this is always true when X is regular. The following result will clarify the proof of Theorem 1.

Proposition 1. *Let* X *and* Y *be dual-isomorphic Banach spaces. If* X *is regular then so is* Y*.*

Proof. Let $\mathcal{B}$ denote bounded operators and $\mathcal{W}$ weakly compact operators. It clearly suffices to see that $\mathcal{B}(Y, X') = \mathcal{W}(Y, X')$, which follows from the regularity of X ($\mathcal{B}(X, Y') = \mathcal{W}(X, Y')$) together with the natural isomorphism $\mathcal{B}(Y, X') = \mathcal{B}(X, Y')$ and Gantmacher's theorem ($\mathcal{W}(Y, X') = \mathcal{W}(X, Y')$). □

Our immediate objective is the following representation of Nicodemi operators.

Lemma 2. *Let* $\Phi: X' \to Y'$ *be a bounded operator. For every* $A \in \mathcal{L}(^n X)$ *and all* $y_i \in Y$ *one has*

$$\Phi^{(n)}(A)(y_1, \ldots, y_n) = \lim_{x_1 \to \Phi'(y_1)} \ldots \lim_{x_n \to \Phi'(y_n)} A(x_1, \ldots, x_n),$$

where the iterated limits are taken for $x_i \in X$ *converging to* $\Phi'(y_i)$ *in the weak* topology of* X''*.*

Proof. Let $B \in \mathcal{L}(X, \overset{(i)}{\ldots}, X, Y, \overset{(n-i)}{\ldots}, Y)$. Then

$$\begin{aligned}
\Phi_i^{(n)} B(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n) &= (\Phi \circ B_i)^i (x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n) \\
&= \langle (\Phi \circ B_i)(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n), y_i \rangle \\
&= \langle \Phi(B_i(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n)), y_i \rangle \\
&= \langle B_i(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n), \Phi'(y_i) \rangle \\
&= \lim_{x_i \to \Phi'(y_i)} \langle B_i(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n), x_i \rangle \\
&= \lim_{x_i \to \Phi'(y_i)} B(x_1, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n),
\end{aligned}$$

from which the result follows. □

It is apparently a well-known fact that if X is a regular space, the iterated limit in the preceding lemma does not depend on the order of the involved variables.

Lemma 3. *Suppose that X is regular. Then, for every $A\in\mathcal{L}(^nX)$ and every permutation π of $\{1,\dots,n\}$, one has*

$$\lim_{x_1\to x_1''}\dots\lim_{x_n\to x_n''}A(x_1,\dots,x_n)=\lim_{x_{\pi(1)}\to x_{\pi(1)}''}\dots\lim_{x_{\pi(n)}\to x_{\pi(n)}''}A(x_1,\dots,x_n)$$

for all $x_i''\in X''$, where the iterated limits are taken for $x_i\in X$ converging to x_i'' in the weak topology of X''.*

We refer the reader to [2, Section 8] for a simple proof. It will be convenient to write the limit appearing in Lemma 3 in a more compact form. Thus, given $A\in\mathcal{L}(^nX)$, consider the multilinear form $\alpha\beta(A)$ given on X'' by

$$\alpha\beta(A)(x_1'',\dots,x_n'')=\lim_{x_1\to x_1''}\dots\lim_{x_n\to x_n''}A(x_1,\dots,x_n).$$

This is the Aron–Berner extension of A (see [1, Proposition 2.1], or [2, Section 8]. Actually the extension operator $\alpha\beta\colon\mathcal{L}(^nX)\to\mathcal{L}(^nX'')$ is nothing but the Nicodemi operator induced by the natural inclusion $X'\to X'''$. In this setting, it is clear that if $\Phi\colon X'\to Y'$ is an operator, then

$$\Phi^{(n)}(A)(y_1,\dots,y_n)=\alpha\beta(A)(\Phi'(y_1),\dots,\Phi'(y_n)).$$

From this, we obtain the following lemma.

Lemma 4. *Let X be a regular space and let $\Phi\colon X'\to Y'$ be an operator. Then, for each $n\geq 1$, the restriction of $\Phi^{(n)}$ to $\mathcal{L}_s(^nX)$ takes values in $\mathcal{L}_s(^nY)$.*

Proof. It obviously suffices to see that $\alpha\beta(A)$ belongs to $\mathcal{L}_s(^nX'')$ for every symmetric $A\in\mathcal{L}(^nX)$. If $\pi\in S_n$, then

$$\begin{aligned}\alpha\beta(A)(x_1'',\dots,x_n'')&=\lim_{x_1\to x_1''}\dots\lim_{x_n\to x_n''}A(x_1,\dots,x_n)\\&=\lim_{x_1\to x_1''}\dots\lim_{x_n\to x_n''}A(x_{\pi(1)},\dots,x_{\pi(n)})\\&=\lim_{x_{\pi(1)}\to x_{\pi(1)}''}\dots\lim_{x_{\pi(n)}\to x_{\pi(n)}''}A(x_{\pi(1)},\dots,x_{\pi(n)})\\&=\alpha\beta(A)(x_{\pi(1)}'',\dots,x_{\pi(n)}''),\end{aligned}$$

as desired. □

End of the proof of Theorem 1. If $\Phi: X' \to Y'$ is an isomorphism and X is a regular space, then, by the lemma just proved, for every $n \geq 1$ the Nicodemi operator $\Phi^{(n)}$ yields an isomorphism from $\mathcal{L}_s(^n X)$ to $\mathcal{L}_s(^n Y)$. It remains to prove that this map is surjective. This is an obvious consequence of Proposition 1, Lemma 4 and the following result which shows the (covariant) functorial character of Nicodemi's procedure on the class of regular spaces. □

Proposition 2. *Let X, Y and Z be regular spaces and let $\Phi: X' \to Y'$ and $\Psi: Y' \to Z'$ be arbitrary operators. Then $(\Psi \circ \Phi)^{(n)} = \Psi^{(n)} \circ \Phi^{(n)}$ for every $n \geq 1$.*

Proof. We only need the regularity of X. It is plain from the definition that for every $A \in \mathcal{L}(^n X)$ the multilinear form $\alpha\beta(A)$ is separately weakly* continuous in the first variable. If X is regular, Lemma 3 implies that $\alpha\beta(A)$ is separately weakly* continuous in each variable. Thus,

$$\begin{aligned}
(\Psi^{(n)} \circ \Phi^{(n)})(A)(z_1, \dots, z_n) &= \Psi^{(n)}(\Phi^{(n)}(A))(z_1, \dots, z_n) \\
&= \lim_{y_1 \to \Psi'(z_1)} \dots \lim_{y_n \to \Psi'(z_n)} \Phi^{(n)}(A)(y_1, \dots, y_n) \\
&= \lim_{y_1 \to \Psi'(z_1)} \dots \lim_{y_n \to \Psi'(z_n)} \alpha\beta(A)(\Phi'(y_1), \dots, \Phi'(y_n)) \\
&= \alpha\beta(A)(\Phi'(\Psi'(y_1)), \dots, \Phi'(\Psi'(y_n))) \\
&= (\Psi \circ \Phi)^{(n)}(A)(z_1, \dots, z_n),
\end{aligned}$$

and the proof is complete. □

Remark 1. In general, $(\Psi \circ \Phi)^{(n)}$ may differ from $\Psi^{(n)} \circ \Phi^{(n)}$; see the instructive counterexample in [6, Section 9].

Corollary 2. *Let X and Y be dual-isomorphic complex spaces. If X is regular, then the Fréchet algebras of holomorphic maps of bounded type $\mathcal{H}_b(X)$ and $\mathcal{H}_b(Y)$ are isomorphic.*

Proof. (See [6] for unexplained terms.) Let $\Phi: X' \to Y'$ be an isomorphism. It is easily seen that the Nicodemi operators have the following property: for all $A \in \mathcal{L}(^n X)$ and all $B \in \mathcal{L}(^k X)$ one has $\Phi^{(n+k)}(A \otimes B) = \Phi^{(n)}(A) \otimes \Phi^{(k)}(B)$. Taking into account that the norm of $\Phi^{(n)}$ is at most $\|\Phi\|^n$, it is not hard to see that the map $\bigoplus_{n=1}^{\infty} \Phi^{(n)}: \mathcal{H}_b(X) \to \mathcal{H}_b(Y)$ given by $\bigoplus_{n=1}^{\infty} \Phi^{(n)}(f) = \sum_{n=1}^{\infty} \Phi^{(n)} d^n f(0)/n!$ is an isomorphism of Fréchet algebras. □

2. The examples

Example 2 can be obtained taking $X = C[0,1]$ and $Y = c_0(J, C[0,1])$, where J is a set having the power of the continuum. Clearly, X and Y are regular spaces.

Moreover, by general representation theorems, one has isometries

$$X' = l_1(J, l_1(\mathbf{N}) \oplus_1 L_1[0,1]) = l_1(J \times J, l_1(\mathbf{N}) \oplus_1 L_1[0,1]) = l_1(J, X') = Y',$$

so Theorem 1 applies.

The space X of Example 3 is Bourgain's example [3] of an l_2-hereditary space having the Radon–Nikodym property and such that X' is isomorphic to l_1 (which obviously implies that X is regular).

Finally, Example 1 is obtained from Theorem 2 taking $X = l_1(l_2^n)$ and $Y = l_1(l_2^n) \oplus l_2$. Clearly, X has the Schur property (weakly convergent sequences converge in norm), and therefore all polynomials on X are weakly sequentially continuous. That Y admits 2-polynomials that are not weakly sequentially continuous is trivial. We want to see that X is stable (this clearly implies that Y is stable too) and that X' and Y' are isomorphic. Let $(e_n)_{n=1}^{\infty}$ be the obvious basis of X and consider the following subspaces of X

$$X_1 = [e_1, e_3, e_4, e_7, e_8, e_9, e_{13}, e_{14}, e_{15}, e_{16}, ...],$$
$$X_2 = [e_2, e_5, e_6, e_{10}, e_{11}, e_{12}, e_{17}, e_{18}, e_{19}, e_{20}, ...].$$

It is easily verified that $X = X_1 \oplus X_2$ and also that $X \cong X_1 \cong X_2$, so that X and Y are stable. To finish, let us prove that X' and Y' are isomorphic. Since $Y' = X' \oplus l_2$ the proof will be complete if we show that l_2 is complemented in X'. (This was first observed by Stegall who gave a rather involved proof; for the sake of completeness we include a simple proof which essentially follows [4].) Let $Q: X = l_1(l_2^n) \to l_2$ be given by $Q((x_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} x_n$. Clearly, Q is a quotient map and therefore $Q': (l_2)' = l_2 \to X'$ is an isomorphic embedding. For each $k \geq 1$, consider the local selection $S_k: l_2 \to l_1(l_2^n)$ given by $S_k = I_k \circ P_k$, where P_k denotes the projection of l_2 onto the subspace spanned by the first k elements of the standard basis and $I_k: l_2^k \to l_1(l_2^n)$ is the inclusion map. Now, take a free ultrafilter U on $\mathbf{N}$ and define $T: X' \to (l_2)'$ by

$$Tx'(x) = \lim_U x'(S_k x)$$

for $x' \in X'$ and $x \in l_2$. Then T is a left inverse for Q'. Indeed, let $f \in (l_2)'$ and take $x \in l_2$. One has $T(Q'(f))(x) = \lim_U Q'(f)(S_k x) = \lim_U f(QS_k x) = f(x)$ since $QS_k x$ converges in norm to x. This completes the proof.

Remark 2. In view of [5, Lemma 3], the following result may be interesting: Let X be a regular space whose dual is stable. Then, for every $n \geq 1$, the spaces $\mathcal{L}_s(^n X)$ and $\mathcal{L}(^n X)$ are isomorphic. (This can be proved by the methods of [5], taking into account that since X^2 is a predual of X', Theorem 1 yields isomorphisms $\mathcal{L}(^n X^2) \cong \mathcal{L}(^n X)$ and $\mathcal{L}_s(^n X^2) \cong \mathcal{L}_s(^n X)$. We refrain from giving the details.)

Remark 3. An operator $T: X \to X'$ is said to be symmetric if $Tx(y)=Ty(x)$ holds for all $x,y \in X$. A Banach space X is said to be symmetrically regular if every symmetric operator $X \to X'$ is weakly compact. Observe that Theorem 1 and Corollary 1 remain valid (with the same proof) replacing "X regular" by "X and Y symmetrically regular". This observation is pertinent since Leung [9] showed that there are symmetrically regular spaces (the duals of certain James-type spaces) which are not regular. On the other hand, l_1 seems to be (essentially) the only known non-symmetrically regular space (see [2, Section 8]). In this way, although the starting question of Díaz and Dineen remains open, the results in this paper show that no available spaces seem to be reasonable candidates for a counterexample (one of the spaces should be non-stable and non-symmetrically regular simultaneously). We do not know if a symmetrically regular space and a non-symmetrically regular space can be dual isomorphic. Again, observe that no predual of l_∞ is symmetrically regular.

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