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Diameter preserving linear maps and isometries, II

FÉLIX CABELLO SÁNCHEZ

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas 06071-Badajoz, Spain
E-mail address: fcabello@unex.es

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Abstract. We study linear bijections of simplex spaces $\mathcal{A}(S)$ which preserve the diameter of the range, that is, the seminorm $\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in S\}$.

Keywords. Isometry; simplex space; linear preserver problem.

1. Introduction and statement of the results

In this paper we study diameter preserving mappings on spaces of affine functions. Precisely, let S be a compact convex set in a locally convex Hausdorff space and let $\mathcal{A}(S)$ be the space of all (real or complex) continuous affine functions on S. We are interested in linear bijections on $\mathcal{A}(S)$ which preserve the diameter of the range, that is, the seminorm

$$\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in S\}.$$

Our main result reads as follows:

Theorem 1. *Let S be a simplex. A linear bijection $T : \mathcal{A}(S) \to \mathcal{A}(S)$ is diameter preserving if and only if there is an affine automorphism $\varphi : S \to S$, a linear functional $\mu : \mathcal{A}(S) \to \mathbb{K}$ and a number τ with $|\tau| = 1$ and $\mu(1_S) + \tau \neq 0$ such that $Tf = \tau f \circ \varphi + \mu(f)1_S$ for every $f \in \mathcal{A}(S)$.*

Simplexes constitute the simplest class of convex sets (see [6, 1] or [9] for precise definitions). In finite dimensional spaces, simplexes are the usual objects (segments, triangles, etc.), but an infinite dimensional simplex may be a very complex object (see the monster constructed by Poulsen in [7]).

Diameter preserving bijections on spaces of continuous functions have been recently studied by a number of authors (see the papers [4, 3, 2]; for an introduction to linear preserver problems we suggest the survey paper [5]). Of course, by a diameter preserving mapping on $C(X)$ (the space of real or complex continuous functions on the compact Hausdorff space X) we mean a mapping which preserves the seminorm

$$\varrho_X(f) = \sup\{|f(x) - f(y)| : x, y \in X\}.$$

The basic result for $C(X)$ spaces is the following ([4], Theorem, [2], Theorem 1, [3], Theorem 5.1):

Theorem 2. *Let X be a compact Hausdorff space. A linear bijection T of $C(X)$ is diameter preserving if and only if there is a homeomorphism ϕ of X, a linear functional*

$\mu : C(X) \to \mathbb{K}$ *and a number* τ *with* $|\tau| = 1$ *and* $\mu(1_X) + \tau \neq 0$ *such that* $Tf = \tau f \circ \phi + \mu(f)1_X$ *for every* $f \in C(X)$.

We feel that Theorem 1 puts this result in its proper setting. Let us explain why. It is well-known (see [9] or [1]) that the space $C(X)$ can be regarded as the space of continuous affine functions on the set of all regular Borel probabilities on X

$$S = \{\mu \in M(X) : \mu(X) = |\mu|(X) = 1\}$$

endowed with the relative weak* topology of $M(X)$ viewed as the dual space of $C(X)$. Of course, the value of $f \in C(X)$ at $\mu \in S$ is given by $\mu(f) = \int_X f d\mu$. It is not hard to see that S is a simplex and, in fact, it is even a regular (Bauer in [1]) simplex. Moreover, identifying each $x \in X$ with the point measure $\delta_x \in S$, the space X becomes (homeomorphic to) the set ∂S of all extreme points of S.

Now, observe that the diameter of the range of f viewed as an element of $C(X)$ equals the diameter of the range of f viewed as an affine map on S. That $\varrho_X(f) \leq \varrho_S(f)$ is obvious. As for the reverse inequality, fix $f \in C(X)$ and define a mapping $g : S \times S \to \mathbb{K}$ as $g(\mu, \lambda) = \mu(f) - \lambda(f)$. Clearly, $S \times S$ is a compact convex set and g is affine and continuous. By the maximum principle for affine functions ([9], Proposition 23.1.10) the maximum value of $|g|$ is attained at some extreme point of $S \times S$. Since $\partial(S \times S) = \partial S \times \partial S$, it follows that

$$\varrho_S(f) = \sup_{\mu,\lambda \in S} |g(\mu, \lambda)| = \max_{x,y \in X} |g(\delta_x, \delta_y)| = \max_{x,y \in X} |f(x) - f(y)| = \varrho_X(f),$$

as desired.

On the other hand, each affine automorphism φ of S induces by restriction a homeomorphism ϕ of $X = \partial S$. (Actually, this restriction determines φ, by the Krein–Milman theorem. Conversely, if ϕ is a homeomorphism of X, then the map $\varphi : S \to S$ given by $(\varphi(\mu))(A) = \mu(\phi^{-1}(A))$ is the unique affine mapping whose restriction to X coincides with ϕ.) Clearly, $f \circ \phi \in C(X)$ and $f \circ \varphi \in \mathcal{A}(S)$ coincide via the identification between $C(X)$ and $\mathcal{A}(S)$ described above. It is now apparent that Theorem 2 is nothing but a particular case of Theorem 1.

Returning to affine functions, let $\mathcal{A}_\varrho(S)$ denote the quotient of the space $\mathcal{A}(S)$ by the kernel of ϱ. Clearly, $\mathcal{A}_\varrho(S)$ is a Banach space under the norm

$$\|\pi(f)\|_\varrho = \varrho(f),$$

where $\pi : \mathcal{A}(S) \to \mathcal{A}(S)/\ker\varrho$ is the natural quotient map.

Suppose that $T : \mathcal{A}(S) \to \mathcal{A}(S)$ is a diameter preserving linear bijection. Then there exists a (unique) isometry T_ϱ of $\mathcal{A}_\varrho(S)$ making commute the following diagram

$$\begin{array}{ccc} \mathcal{A}(S) & \xrightarrow{T} & \mathcal{A}(S) \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{A}_\varrho(S) & \xrightarrow{T_\varrho} & \mathcal{A}_\varrho(S) \end{array}.$$

The main step in the proof of Theorem 1 is the following characterization of the isometries of $\mathcal{A}_\varrho(S)$.

Theorem 3. *Let* S *be a simplex. A linear map* $T_\varrho : \mathcal{A}_\varrho(S) \to \mathcal{A}_\varrho(S)$ *is a surjective isometry if and only if there is an affine automorphism* φ *of* S *and* $\tau \in \mathbb{K}$, *with* $|\tau| = 1$, *such that* $T_\varrho(\pi(f)) = \pi(\tau f \circ \varphi)$, *for all* $f \in \mathcal{A}(S)$.

Remark 1. Our main result has been independently obtained, in a more general setting, by Rao and Roy in [8]. This paper contains other interesting results about function algebras and vector-valued maps.

2. Proofs

We shall keep the organisation of [2]. First, we derive Theorem 1 from Theorem 3.

Proof of Theorem 1. Let T be a diameter preserving bijection of $\mathcal{A}(S)$ and let $T_\varrho : \mathcal{A}_\varrho(S) \to \mathcal{A}_\varrho(S)$ be the corresponding isometry. According to Theorem 3, one has $T_\varrho(\pi(f)) = \pi(\tau f \circ \varphi)$, for suitable φ and τ. Since $T_\varrho \circ \pi = \pi \circ T$ one has

$$\pi(Tf) = \pi(\tau f \circ \varphi),$$

so that $f \to Tf - \tau f \circ \varphi$ takes values in the subspace of constant functions of $\mathcal{A}(S)$ (which is the kernel of π). This obviously implies that there is $\mu : \mathcal{A}(S) \to \mathbb{K}$ such that

$$Tf = \tau f \circ \varphi + \mu(f) 1_S$$

for every $f \in \mathcal{A}(S)$. □

Remark 2. Observe that T need not be continuous. In fact, T is continuous if and only if μ is.

For the proof of Theorem 3 we need a description of the extreme points of U^*, the unit ball of $\mathcal{A}_\varrho(S)^*$.

Lemma 1. *Let $\mu \in \mathcal{A}_\varrho(S)^*$. Then μ is an extreme point of U^* if and only if $\mu = \sigma(\delta_x - \delta_y)$, where x and y are distinct extreme points of S and $|\sigma| = 1$.*

Proof. First note that for any compact convex set K the extreme points of the unit ball of $\mathcal{A}(K)^*$ (the dual space of $\mathcal{A}(K)$ which is equipped with the usual supremum norm $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$ unless otherwise stated) have the form $\sigma\delta_e$ for some $e \in \partial K$ and $|\sigma| = 1$.

Necessity. Consider the linear operator $L : \mathcal{A}_\varrho(S) \to \mathcal{A}(S \times S)$ given by

$$L(\pi(f))(x, y) = f(x) - f(y).$$

Obviously, $\|\pi(f)\|_\varrho = \|Lf\|_\infty$, so that L is an isometric embedding. The Hahn–Banach theorem implies that μ is an extreme point of U^*, then μ is the restriction to $\mathcal{A}_\varrho(S)$ of some extreme point on the ball of $\mathcal{A}(S \times S)^*$, so $\mu = L^*(\sigma\delta_{(x,y)})$, where $|\sigma| = 1$ and $(x, y) \in \partial(S \times S)$ (that is $x, y \in \partial S$). Hence,

$$\mu = \sigma L^* \delta_{(x,y)} = \sigma(\delta_x - \delta_y)$$

with $x \neq y$. This proves the 'only if' part.

Sufficiency. Let us assume for a moment that $\mathbb{K} = \mathbb{R}$. One then has

$$\varrho(f) = 2 \inf\{\|f - \lambda 1_S\|_\infty : \lambda \in \mathbb{R}\}$$

for all $f \in \mathcal{A}(S)$. This means that $\mathcal{A}_\varrho(S)$ is, up to a constant factor 2, isometric to the quotient of $(\mathcal{A}(S), \|\cdot\|_\infty)$ by the subspace of constant functions (which is not true for

$\mathbb{K} = \mathbb{C}$). Therefore, the space $\mathcal{A}_\varrho(S)^*$ is, up to a factor 1/2, isometric (and not only isomorphic) to a subspace of $\mathcal{A}(S)^*$.

In fact, we have $\mathcal{A}_\varrho(S)^* = \{\mu \in \mathcal{A}(S)^* : \mu(1_S) = 0\}$, with

$$2\|\mu\|_{\mathcal{A}_\varrho(S)^*} = \|\mu\|_{\mathcal{A}(S)^*}.$$

So, one can work with $\|\cdot\|_{\mathcal{A}(S)^*}$ instead of the original norm of $\mathcal{A}_\varrho(S)^*$. This is the point where the hypothesis that S is a simplex (and not merely a compact convex set) appears.

Being S a simplex, the dual space $(\mathcal{A}(S)^*, \|\cdot\|_{\mathcal{A}(S)^*})$ which is always an ordered space (define $\mu \in \mathcal{A}(S)^*$ to be non-negative provided $\mu(f) \geq 0$ for every non-negative $f \in \mathcal{A}(S)$) becomes a Banach lattice. In fact, $\mathcal{A}(S)^*$ is an abstract L-space (that is, a Banach lattice where the norm is additive in the positive cone) and, by an old result of Kakutani, it is order isometric to some concrete L_1-space (see [9]). In particular, if λ^+ and λ^- denote respectively the positive and negative part of $\lambda \in \mathcal{A}(S)^*$, then

$$\|\lambda\|_{\mathcal{A}(S)^*} = \|\lambda^+\|_{\mathcal{A}(S)^*} + \|\lambda^-\|_{\mathcal{A}(S)^*}.$$

Moreover, it is easily seen that if $\lambda = \lambda_1 - \lambda_2$ is a decomposition of λ with λ_1 and λ_2 non-negative and $\|\lambda\|_{\mathcal{A}(S)^*} = \|\lambda_1\|_{\mathcal{A}(S)^*} + \|\lambda_2\|_{\mathcal{A}(S)^*}$, then $\lambda^+ = \lambda_1$ and $\lambda^- = \lambda_2$. Also note that $\mu \in \mathcal{A}(S)^*$ is non-negative if and only if $\|\mu\|_{\mathcal{A}(S)^*} = \mu(1_S)$.

After these preparatives, let $x, y \in \partial S$ with $x \neq y$. Note that $\|\delta_x - \delta_y\|_{\mathcal{A}(S)^*} = 2$ since δ_x and δ_y are distinct extreme points in the unit ball of an abstract L-space. (Actually it is not hard to find a continuous affine map $f : S \to [-1, 1]$ such that $f(x) = 1$ and $f(y) = -1$.) Thus, $\delta_x = (\delta_x - \delta_y)^+$ and $\delta_y = (\delta_x - \delta_y)^-$. Now, suppose $\mu, \nu \in \mathcal{A}_\varrho(S)^*$ are such that $\delta_x - \delta_y = \mu + \nu$, with $\|\delta_x - \delta_y\|_{\mathcal{A}_\varrho(S)^*} = \|\mu\|_{\mathcal{A}_\varrho(S)^*} + \|\nu\|_{\mathcal{A}_\varrho(S)^*}$. Writing $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$, one obtains

$$\begin{aligned}\|\delta_x - \delta_y\|_{\mathcal{A}(S)^*} &= \|\mu\|_{\mathcal{A}(S)^*} + \|\nu\|_{\mathcal{A}(S)^*} \\ &= \|\mu^+\|_{\mathcal{A}(S)^*} + \|\mu^-\|_{\mathcal{A}(S)^*} + \|\nu^+\|_{\mathcal{A}(S)^*} + \|\nu^-\|_{\mathcal{A}(S)^*} \\ &= \|\mu^+ + \nu^+\|_{\mathcal{A}(S)^*} + \|\mu^- + \nu^-\|_{\mathcal{A}(S)^*}.\end{aligned}$$

It follows that $\delta_x = (\delta_x - \delta_y)^+ = \mu^+ + \nu^+$ and $\delta_y = (\delta_x - \delta_y)^- = \mu^- + \nu^-$. Hence $\|\delta_x\|_{\mathcal{A}(S)^*} = \|\mu^+\|_{\mathcal{A}(S)^*} + \|\nu^+\|_{\mathcal{A}(S)^*}$ and $\|\delta_y\|_{\mathcal{A}(S)^*} = \|\mu^-\|_{\mathcal{A}(S)^*} + \|\nu^-\|_{\mathcal{A}(S)^*}$. Since δ_x and δ_y are extreme points in the unit ball of $\mathcal{A}(S)^*$, one obtains

$$\begin{aligned}&\mu^+ = \mu^+(1_S)\delta_x, \quad \nu^+ = \nu^+(1_S)\delta_x \\ &\mu^- = \mu^-(1_S)\delta_y, \quad \nu^- = \nu^-(1_S)\delta_y.\end{aligned}$$

But μ and ν belong to $\mathcal{A}_\varrho(S)^*$, so we have $\mu^+(1_S) = \mu^-(1_S)$ and $\nu^+(1_S) = \nu^-(1_S)$ and, therefore,

$$\begin{aligned}\mu &= \mu^+(1_S)(\delta_x - \delta_y), \\ \nu &= \nu^+(1_S)(\delta_x - \delta_y).\end{aligned}$$

This shows that $\delta_x - \delta_y$ is an extreme point of U^* in the real case.

To end with the proof of the Lemma, let $\mathbb{K} = \mathbb{C}$. It obviously suffices to see that $\delta_x - \delta_y$ is an extreme point of the unit ball of the complex $\mathcal{A}_\varrho(S)^*$. Suppose that

$$\delta_x - \delta_y = \mu + \nu \quad \text{and} \quad \|\delta_x - \delta_y\|_{\mathcal{A}_\varrho(S)^*} = \|\mu\|_{\mathcal{A}_\varrho(S)^*} + \|\nu\|_{\mathcal{A}_\varrho(S)^*}.$$

Thinking $\mathcal{A}_\varrho(S)$ as a subspace of $C(S\times S)$ via the map L constructed above, the Hahn–Banach theorem implies the existence of extensions $\tilde{\mu},\tilde{\nu}\in M(S\times S)$ so that

$$T^*\tilde{\mu}=\mu \quad \text{with} \quad \|\tilde{\mu}\|_1=\|\mu\|_{\mathcal{A}_\varrho(S)^*}$$
$$T^*\tilde{\nu}=\nu \quad \text{with} \quad \|\tilde{\nu}\|_1=\|\nu\|_{\mathcal{A}_\varrho(S)^*}.$$

Since $\|\Re(\eta)\|_1\leq\|\eta\|_1$ for every $\eta\in M(S\times S)$, with equality only if η is real, it follows that $\tilde{\mu}$ and $\tilde{\nu}$ are real measures. Hence $\mu(\pi(f))$ and $\nu(\pi(f))$ are real for every real-valued $f\in\mathcal{A}(S)$ and

$$(\delta_x-\delta_y)|_{\mathcal{A}_\varrho(S,\mathbb{R})}=\mu|_{\mathcal{A}_\varrho(S,\mathbb{R})}+\nu|_{\mathcal{A}_\varrho(S,\mathbb{R})}$$

with

$$\|(\delta_x-\delta_y)|_{\mathcal{A}_\varrho(S,\mathbb{R})}\|_{\mathcal{A}_\varrho(S,\mathbb{R})^*}=\|\mu|_{\mathcal{A}_\varrho(S,\mathbb{R})}\|_{\mathcal{A}_\varrho(S,\mathbb{R})^*}+\|\nu|_{\mathcal{A}_\varrho(S,\mathbb{R})}\|_{\mathcal{A}_\varrho(S,\mathbb{R})^*},$$

since obviously $\|(\delta_x-\delta_y)|_{\mathcal{A}_\varrho(S,\mathbb{R})}\|_{\mathcal{A}_\varrho(S,\mathbb{R})^*}=2$. On the other hand $(\delta_x-\delta_y)|_{\mathcal{A}_\varrho(S,\mathbb{R})}$ is an extreme point of the unit ball of $\mathcal{A}_\varrho(S,\mathbb{R})^*$ and, therefore, μ and ν are proportional to $\delta_x-\delta_y$ when restricted to real functions. By complex linearity one obtains that μ and ν also are proportional to $\delta_x-\delta_y$ as complex functionals. This completes the proof of Lemma 1. □

Beginning of the proof of Theorem 3. Let T be a surjective isometry of $\mathcal{A}_\varrho(S)$. Then the adjoint map $T^*:\mathcal{A}_\varrho(S)^*\to\mathcal{A}_\varrho(S)^*$ is an isometry as well and, therefore, it sends the set of extreme points of U^* into itself. Taking Lemma 1 into account, it is clear that, given $x,y\in\partial S$ with $x\neq y$, there are $u,v\in\partial S$, $u\neq v$ and $\sigma\in\mathbb{K}$ with $|\sigma|=1$ such that

$$T^*(\delta_x-\delta_y)=\sigma(\delta_u-\delta_v).$$

Let ∂S_2 stand for the collection of all subsets of ∂S having exactly two elements. Plainly, T induces a bijection $\partial S_2\to\partial S_2$ by

$$\Phi\{x,y\}=\operatorname{supp}(T^*(\delta_x-\delta_y)).$$

The definition of Φ makes sense because if x,y,z,w are extreme points of S with $x\neq y$, $z\neq w$ and $\delta_x-\delta_y=\sigma(\delta_z-\delta_w)$ for some unimodular σ, then $\{x,y\}=\{z,w\}$. This follows from the fact that if e,f,g,h are positive extreme points of the unit ball of an abstract L-space with $e\neq f,g\neq h$ and $e-f=h-g$, then $e=h$ and $f=g$.

Let $|A|$ denote the cardinality of the set A.

Lemma 2. *For all $\{x,y\},\{u,v\}\in\partial S_2$, one has $|\{x,y\}\cap\{u,v\}|=|\Phi\{x,y\}\cap\Phi\{u,v\}|$.*

Proof of Lemma 2. Simply observe that if $\{x,y\}\neq\{u,v\}$, then $\{x,y\}\cap\{u,v\}$ is non-empty if and only if there is a nontrivial linear combination of $\delta_x-\delta_y$ and $\delta_u-\delta_v$ that is an extreme point of the unit ball of $\mathcal{A}_\varrho(S)^*$. □

Lemma 3. (see [2], Lemma 3). *There is a bijection $\phi:\partial S\to\partial S$ such that $\Phi\{x,y\}=\{\phi(x),\phi(y)\}$ for every $x,y\in\partial S$.* □

End of the proof of Theorem 3. Let $\phi:\partial S\to\partial S$ be the (obviously bijective) map of the preceding lemma. Clearly

$$T^*(\delta_x-\delta_y)=\sigma(x,y)(\delta_{\phi(x)}-\delta_{\phi(y)}),$$

where $|\sigma(x,y)| = 1$. We want to see that $\sigma(x,y)$ does not depend on x,y. Let $z \notin \{x,y\}$. Then

$$\begin{aligned}\sigma(x,y)(\delta_{\phi(x)} - \delta_{\phi(y)}) &= T^*(\delta_x - \delta_y)\\ &= T^*(\delta_x - \delta_z + \delta_z - \delta_y)\\ &= T^*(\delta_x - \delta_z) + T^*(\delta_z - \delta_y)\\ &= \sigma(x,z)(\delta_{\phi(x)} - \delta_{\phi(z)}) + \sigma(z,y)(\delta_{\phi(z)} - \delta_{\phi(y)}),\end{aligned}$$

so that

$$\sigma(x,y) = \sigma(x,z) = \sigma(z,y).$$

Since x,y and z are arbitrary, the equality $\sigma(x,y) = \sigma(z,y)$ means that $\sigma(\cdot,\cdot)$ does not depend on the first variable, while $\sigma(x,y) = \sigma(x,z)$ implies that the same occurs with the second one. Hence $\sigma(x,y) = \tau$ for some unimodular τ.

Our next objective is to extend $\phi : \partial S \to \partial S$ to an automorphism ψ of the simplex S. (Notice that if S were a regular simplex, that is, with closed extreme boundary, this would be automatic.)

Without loss of generality, assume that $\tau = 1$. Fix $y \in \partial S$ and define an affine mapping $\psi : S \to \mathcal{A}_\varrho(S)^*$ by

$$\psi(x) = T^*(\delta_x - \delta_y) + \delta_{\phi(y)}.$$

Observe that ψ is continuous when $\mathcal{A}_\varrho(S)^*$ is endowed with the weak* topology. Since $\psi(x) = \delta_{\phi(x)}$ for every $x \in \partial S$, it follows that ψ takes values in S (that is, in the canonical image of S in $\mathcal{A}_\varrho(S)^*$).

Thus, we can define a continuous affine map $\varphi : S \to S$ by the condition

$$\delta_{\varphi(x)} - \delta_{\psi(y)} = T^*(\delta_x - \delta_y).$$

Notice that φ is in fact an affine automorphism (whose inverse can be obtained from T^{-1}). Finally, define $T_{(\tau,\varphi)} : \mathcal{A}_\varrho(S) \to \mathcal{A}_\varrho(S)$ as $T_{(\tau,\varphi)}(\pi(f)) = \pi(\tau f \circ \varphi)$. Since

$$T^*(\delta_x - \delta_y) = T^*_{(\tau,\varphi)}(\delta_x - \delta_y)$$

for all $x,y \in \partial S$, the Krein–Milman theorem implies that $T = T_{(\tau,\varphi)}$. This completes the proof of Theorem 3. □

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