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Maximal norms on Banach spaces of continuous functions

CORRIGENDUM

'Orthonormal systems in Banach spaces and their applications'
by N. J. Kalton and G. V. Wood
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BY

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The purpose of this short note is to reformulate theorem 9.3 in [5] which is not correct as stated. We note that all other results in [5] are independent of that statement.

The notation is the same as [5] with the sole exception that $C_0(S)$ will always denote the space of all real-valued continuous functions on the locally compact space S vanishing at infinity. As usual, αS stands for the one-point compactification of S. Recall that a norm $\|\cdot\|$ on a Banach space X is said to be maximal if there is no equivalent norm on X whose isometry group contains properly that of $\|\cdot\|$.

We begin with the following somewhat surprising result:

Theorem 1. *Let S be a connected, locally compact space. If there exists a homeomorphism ϕ of αS such that $\phi(\infty) \neq \infty$, then the supremum norm is not maximal on the real space $C_0(S)$.*

This yields a large class of manifolds S for which the usual supremum norm fails to be maximal on $C_0(S)$ (compare to [5, theorem 9.3]):

Corollary 1. *Let S be a connected non-compact space whose one-point compactification is a manifold without boundary. Then the supremum norm fails to be maximal on the space $C_0(S)$. In particular $C_0(\mathbb{R}^n)$ lacks maximal norm for every $n \geq 1$.* □

Proof of Theorem 1. Given $f \in C_0(S)$, define

$$\varrho(f) = \sup\{|f(x) - f(y)| : x, y \in S\},$$

that is, $\varrho(f)$ is the diameter of the range of f. If S is not compact, then $\varrho(\cdot)$ is a norm on $C_0(S)$ which shall be referred to as the diameter norm. In fact one has

$$\|f\|_\infty \leq \varrho(f) \leq 2\|f\|_\infty$$

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for all $f \in C_0(S)$. We prove that $(C_0(S), \varrho(\cdot))$ has a strictly larger group of isometries than $(C_0(S), \|\cdot\|_\infty)$.

Since S is a connected space every isometry of $(C_0(S), \|\cdot\|_\infty)$ is an isometry under the diameter norm as well (this is not true if S is not connected or if one allows complex functions).

Consider now $C_0(S)$ as a subspace of $C(\alpha S)$ in the obvious way. Given a homeomorphism $\phi : \alpha S \to \alpha S$ one can define a surjective isometry of $(C_0(S), \varrho(\cdot))$ by

$$U(f) = f \circ \phi - f(\phi(\infty))1_{\alpha S}.$$

It is clear that U preserves the supremum norm if and only if ϕ leaves fixed the infinity point of αS. Hence if $\phi(\infty) \neq \infty$ for some homeomorphism ϕ of αS the isometry group of $\varrho(\cdot)$ contains properly that of $\|\cdot\|_\infty$, which completes the proof. □

Remark 1. As far as we know, the diameter (semi) norm was first studied by Győry and Molnár in [4], where the general form of the diameter preserving bijections of the (real or complex) spaces $C(S)$ for S compact metric was given. Later, González and Uspenskij [3] and the second named author [1] got similar results for arbitrary compact S. The paper [1] contains a complete description of the isometries of the space $(C_0(S), \varrho(\cdot))$.

The following result covers all spaces considered in [5, theorem 9.3]. The somewhat involved proof depends heavily on [5, lemma 9.2] and also on the argument given by Kalton and Wood on p. 509. Note that even in the compact case the proof requires some extra work.

Theorem 2. *Let S be a connected manifold without boundary of dimension greater than one.*

(a) *If S is compact, then the supremum norm is maximal on $C(S)$.*
(b) *If S is locally compact but not compact, then the diameter norm is maximal on $C_0(S)$.*

Corollary 2. *Let S be a connected non-compact manifold without boundary of dimension greater than one. If every homeomorphism of αS leaves fixed the infinity point, then the supremum norm is maximal on $C_0(S)$.*

Proof. The hypothesis about S together with [1, theorem 4] implies that the supremum norm and the diameter norm have the same isometry group. Hence the result follows from the part (b) of Theorem 2. □

Beginning of the proof of Theorem 2. Let $\|\cdot\|$ be any norm on $C_0(S)$ whose isometry group contains all isometries induced by homeomorphisms of S. Let K^* denote the unit ball of $(C_0(S), \|\cdot\|)^*$. By [5, lemma 9.2] every extreme point of K^* has the form $\alpha\delta_x - \beta\delta_y$ for some $x, y \in S$ with $x \neq y$ and some $\alpha, \beta \geq 0$.

Now, let U be an isometric automorphism of $(C_0(S), \|\cdot\|)$. We must prove that U is an isometry for the supremum norm if S is compact and that U is diameter preserving for locally compact, noncompact S.

Let U^* be the adjoint isometry. Reasoning as in [5], one sees that for every $x \in S$ the support of the measure $U^*\delta_x$ contains at most two points. Write $S = S_1 \oplus S_2$, where

$$S_i = \{x \in S : |\operatorname{supp} U^*\delta_x| = i\}.$$

If S_2 is empty, it is easily seen that U is induced by a homeomorphism of S and there is nothing to prove.

From now on, we assume that S_2 is nonempty (and hence it is infinite because S_2 is always open in S). This implies that there exists $\alpha, \beta > 0$ (which we consider fixed in what follows) and $x, y \in S$ with $x \neq y$ such that $\alpha\delta_x - \beta\delta_y$ is an extreme point of K^*, since otherwise $\|\cdot\|$ is simply a multiple of $\|\cdot\|_\infty$. By homogeneity of S one actually has that $\alpha\delta_x - \beta\delta_y$ is an extreme point of K^* for any distinct points $x, y \in S$.

Step 1. There is $p \in S$ such that $p \in \operatorname{supp} U^*\delta_x$ for all $x \in S_2$.

Proof. Let $x, y \in S_2$. Since $\alpha\delta_x - \beta\delta_y$ is an extreme point of K^* so is $U^*(\alpha\delta_x - \beta\delta_y)$. Hence

$$\operatorname{supp} U^*\delta_x \cap \operatorname{supp} U^*\delta_y \neq \emptyset. \tag{1}$$

Let $P_2(S)$ stand for the collection of all subsets of S having exactly two points. Then one can define a mapping $\Phi : S_2 \to P_2(S)$ by

$$\Phi(x) = \operatorname{supp} U^*\delta_x.$$

Now take a subset $A \subset S_2$ such that for every $x \in S_2$ there is exactly one $a \in A$ for which $\Phi(a) = \Phi(x)$. Since $\cap_{a\in A}\Phi(a) = \cap_{x\in S_2}\Phi(x)$ our claim will be proved if we show that $\cap_{a\in A}\Phi(a)$ is nonempty. To see this, first note that A is infinite (this follows from the facts that S_2 is itself infinite and that, since the set $\{U^*\delta_x : x \in S\}$ is linearly independent, no more than two points of S_2 can have the same image in $P_2(S)$).

Take $a_1, a_2 \in A$ with $a_1 \neq a_2$ and let p be the only element of $\Phi(a_1) \cap \Phi(a_2)$, so that $\Phi(a_1) = \{p, s_1\}$ and $\Phi(a_2) = \{p, s_2\}$, with $s_1 \neq s_2$. We show that $p \in \Phi(a)$ for all $a \in A$.

Suppose on the contrary that there exists $a_3 \in A$ such that $p \notin \Phi(a_3)$. Then by (1) one has $\Phi(a_3) = \{s_1, s_2\}$. Let $a_4 \in A$ be a point different from a_1, a_2 and a_3. Then $p \in \Phi(a_4)$ (for if not $\Phi(a_4) = \{s_1, s_2\} = \Phi(a_3)$ and Φ cannot be injective on A) and since $\Phi(a_3)$ and $\Phi(a_4)$ must have a common point one has either $\Phi(a_4) = \Phi(a_1)$ or $\Phi(a_4) = \Phi(a_2)$, a contradiction. □

It is now clear that there are maps a and b from S_2 to $\mathbb{R}\backslash\{0\}$ and $\varphi : S_2 \to S$ such that

$$U^*\delta_x = a(x)\delta_{\varphi(x)} + b(x)\delta_p$$

for all $x \in S_2$.

Step 2. The maps a, b and φ are continuous on S_2.

Proof. The continuity of b easily follows from weak* continuity of U^*. The continuity of a and φ then follows from that of b and weak* continuity of U^*. □

Step 3. The map b is constant on S_2. Moreover, if $\alpha\delta_x - \beta\delta_y$ is an extreme point of K^* with $x \neq y$ and $\alpha, \beta \neq 0$, then $\alpha = \beta$.

Proof. Let us fix $x \in S_2$. Note that the set $\Sigma_x = \{z \in S_2 : \varphi(z) = \varphi(x)\}$ contains at most two points. Take $y \in S_2 \backslash \Sigma_x$ and choose nonzero α, β such that $\alpha\delta_x - \beta\delta_y$ is an extreme point of K^*. It follows that the support of the measure

$$U^*(\alpha\delta_x - \beta\delta_y) = \alpha a(x)\delta_{\varphi(x)} - \beta a(y)\delta_{\varphi(y)} + (\alpha b(x) - \beta b(y))\delta_p$$

cannot contain more than two points. But $\varphi(x), \varphi(y)$ and p are different points of S. Hence $\alpha b(x) = \beta b(y)$ holds for every $y \in S_2 \backslash \Sigma_x$. Now, the fact that $S_2 \backslash \Sigma_x$ is dense in S_2, the continuity of b and a moment of reflection show that b is constant on S_2 and also that $\alpha = \beta$. □

Step 4. The map a is constant on S_2.

Proof. Again, fix $x \in S_2$ and let Σ_x be as before. Let $y \in S_2 \backslash \Sigma_x$. Since $\alpha(\delta_x - \delta_y)$ in an extreme point of K^* so is

$$U^*(\alpha(\delta_x - \delta_y)) = \alpha(a(x)\delta_{\varphi(x)} - a(y)\delta_{\varphi(y)}),$$

so that $a(x) = a(y)$ for all $y \in S_2 \backslash \Sigma_x$ and the result follows. □

Step 5. The set S_1 is nonempty.

Proof. Suppose on the contrary that S_1 is empty. Then $S = S_2$ and one has

$$U^*\delta_x = a(x)\delta_{\varphi(x)} + b\delta_p$$

for every $x \in S$.

If S is compact, the range of φ is a compact subset of S which does not contain p and it is straightforward that U^* cannot be surjective. This proves the claim for compact S.

Now assume that S is locally compact, but not compact. Since $\delta_x \to 0$ weakly* as $x \to \infty$, using the weak* continuity of U^*, we obtain that $\varphi(x) \to p$ as $x \to \infty$. This shows that the range of φ lies in an compact subset of S. Since S is not compact U^* cannot be surjective, a contradiction. □

Step 6. The set S_1 is a singleton.

Proof. Write $U^*\delta_z = c(z)\delta_{\psi(z)}$ for $z \in S_1$, where $\psi : S_1 \to S$ is a suitable map and $c(z) = \pm 1$. Take $x \in S_2$ and $z \in S_1$. Since $\alpha(\delta_x - \delta_z)$ in an extreme point of K^* the support of the measure

$$U^*(\alpha(\delta_x - \delta_y)) = \alpha(a\delta_{\varphi(x)} + b\delta_p - c(z)\delta_{\psi(z)})$$

contains at most two points. Hence

$$\psi(z) \in \bigcap_{x \in S_2} \{\varphi(x), p\}$$

and, therefore, $\psi(z) = p$ for every $z \in S_1$. Since ψ must be injective, the result follows. □

Let q denote the only point of S_1, so that $S_1 = \{q\}$ and $U^*\delta_q = c\delta_p$, where $c = \pm 1$. Moreover, we may assume (and do) that $c = -1$.

Thus, we can summarize the information about U obtained so far as follows: there exist nonzero numbers a, b, two points $p, q \in S$ and a continuous mapping $\varphi : S\backslash\{q\} \to S$ for which

$$\begin{aligned} &U^*\delta_x = a\delta_{\varphi(x)} + b\delta_p \qquad (x \neq q) \\ &U^*\delta_q = -\delta_p \end{aligned} \tag{2}$$

End of the proof of (a). Letting $x \to q$ in (2) and using the weak* continuity of U^* one obtains that $a\delta_{\varphi(x)} + b\delta_p \to -\delta_p$ in the weak* topology of $C(S)^*$. By compactness of S this implies that $\varphi(x) \to p$ as $x \to q$ and also that $a + b = -1$. On the other hand, it is clear that U generates a bounded group of operators only if $|a| = 1$. Since $b \neq 0$, this yields $a = 1$ and $b = -2$ and U cannot have uniformly bounded powers. This contradiction shows that U is an isometry of the supremum norm and ends the proof of part (a). □

End of the proof of (b). In this case since $\delta_x \to 0$ weakly* as $x \to \infty$ we obtain from (2) $a+b = 0$ and $\varphi(x) \to p$ as $x \to \infty$. Therefore $a(\delta_{\varphi(x)} - \delta_p) \to -\delta_p$ weakly* as $x \to q$. This clearly implies that $\varphi(x) \to \infty$ as $x \to q$ and also that $a = 1$. Thus, we can extend φ to a continuous mapping $\phi : \alpha S \to \alpha S$ taking $\phi(\infty) = p$ and $\phi(q) = \infty$. It is easily seen that U is given by

$$(U(f))(x) = f(\phi(x)) - f(\phi(\infty))$$

for all $x \in \alpha S$ and therefore U is an isometry with respect to the diameter norm. This completes the proof. □

We now discuss some examples in the light of Corollaries 1 and 2.

The supremum norm fails to be maximal on the real space $C_0(S)$ when $S = K\backslash\{p\}$, K being a connected, compact manifold and $p \in K$. In particular the conclusion holds for $S = \mathbb{R}^n$ for every $n \geq 1$ taking $K = \mathbb{S}^n$ (the n-dimensional sphere) and when S is the Möbius strip taking $K = \mathbb{P}_2$ (the projective plane). This is straightforward from Corollary 1.

On the positive side, the supremum norm is maximal on the real space $C_0(S)$ whenever S can be decomposed as $S = K \times \mathbb{R}^n$, K being a connected, compact manifold of positive dimension k and $n \geq 1$ or when S admits a two-point compactification (for instance if S is a region of the Riemann sphere whose complement is disconnected).

In the last case, it is clear that the infinity point is the only point of αS having disconnected punctured neighbourhoods, so that Corollary 2 applies.

As for the former one, it is not hard to see that $\pi_{n-2}(U\backslash\{\infty\})$ does not vanish if U is a sufficiently small neighbourhood of the infinity point in αS (here π stands for the corresponding free homotopy group) while, obviously, $\pi_{n-2}(U\backslash\{p\}) = \pi_{n-2}(\mathbb{R}^{n+k}\backslash\{0\})$ vanishes on a neighbourhood base at any $p \in S$. Thus, every homeomorphism of αS fixes the infinity point.

Remark 2. The hypotheses about S in Theorem 2 (and also in its Corollary 2) can be considerably relaxed: one only needs, apart from the connectedness and locally compactness, the following properties of S.

(3) Given two ordered sets $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of distinct points of S there exists a homeomorphism φ of S such that $\varphi(x_i) = y_i$ for every $1 \leq i \leq n$.

(4) Each $x \in S$ has a neighbourhood base V of open sets such that, given $B \in V$, there is a sequence (φ_n) of homeomorphisms of S leaving fixed $S \backslash B$ while $\lim_n \varphi_n(y) = x$ for all $y \in B$.

Actually, an examination of the proofs of [5, lemma 9.2] and our Theorem 2 above yields the following characterization of the diameter norm for sufficiently homogeneous spaces.

Theorem 3. *Let S be a connected, locally compact space satisfying* (3) *and* (4). *Suppose $\|\cdot\|$ is an equivalent norm on the real space $C_0(S)$ whose isometry group contains properly that of the supremum norm. Then $\|\cdot\|$ is, up to a constant factor, the diameter norm.*

Thus, the diameter norm is not only maximal but also "uniquely maximal" and hence convex-transitive (see [2]).

The proof of Theorem 3 is as follows: assuming that U is an isometry for the norm $\|\cdot\|$ but not for $\|\cdot\|_\infty$ and taking into account (2) and the argument given in the end of the proof of (b) we obtain that the adjoint map U^* has the form

$$\begin{aligned} U^*\delta_x &= \delta_{\varphi(x)} - \delta_p \qquad (x \neq q) \\ U^*\delta_q &= -\delta_p \end{aligned}$$

for suitable x, p, q and φ. This implies (normalizing $\|\cdot\|$ by the condition $\|\delta_p\| = 1$ if necessary) that the set of extreme points of K^* is exactly $\{\pm\delta_x : x \in S\} \cup \{\pm(\delta_y - \delta_z) : y, z \in S\}$ and hence $\|\cdot\|$ and ϱ coincide on $C_0(S)$.

We leave the details to the reader.

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