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Pseudo-Characters and Almost Multiplicative Functionals

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INTRODUCTION

Let $\mathscr{G}$ be a (multiplicative) group. An ϵ-character on $\mathscr{G}$ is a mapping ϕ: $\mathscr{G} \to \mathbb{T}$ satisfying $|\phi(xy) - \phi(x)\phi(y)| \leq \epsilon$ for all $x, y \in \mathscr{G}$. By a pseudo-character we mean an ϵ-character for some (usually small) $\epsilon > 0$. It should be noted that this term has been used with a slightly different meaning by a number of authors (see [19] and references therein). A very natural question which, as far as we know, goes back to Ulam [21–24], is whether a given pseudo-character on $\mathscr{G}$ must be near to a true character χ: $\mathscr{G} \to \mathbb{T}$. We refer the reader to the survey papers [5, 10] for information on Ulam's problem. Also, see the new book by J. H. Hyers, G. Isac, and Th. M. Rassias, "Stability of Functional Equations in Several Variables," Birkhäuser, Basel, 1999. It should be noted that these works deal mainly with mappings f: $\mathscr{G} \to \mathbb{R}$ which are nearly additive, that is, mappings satisfying $|f(xy) - f(x) - f(y)| \leq \epsilon$ for all $x, y \in \mathscr{G}$. (Despite the importance of characters in harmonic analysis pseudo-characters received much less attention in recent years.)

Following [5], let us say that characters are stable on $\mathscr{G}$ if, for every $\delta > 0$, there is $\epsilon > 0$ such that to each ϵ-character ϕ: $\mathscr{G} \to \mathbb{T}$ there corresponds a character χ: $\mathscr{G} \to \mathbb{T}$ fulfilling $|\phi(x) - \chi(x)| \leq \delta$ for all $x \in \mathscr{G}$. (Similarly, we say that $\mathscr{G}$ has stable real characters if, for every $\delta > 0$, there is $\epsilon > 0$ such that to each mapping f: $\mathscr{G} \to \mathbb{R}$ satisfying the estimate $|f(xy) - f(x) - f(y)| \leq \epsilon$ there corresponds a real-character a: $\mathscr{G} \to \mathbb{R}$ fulfilling $|f(x) - a(x)| \leq \delta$ for all $x \in \mathscr{G}$.)

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We describe now the results of the paper. In the first section, we study pseudo-characters on amenable groups. We prove in a very simple way that amenable groups have stable characters. The proof (an adaptation of Pełczyński's method in [17]; also see [4, 20]) also shows that every measurable ϵ-character on a compact group can be approximated by a continuous character.

Section 2 develops a rudimentary "theory" of pseudo-characters. For instance, we show that if $\mathscr{G}$ have stable characters, then one can take $\epsilon = \delta$ in the definition for δ small enough. Also, we prove that if $\mathscr{S}$ is a normal subgroup of $\mathscr{G}$ and $\mathscr{G}$ has stable characters, then so does $\mathscr{G}/\mathscr{S}$. As a partial converse, if $\mathscr{S}$ is dually embedded and both $\mathscr{S}$ and $\mathscr{G}/\mathscr{S}$ have stable characters, then so does $\mathscr{G}$. Next, we characterize "approximable" pseudo-characters as those pseudo-characters whose values on the commutator subgroup are near 1. As an application, it is proved that if a group has stable characters then it must have stable real-characters.

The examples of Section 3 borrow from Hyers–Ulam's theory of approximately additive mappings. First, by following an idea of Forti [4], we prove that for each $\epsilon > 0$ there is an ϵ-character on $\mathbb{F}_2$ (the free group with two generators) which is far from any character of $\mathbb{F}_2$. Also, using an argument of Giudici [5], we prove that some classical groups such as $GL_2(\mathbb{R})$ and $O(3, \mathbb{R})$ have stable characters, in spite of being non-amenable.

In Section 4 we give some applications to almost multiplicative functionals (on complex Banach algebras). These are linear functionals $\Phi: \mathscr{A} \to \mathbb{C}$ satisfying

$$|\Phi(f \cdot g) - \Phi(f)\Phi(g)| \leq \epsilon \|f\| \, \|g\|$$

for some $\epsilon > 0$ and all $f, g \in \mathscr{A}$. Almost multiplicative functionals were studied by Johnson [12, 13] for several Banach algebras. He introduced AMNM algebras (an abbreviation for "algebras on which almost multiplicative functionals are near to multiplicative functionals") as those algebras $\mathscr{A}$ such that, for every $\delta > 0$, there is $\epsilon > 0$ such that for each ϵ-multiplicative functional $\Phi: \mathscr{A} \to \mathbb{C}$ there is a multiplicative functional $\Psi: \mathscr{A} \to \mathbb{C}$ satisfying $\|\Phi - \Psi\| \leq \delta$.

Here, we are interested in "discrete" group algebras $l_1(\mathscr{G})$ (the product being convolution). The key point is that every pseudo-character ϕ on $\mathscr{G}$ induces an almost multiplicative functional on the group algebra $l_1(\mathscr{G})$ by

$$\langle \phi, f \rangle = \sum_{x \in \mathscr{G}} \phi(x) f(x),$$

and, conversely, every almost multiplicative functional on $l_1(\mathscr{G})$ comes essentially in this way from a pseudo-character of $\mathscr{G}$. Moreover, ϕ is near to a character if and only if $\langle \phi, \cdot \rangle$ is near to a multiplicative functional $l_1(\mathscr{G}) \to \mathbb{C}$.

Consequently, $l_1(\mathscr{G})$ is AMNM if and only if $\mathscr{G}$ has stable characters. In this way the results of Section 3 lead to a simple example of non-AMNM group algebra (namely $l_1(\mathbb{F}_2)$) and also to non-amenable AMNM group algebras (the group algebra of $GL_2(\mathbb{R})$ or $O(3, \mathbb{R})$).

1. STABILITY OF CHARACTERS ON AMENABLE GROUPS

In this section we investigate Ulam's problem for pseudo-characters on amenable groups. It will be convenient to consider topological groups and not only discrete groups. So, let $\mathscr{G}$ be a locally compact (topological) group. We denote by $L_\infty(\mathscr{G})$ the Banach space of all Borel maps $\mathscr{G} \to \mathbb{C}$ which are essentially bounded (with respect to the Haar measure of $\mathscr{G}$) with the usual convention about identifying functions equal almost everywhere. The group $\mathscr{G}$ is said to be amenable if $L_\infty(\mathscr{G})$ admits an invariant mean. For instance, compact or abelian groups are amenable. (We refer the reader to [7] for background on amenability.)

Note that every (not necessarily topological) group can be regarded as a locally compact group equipped with the discrete topology. Thus, *discrete* amenable groups are those groups $\mathscr{G}$ admitting an invariant mean on the space $l_\infty(\mathscr{G})$ of all bounded functions on $\mathscr{G}$. This clearly implies that $\mathscr{G}$ is amenable under any locally compact topology. The converse is not true: the orthogonal group $O(3, \mathbb{R})$ is compact, hence amenable, with the usual Lie topology but fails to be amenable as a discrete group (this is the Banach–Tarski "paradox" [1]).

Our main result in this section is that characters are always stable on amenable groups. To be more precise, let us introduce a new metric on $\mathbb{T}$ by means of

$$d(z, w) = \left| \operatorname{Arg}\left(\frac{z}{w} \right) \right|,$$

where we take $-\pi < \operatorname{Arg}(\zeta) \leq \pi$. Note that $d(z, w)$ is the arc length between z and w, while $|z - w|$ corresponds to the chord length. It is clear that $d(\cdot, \cdot)$ is rotation-invariant (that is, it is an invariant metric for $\mathbb{T}$) and also that $d(z, w)$ determines completely (and is completely determined by) $|z - w|$. In fact, $|z - w| = |1 - e^{id(z,w)}|$. So, we consider maps $\phi: \mathscr{G} \to \mathbb{T}$ satisfying

$$d(\phi(xy), \phi(x)\phi(y)) \leq \epsilon \qquad (x, y \in \mathscr{G}),$$

which shall be referred to as ϵ^*-characters. (Notice that ϕ is an ϵ^*-character in this sense if and only if it is an ϵ^*-character in the usual sense for $\epsilon^* = |1 - e^{i\epsilon}|$.)

THEOREM 1. *Suppose that $\mathscr{G}$ is an amenable locally compact group and that ϕ: $\mathscr{G} \to \mathbb{T}$ is a Borel ϵ^*-character, with $\epsilon < \pi/3$. Then there exists a unique character χ: $\mathscr{G} \to \mathbb{T}$ such that $d(\chi(x), \phi(x)) \le \epsilon$ for all $x \in \mathscr{G}$. Moreover χ is continuous.*

COROLLARY 1. *Let $\mathscr{G}$ be an amenable locally compact group and let $\epsilon < 1$. Then for every measurable ϵ-character ϕ: $\mathscr{G} \to \mathbb{T}$ there exists a unique continuous character χ: $\mathscr{G} \to \mathbb{T}$ such that $|\chi(x) - \phi(x)| \le \epsilon$ for all $x \in \mathscr{G}$. In particular, if $\mathscr{G}$ is a discrete amenable group, then for every ϵ-character ϕ: $\mathscr{G} \to \mathbb{T}$ there exists a unique character χ: $\mathscr{G} \to \mathbb{T}$ such that $|\chi(x) - \phi(x)| \le \epsilon$ for all $x \in \mathscr{G}$.*

Proof of the corollary. For the first part, simply note that the chord length equals 1 if and only if the arc length equals $\pi/3$. The second part is obvious. ∎

Proof of Theorem 1. We want to see that there exists a map β: $\mathscr{G} \to \mathbb{T}$ such that

$$\frac{\phi(x)\phi(y)}{\phi(xy)} = \frac{\beta(x)\beta(y)}{\beta(xy)} \tag{1}$$

for all $x, y \in \mathscr{G}$ with $|\text{Arg } \beta(x)| = d(\beta(x), 1) \le \epsilon$. In this case it is clear that $\chi = \phi \cdot \beta^{-1}$ is the desired character since $d(\chi(x), \phi(x)) = d(\beta^{-1}(x), 1) = d(1, \beta(x)) \le \epsilon$.

Let dz denote a (say left) invariant mean for $\mathscr{G}$ and define

$$\beta(x) = \exp\left(i \int_G \text{Arg } \frac{\phi(x)\phi(z)}{\phi(xz)}\, dz \right).$$

The definition of β makes sense because, for every $x \in \mathscr{G}$, the map $z \in \mathscr{G} \mapsto \phi(x)\phi(y)/\phi(xy) \in \mathbb{C}$ is (Borel) measurable. That $|\text{Arg } \beta(x)| \le \epsilon$ for all $x \in \mathscr{G}$ is trivial. To verify (1), fix $x, y \in \mathscr{G}$. Using invariance of dz, we obtain

$$\beta(x) = \exp\left(i \int_G \text{Arg } \frac{\phi(x)\phi(yz)}{\phi(xyz)}\, dz \right).$$

Hence,

$$\frac{\beta(x)\beta(y)}{\beta(xy)}$$

$$= \frac{\exp\left(i\int_G \operatorname{Arg} \phi(x)\phi(yz)\phi(xyz)^{-1}\,dz\right) \times \exp\left(i\int_G \operatorname{Arg} \phi(y)\phi(z)\phi(yz)^{-1}\,dz\right)}{\exp\left(i\int_G \operatorname{Arg} \phi(xy)\phi(z)\phi(xyz)^{-1}\,dz\right)}$$

$$= \exp\left(i\int_G \left(\operatorname{Arg} \frac{\phi(x)\phi(yz)}{\phi(xyz)} + \operatorname{Arg} \frac{\phi(y)\phi(z)}{\phi(yz)} - \operatorname{Arg} \frac{\phi(xy)\phi(z)}{\phi(xyz)}\right) dz\right)$$

$$= \exp\left(i\int_G \operatorname{Arg}\left(\frac{\phi(x)\phi(yz)}{\phi(xyz)} \cdot \frac{\phi(y)\phi(z)}{\phi(yz)} \cdot \frac{\phi(xyz)}{\phi(xy)\phi(z)}\right) dz\right)$$

$$= \frac{\phi(x)\phi(y)}{\phi(xy)}.$$

The uniqueness of χ follows from the fact that if χ and ψ are characters such that $d(\chi(x), \psi(x)) < \pi/3$ for all $x \in \mathscr{G}$, then $\psi^{-1} \cdot \chi$ is a character with $d((\psi^{-1} \cdot \chi)(x), 1) < 2\pi/3$ for all $x \in \mathscr{G}$ and therefore $\psi = \chi$.

It remains to show that χ is continuous. But a character on a locally compact group is continuous if and only if it is measurable on some set of positive Haar measure (see, for instance, [6, Theorem 1.1.4.1] for a simple proof). Thus, the proof will be complete if we show that β is measurable. To this end, consider the mapping $\mathscr{G} \times \mathscr{G} \to \mathbb{C}$ given by

$$(x, z) \mapsto \operatorname{Arg}\left(\frac{\phi(x)\phi(z)}{\phi(xz)}\right).$$

Obviously it is a Borel map on $\mathscr{G} \times \mathscr{G}$. Now Fubini's theorem (as presented in [18, Theorem 7.3.1]) implies that the map $\alpha\colon \mathscr{G} \to \mathbb{C}$ defined by

$$\alpha(x) = \int_G \operatorname{Arg} \frac{\phi(x)\phi(z)}{\phi(xz)}\,dz$$

is a Borel map. Therefore $\beta = \exp(i\alpha)$ is a Borel map too. This completes the proof. ▮

Remark 1. Approximate representations of groups in the unitary group of (the algebra of all bounded operators on) a Hilbert space have been studied by a number of authors [8, 9, 15, 19]. Since $\mathbb{T}$ can be regarded as the unitary group of a unidimensional Hilbert space our Theorem 1 is a consequence of the main result in [15] (see also [9]) for continuous ϕ. Nevertheless, Theorem 1 in its full generality seems to be new even for $\mathcal{G} = \mathbb{T}$ (see the results of Cenzer in [2, 3]).

Moreover, it should be noted that our proof actually *constructs* a continuous character near ϕ if ϕ is a Borel pseudo-character on a compact group whose Haar measure is known, for instance, if $\mathcal{G} = \mathbb{T}^n$ is a torus.

2. SOME THEORY FOR PSEUDO-CHARACTERS

This section develops a rudimentary "theory" of pseudo-characters. The following result implies that if $\mathcal{G}$ has stable characters, then one can take $\epsilon = \delta$ in the definition for δ small enough. (Compare to [3, 4].) This will simplify some computations.

PROPOSITION 1. *Let $\phi: \mathcal{G} \to \mathbb{T}$ be an ϵ^*-character. Suppose $\chi: \mathcal{G} \to \mathbb{T}$ is a character such that $d(\phi(x), \chi(x)) < \pi/3$ for all $x \in \mathcal{G}$. Then $d(\phi(x), \chi(x)) \leq \epsilon$ for all $x \in \mathcal{G}$.*

Proof. The hypothesis implies the existence of a mapping $\beta: \mathcal{G} \to \mathbb{T}$ with $|\operatorname{Arg} \beta(x)| < \pi/3$ for all $x \in \mathcal{G}$ and such that

$$\frac{\phi(x)\phi(y)}{\phi(xy)} = \frac{\beta(x)\beta(y)}{\beta(xy)}$$

for all x, y. Define $f: \mathcal{G} \to \mathbb{R}$ by $f(x) = \operatorname{Arg} \beta(x)$. Then f is ϵ-additive:

$$\begin{aligned} |f(x) + f(y) - f(xy)| &= |\operatorname{Arg} \beta(x) + \operatorname{Arg} \beta(y) - \operatorname{Arg} \beta(xy)| \\ &= \left|\operatorname{Arg}\left(\beta(x)\beta(y)\beta(xy)^{-1}\right)\right| \\ &= \left|\operatorname{Arg}\left(\phi(x)\phi(y)\phi(xy)^{-1}\right)\right| \\ &\leq \epsilon. \end{aligned}$$

It follows that for every $x \in \mathcal{G}$ and all $n \geq 1$ one has $|f(x^n) - nf(x)| \leq (n - 1)\epsilon$. Hence

$$\left|\frac{1}{n} f(x^n) - f(x)\right| \leq \frac{n-1}{n}\epsilon$$

and the boundedness of f implies that $|f(x)| \leq \epsilon$ for all $x \in \mathscr{G}$ which completes the proof. ∎

Next, we prove the following useful lemma.

LEMMA 1. *Let $\mathscr{S}$ be a normal subgroup of $\mathscr{G}$.*

(a) *If $\mathscr{G}$ has stable characters, then so does $\mathscr{G}/\mathscr{S}$.*

(b) *If $\mathscr{S}$ is dually embedded (that is, every character of $\mathscr{S}$ extends to a character of $\mathscr{G}$) and both $\mathscr{S}$ and $\mathscr{G}/\mathscr{S}$ have stable characters, then $\mathscr{G}$ has stable characters.*

(c) *Consequently, a direct product of finitely many groups has stable characters if and only if each factor has stable characters.*

Proof. (a) Let π denote the natural quotient map $\mathscr{G} \to \mathscr{G}/\mathscr{S}$. Suppose ϕ is an ϵ^*-character on $\mathscr{G}/\mathscr{S}$. Then $\phi \circ \pi$ is an ϵ^*-character on $\mathscr{G}$ and there is χ: $\mathscr{G} \to \mathbb{T}$ such that $d(\phi(\pi(x)), \chi(x)) \leq \epsilon$ for all $x \in \mathscr{G}$. Since $d(1, \phi(e)) = d(\phi(e), \phi(e)^2) \leq \epsilon$, it follows that $d(\chi(y), 1) \leq 2\epsilon$ for all $y \in \mathscr{S}$. Assuming $\epsilon < \pi/3$ if necessary, we infer that $\ker \chi$ contains $\mathscr{S}$. Thus, there is a character ξ: $\mathscr{G}/\mathscr{S} \to \mathbb{T}$ so that $\chi = \xi \circ \pi$. This obviously implies that $d(\xi(x), \phi(x)) \leq \epsilon$, as desired.

(b) Let ϕ be an ϵ^*-character on $\mathscr{G}$. The restriction to $\mathscr{S}$ is an ϵ^*-character and the hypothesis about $\mathscr{S}$ yields a character η_0: $\mathscr{S} \to \mathbb{T}$ such that $d(\phi(y), \eta_0(y)) \leq \epsilon$. Let η be a character of $\mathscr{G}$ extending η_0 and set $\varphi = \phi \cdot \eta^{-1}$. Then φ is an ϵ^*-character on $\mathscr{G}$ which can be approximated by a true character if and only if ϕ does. We want to see that there exists some character near φ.

First, note that $d(\varphi(y), 1) \leq \epsilon$ for all $y \in \mathscr{S}$. Take a selection S: $\mathscr{G}/\mathscr{S} \to \mathscr{G}$ for the quotient and define ψ: $\mathscr{G}/\mathscr{S} \to \mathbb{T}$ by $\psi(w) = \varphi(S(w))$. We claim that ψ is a pseudo-character. Indeed, let $u, v \in \mathscr{G}/\mathscr{S}$. Note that $(S(u)S(v))^{-1}S(uv)$ lies in $\mathscr{S}$ and so, $d(\varphi((S(u)S(v))^{-1}S(uv)), 1) \leq \epsilon$. Hence one has

$$\begin{aligned}
&d(\psi(uv), \psi(u)\psi(v)) \\
&\qquad = d(\varphi(S(uv)), \varphi(S(u))\varphi(S(v))) \\
&\qquad \leq d(\varphi(S(uv)), \varphi(S(u)S(v))) \\
&\qquad\qquad + d(\varphi(S(u)S(v)), \varphi(S(u))\varphi(S(v))) \\
&\qquad \leq d\Big(\varphi(S(uv)), \varphi(S(u)S(v))\varphi\big((S(u)S(v))^{-1}S(uv)\big)\Big) + \epsilon + \epsilon \\
&\qquad \leq 3\epsilon.
\end{aligned}$$

Now, the hypothesis on $\mathscr{S}$ together with Proposition 1 yields a character $\xi\colon \mathscr{G}/\mathscr{S} \to \mathbb{T}$ such that

$$d(\psi(\pi(x)), \xi(\pi(x))) \leq 3\epsilon.$$

Define $\chi\colon \mathscr{G} \to \mathbb{T}$ by $\chi = \xi \circ \pi$. We show that χ approximates φ. Take $x \in \mathscr{G}$. Write $x = (x \cdot (S(\pi(x)))^{-1}) \cdot S(\pi(x))$. Clearly, $x \cdot (S(\pi(x)))^{-1} \in \mathscr{S}$. Thus,

$$\begin{aligned} d(\varphi(x), \chi(x)) \leq\ & d\Big(\varphi\big(x \cdot (S(\pi(x)))^{-1} \cdot S(\pi(x))\big), \\ & \qquad \varphi\big(x \cdot (S(\pi(x)))^{-1}\big)\varphi(S(\pi(x)))\Big) \\ & \quad + d\Big(\varphi\big(x \cdot (S(\pi(x)))^{-1}\big)\varphi(S(\pi(x)), \xi(\pi(x)))\Big) \\ \leq\ & \epsilon + \epsilon + 3\epsilon. \end{aligned}$$

This completes the proof of (b). Part (c) follows from (a) and (b). ∎

In a sense, the behaviour of a pseudo-character depends only on its restriction to the commutator subgroup. Recall that the commutator subgroup $\mathscr{G}_0$ of $\mathscr{G}$ is the subgroup generated by the set of "commutators" $\{aba^{-1}b^{-1} : a, b \in \mathscr{G}\}$. The commutator subgroup is normal and the corresponding quotient $\mathscr{G}/\mathscr{G}_0$ is abelian. Actually, the quotient map $\pi\colon \mathscr{G} \to \mathscr{G}/\mathscr{G}_0$ has the following universal property: for every group homomorphism Φ from $\mathscr{G}$ to an abelian group $\mathscr{H}$ there exists a homomorphism $\Psi\colon \mathscr{G}/\mathscr{G}_0 \to \mathscr{H}$ such that $\Phi = \Psi \circ \pi$. This obviously implies that the kernel of Φ contains $\mathscr{G}_0$.

The following result asserts that a pseudo-character can be approximated by some character if and only if its values on $\mathscr{G}_0$ are near 1. (Compare to [5, Lemma 1].)

PROPOSITION 2. *Let $\phi\colon \mathscr{G} \to \mathbb{T}$ be an ϵ^*-character.*

(a) *If a character χ such that $d(\phi(x), \chi(x)) < \pi/3$ $(x \in \mathscr{G})$ exists, then $d(\phi(y), 1) \leq \epsilon$ for all $y \in \mathscr{G}_0$.*

(b) *Conversely, suppose $d(\phi(y), 1) \leq \delta$ holds for every $y \in \mathscr{G}_0$, with $2\delta + 3\epsilon < \pi/3$. Then there exists a character χ fulfilling $d(\phi(x), \chi(x)) \leq \epsilon$ for all $x \in \mathscr{G}$.*

Proof. (a) Since $\chi(y) = 1$ for all $y \in \mathscr{G}_0$ this clearly follows from Proposition 1.

(b) Take a selection $S\colon \mathscr{G}/\mathscr{G}_0 \to \mathscr{G}$ for the natural quotient map $\pi\colon \mathscr{G} \to \mathscr{G}/\mathscr{G}_0$ and define $\psi\colon \mathscr{G}/\mathscr{G}_0 \to \mathbb{T}$ by $\psi(w) = \phi(S(w))$. Just as in the proof of part (b) of the preceding proposition, ψ is a pseudo-character.

More precisely, for $u, v \in \mathscr{G}/\mathscr{G}_0$, one has

$$\begin{aligned}
&d(\psi(uv), \psi(u)\psi(v)) \\
&\quad = d(\phi(S(uv)), \phi(S(u))\phi(S(v))) \\
&\quad \le d(\phi(S(uv)), \phi(S(u)S(v))) \\
&\qquad + d(\phi(S(u)S(v)), \phi(S(u))\phi(S(v))) \\
&\quad \le d\Big(\phi(S(uv)), \phi(S(u)S(v))\phi\big((S(u)S(v))^{-1}S(uv)\big)\Big) + \delta + \epsilon \\
&\quad \le \delta + 2\epsilon \\
&\quad < \pi/3.
\end{aligned}$$

But $\mathscr{G}/\mathscr{G}_0$ is abelian and by Theorem 1 there is a character $\xi\colon \mathscr{G}/\mathscr{G}_0 \to \mathbb{T}$ such that

$$d(\psi(\pi(x)), \xi(\pi(x))) \le \delta + 2\epsilon.$$

Define $\chi\colon \mathscr{G} \to \mathbb{T}$ by $\chi = \xi \circ \pi$. We show that χ approximates ϕ. Take $x \in \mathscr{G}$. Write $x = (x \cdot (S(\pi(x)))^{-1}) \cdot S(\pi(x))$. Clearly, $x \cdot (S(\pi(x)))^{-1} \in \mathscr{G}_0$. Thus,

$$\begin{aligned}
&d(\phi(x), \chi(x)) \\
&\quad \le d\Big(\phi\big(x \cdot (S(\pi(x)))^{-1} \cdot S(\pi(x))\big), \\
&\qquad\quad \phi\big(x \cdot (S(\pi(x)))^{-1}\big)\phi(S(\pi(x)))\Big) \\
&\qquad + d\Big(\phi\big(x \cdot (S(\pi(x)))^{-1}\big)\phi(S(\pi(x)), \xi(\pi(x)))\Big) \\
&\quad \le 2\delta + 3\epsilon \\
&\quad < \pi/3.
\end{aligned}$$

Now, an appeal to Proposition 1 completes the proof. ∎

The following result shows the connection between our approach to Ulam's problem and the classical (additive) one.

THEOREM 2. *Every group whose characters are stable has stable real-characters.*

Proof. Suppose on the contrary that $\mathscr{G}$ does not have stable real-characters. Then, by a result of Giudici [5, Lemma 1] there is $f\colon \mathscr{G} \to \mathbb{R}$ such that $|f(xy) - f(x) - f(y)|$ remains bounded (by 1, say) on $\mathscr{G}$ while f itself is unbounded on the commutator subgroup $\mathscr{G}_0$. For $\epsilon > 0$, define $\phi_\epsilon\colon \mathscr{G} \to$

$\mathbb{T}$ by

$$\phi_\epsilon(x) = \exp(i\epsilon f(x)).$$

Obviously, ϕ_ϵ is an ϵ^*-character. On the other hand, it is clear that for every $\delta > 0$ there is $0 < \epsilon < \delta$ such that

$$\pi = \sup_{y \in \mathscr{G}_0} d(\phi_\epsilon(x), 1),$$

which implies that for any character $\chi\colon \mathscr{G} \to \mathbb{T}$ one has

$$\pi = \sup_{x \in \mathscr{G}} d(\phi_\epsilon(x), \chi(x)).$$

This completes the proof. ∎

3. SOME EXAMPLES

We now discuss some examples. The following one is taken from [4]. In what follows, $\mathbb{F}_2$ denotes the free group generated by the symbols a and b, the operation being juxtaposition.

EXAMPLE 1 (Forti). For every $\epsilon > 0$ there exists an ϵ^*-character $\phi\colon \mathbb{F}_2 \to \mathbb{T}$ such that, for any character $\chi\colon \mathbb{F}_2 \to \mathbb{T}$, one has

$$\sup_{x \in \mathbb{F}_2} d(\phi(x), \chi(x)) = \pi.$$

Proof. Let $x \in \mathbb{F}_2$ be written in the "reduced" form; that is, x does not contain pairs of the form aa^{-1}, $a^{-1}a$, bb^{-1}, or $b^{-1}b$, and it is written without exponents different from 1 and -1. Let $r(x)$ be the number of pairs of the form ab in x and let $s(x)$ be the number of pairs of the form $b^{-1}a^{-1}$ in x. Now, put $f(x) = r(x) - s(x)$. Clearly, one has $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$ for all $x, y \in \mathbb{F}_2$, but f is unbounded on the commutator subgroup of $\mathbb{F}_2$. Fix $\epsilon > 0$ and define $\phi_\epsilon\colon \mathbb{F}_2 \to \mathbb{T}$ by

$$\phi_\epsilon(x) = \exp(i\epsilon f(x)).$$

Obviously, ϕ_ϵ is an ϵ^*-character. Now, argue as in the proof of Theorem 2. ∎

The following observation is based on a result of Giudici [5, Theorem 3, p. 150] and provides to us with a new class of (not necessarily amenable) groups whose characters are stable.

PROPOSITION 3. *Let $\mathcal{G}$ be a group. Suppose there is k such that every element of the commutator subgroup $\mathcal{G}_0$ is the product of no more than k commutators. Then $\mathcal{G}$ has stable characters.*

Proof. Suppose $\phi: \mathcal{G} \to \mathbb{T}$ satisfies $d(\phi(xy), \phi(x)\phi(y)) \leq \epsilon$ for all $x, y \in \mathcal{G}$. It is easily seen that

$$d\left(\phi\left(\prod_{i=1}^{n} x_i\right), \prod_{i=1}^{n} \phi(x_i)\right) \leq (n-1)\epsilon$$

for all n and x_i.

Let $c \in \mathcal{G}_0$ be a commutator; that is, $c = aba^{-1}b^{-1}$ for some $a, b \in \mathcal{G}$. We estimate $d(\phi(c), 1)$. Since $d(\phi(c), \phi(a)\phi(b)\phi(a^{-1})\phi(b^{-1})) \leq 3\epsilon$, $d(\phi(1), \phi(a)\phi(a^{-1})\phi(b)\phi(b^{-1})) \leq 3\epsilon$, and $d(1, \phi(e)) = d(\phi(e), \phi(e)^2) \leq \epsilon$, it follows that $d(\phi(c), 1) \leq 7\epsilon$.

Suppose $y \in \mathcal{G}_0$ and write $y = \prod_{i=1}^{k} c_i$, where c_i are commutators. Then

$$d(\phi(y), 1) \leq d\left(\phi(y), \prod_{i=1}^{k} \phi(c_i)\right) + d\left(\prod_{i=1}^{k} \phi(c_i), 1\right) \leq (8k-1)\epsilon.$$

Now, apply part (b) of Proposition 2. ∎

Thus, for instance, the general linear group $GL_2(\mathbb{R})$ consisting of all invertible 2×2 real matrices is not amenable, yet it enjoys the property required by the preceding proposition for $k = 2$, and, therefore, it has stable characters. According to [19, Sect. 6] (but this paper contains no proofs) "a connected, locally compact group is amenable if and only if it is stably representable on the Hilbert space." Hence $GL_2(\mathbb{R})$ would admit a strongly continuous approximate representation which is far from any representation.

Our next example concerns the multiplicative group $\mathbb{H}^*$ of nonzero quaternions. It is well known that the commutator of $\mathbb{H}^*$ is the subgroup $\mathbb{H}_1 = \{q \in \mathbb{H} : |q| = 1\}$, so that $\mathbb{H}_1 \to \mathbb{H}^* \to \mathbb{R}^+$ is the "abelianizing" sequence. Moreover, $\mathbb{H}_1$ consists only of "pure" commutators (see [6] for a nice proof). Thus, we obtain from Proposition 3 that $\mathbb{H}^*$ has stable characters (but fails to be amenable as a discrete group).

Finally, since the mapping $q \in \mathbb{H}^* \to q/|q| \in \mathbb{H}_1$ is a surjective homomorphism, Lemma 1 shows that $q \in \mathbb{H}_1$ as stable characters as well. This is an interesting fact since $\mathbb{H}_1$ turns out to be isomorphic to the special orthogonal group $SO(3, \mathbb{R})$. Hence the full orthogonal group $O(3, \mathbb{R}) = \mathbb{Z}_2 \times SO(3, \mathbb{R})$ also has stable characters.

Another consequence is that every group $\mathcal{G}$ is a subgroup of a larger group with stable characters. If $\mathcal{G}$ is finite this is obvious. If $\mathcal{G}$ is infinite, consider $\mathscr{G}$ as a subgroup of $S(\mathscr{G})$, the group of all bijective transformations of $\mathscr{G}$. Ore proved in [16] that all elements of $S(\mathscr{G})$ are commutators; hence $S(\mathscr{G})$ satisfies the hypothesis of Proposition 3 for $k = 1$.

4. PSEUDO-CHARACTERS VERSUS ALMOST MULTIPLICATIVE FUNCTIONALS

In this section we present some applications to almost multiplicative functionals on group algebras. For simplicity, we shall consider only discrete groups. Recall that the group algebra of $\mathscr{G}$ is the Banach space $l_1(\mathscr{G})$ endowed with the convolution product

$$f * g(x) = \sum_{\xi \in \mathscr{G}} f(\xi) g(\xi^{-1} x).$$

The dual space of $l_1(\mathscr{G})$ can be regarded as $l_\infty(\mathscr{G})$, the space of all bounded maps $\phi\colon \mathscr{G} \to \mathbb{C}$ by means of

$$\langle \phi, f \rangle = \sum_{x \in \mathscr{G}} \phi(x) f(x).$$

Moreover, $\|\phi\|_\infty = \sup_{x \in \mathscr{G}} |\phi(x)|$ equals $\|\langle \phi, \cdot \rangle\|$. In this way (nonzero) multiplicative linear functionals $l_1(\mathscr{G}) \to \mathbb{C}$ correspond to characters $\chi\colon \mathscr{G} \to \mathbb{T}$. We begin with the following

LEMMA 2. *For a bounded map $\phi\colon \mathscr{G} \to \mathbb{C}$ the following statements are equivalent*:

(a) $|\phi(xy) - \phi(x)\phi(y)| \leq \epsilon$ *for all* $x, y \in \mathscr{G}$.

(b) *For every* $f, g \in l_1(\mathscr{G})$, *one has* $|\langle \phi, f * g \rangle - \langle \phi, f \rangle \langle \phi, g \rangle| \leq \epsilon \|f\| \, \|g\|$.

Proof. That (b) implies (a) is obvious since

$$\begin{aligned} |\phi(xy) - \phi(x)\phi(y)| &= |\langle \phi, \delta_{xy} \rangle - \langle \phi, \delta_x \rangle \langle \phi, \delta_y \rangle| \\ &= |\langle \phi, \delta_x * \delta_y \rangle - \langle \phi, \delta_x \rangle \langle \phi, \delta_y \rangle| \\ &\leq \epsilon \|\delta_x\| \, \|\delta_y\| \\ &= \epsilon. \end{aligned}$$

For the converse, suppose (a) holds. Let $f, g \in l_1(\mathscr{G})$. Then

$$\langle \phi, f * g \rangle = \sum_{x \in \mathscr{G}} \phi(x)(f * g)(x) = \sum_{x,\, \xi \in \mathscr{G}} \phi(x) f(\xi) g(\xi^{-1} x),$$

while

$$\begin{aligned}\langle \phi, f \rangle \langle \phi, g \rangle &= \sum_{x \in \mathscr{G}} \phi(x)(f * g)(x) \\ &= \sum_{x,\, \xi \in \mathscr{G}} \phi(\xi^{-1} x) \phi(\xi) f(\xi) g(\xi^{-1} x).\end{aligned}$$

Hence,

$$\begin{aligned}&|\langle \phi, f * g \rangle - \langle \phi, f \rangle \langle \phi, g \rangle| \\ &\quad = \left| \sum_{x,\, \xi \in \mathscr{G}} (\phi(x) - \phi(\xi^{-1} x)\phi(\xi)) f(\xi) g(\xi^{-1} x) \right| \\ &\quad \leq \epsilon \left(\sum_{x,\, \xi \in \mathscr{G}} |f(\xi) g(\xi^{-1} x)| \right) \\ &\quad \leq \epsilon \|f\| \, \|g\|,\end{aligned}$$

which completes the proof. ∎

Now suppose $\langle \phi, \cdot \rangle: l_1(\mathscr{G}) \to \mathbb{C}$ is an ϵ-multiplicative functional. Then, by [11] $\langle \phi, \cdot \rangle$ is continuous and, in fact, $\|\langle \phi, \cdot \rangle\| = \|\phi\|_\infty \leq 1 + \epsilon$. We want to see that either $|\phi(x)|$ is very small for all $x \in \mathscr{G}$ or $\phi: \mathscr{G} \to \mathbb{C}$ takes values near $\mathbb{T}$. From $|\phi(e) - \phi(e)^2| \leq \epsilon$, it follows that either

(a) $|\phi(e)| \leq \frac{1}{2}(1 - \sqrt{1 - 4\epsilon})$, or

(b) $\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \leq |\phi(e)| \leq \frac{1}{2}(1 + \sqrt{1 + 4\epsilon})$.

Note that $\frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) = \epsilon + o(\epsilon)$, while $\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) = 1 - \epsilon + o(\epsilon)$ and $\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \leq 1 + \epsilon$.

In case (a) taking $x \in \mathscr{G}$, one has $|\phi(x) - \phi(x)\phi(e)| \leq \epsilon$ and having in mind that $|\phi(x)| \leq \epsilon$ we see that $|\phi(x)| \leq 2\epsilon + o(\epsilon)$.

If (b) holds then, from $|\phi(e) - \phi(x)\phi(x^{-1})| \leq \epsilon$, we have $|\phi(x)|\,|\phi(x^{-1})| \geq |\phi(e)| - \epsilon$. Hence

$$1 - 2\epsilon + o(\epsilon) \leq \frac{|\phi(e)| - \epsilon}{1 + \epsilon} \leq |\phi(x)|.$$

Thus, we have proved the following.

LEMMA 3. *There exists a function* $o(\cdot)$ (*with* $o(\epsilon)/\epsilon \to 0$ *as* $\epsilon \to 0$) *such that*, *for every* $\phi \in l_\infty(\mathscr{G})$ *which represents an* ϵ-*multiplicative functional on* $l_1(\mathscr{G})$ *with* $\epsilon < 1/4$, *exactly one of the following holds*

(a) $|\phi(e)| \le \epsilon$ *and then* $\|\phi\|_\infty \le 2\epsilon + o(\epsilon)$, *or*

(b) $1 - \epsilon + o(\epsilon) \le |\phi(e)| \le 1 + \epsilon$ *and then* $1 - 2\epsilon + o(\epsilon) \le |\phi(x)| \le 1 + \epsilon$ *for every* $x \in \mathscr{G}$.

Moreover $o(\cdot)$ *does not depend on* $\mathscr{G}$.

THEOREM 3. *Let* $\mathscr{G}$ *be a group. Then* $l_1(\mathscr{G})$ *is an AMNM algebra if and only if* $\mathscr{G}$ *has stable characters.*

Proof. We first prove the "only if" part. Suppose $l_1(\mathscr{G})$ is AMNM and let $\phi\colon \mathscr{G} \to \mathbb{T}$ be an ϵ-character. Then $\langle \phi, \cdot \rangle$ is a norm one ϵ-multiplicative functional on $l_1(\mathscr{G})$. Let $\langle \xi, \cdot \rangle\colon l_1(\mathscr{G}) \to \mathbb{C}$ be any multiplicative functional such that $\|\langle \phi, \cdot \rangle - \langle \xi, \cdot \rangle\| < 1$. Then ξ is a character on $\mathscr{G}$ with $|\phi(x) - \xi(x)| < 1$ for every $x \in \mathscr{G}$ and Proposition 1 implies that, in fact, $|\phi(x) - \xi(x)| \le \epsilon$ for every $x \in \mathscr{G}$.

For the converse, let $\phi\colon \mathscr{G} \to \mathbb{C}$ represent an ϵ-multiplicative functional, with $\epsilon < 1$. Then either (a) or (b) in Lemma 3 holds. In case (a) one has $\|\phi\|_\infty \le 2\epsilon + o(\epsilon)$ and ϕ is near zero. If (b) holds, let $\psi\colon \mathscr{G} \to \mathbb{T}$ be given by $\psi(x) = \phi(x)/|\phi(x)|$. Then $|\psi(x) - \phi(x)| \le 2\epsilon + o(\epsilon)$. Moreover ψ is a pseudo-character:

$$\begin{aligned}
&|\psi(xy) - \psi(x)\psi(y)| \\
&\qquad \le |\psi(xy) - \phi(xy)| + |\phi(xy) - \phi(x)\phi(y)| \\
&\qquad\qquad + |\phi(x)\phi(y) - \psi(x)\psi(y)| \\
&\qquad \le 7\epsilon + o(\epsilon).
\end{aligned}$$

Thus, if $\mathscr{G}$ has stable characters, there is a character $\chi\colon \mathscr{G} \to \mathbb{T}$ near ψ and, in fact, with $|\chi(x) - \psi(x)| \le 7\epsilon + o(\epsilon)$ for all $x \in \mathscr{G}$. Therefore,

$$\|\langle \phi, \cdot \rangle - \langle \chi, \cdot \rangle\| \le \|\phi - \psi\|_\infty + \|\psi - \chi\|_\infty \le 8\epsilon,$$

which completes the proof. ∎

In view of the theorem just proved, Example 1 provides to us with a simple example of a non-AMNM group algebra, while the examples discussed after Proposition 3 show that an AMNM group algebra need not be amenable. (For the definition of amenability for algebras, see [14]. For groups algebras amenability is equivalent to amenability in the usual sense for the underlying group.) Also, Theorem 3 implies that Theorem 1 can be obtained as a corollary of the results of Johnson in [12]. Nevertheless, I believe there is some merit in our presentation.

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