Journal of Functional Analysis **175**, 1–16 (2000)
doi:10.1006/jfan.2000.3598, available online at http://www.idealibrary.com on IDEAL®

Duality and Twisted Sums of Banach Spaces[1]

Félix Cabello Sánchez and Jesús M. F. Castillo

Departamento de Matemáticas, Universidad de Extremadura,
Avenida de Elvas, 06071 Badajoz, Spain
E-mail: fcabello@unex.es; castillo@unex.es

Communicated by R. B. Melrose

Received November 7, 1997; revised April 30, 1999; accepted February 27, 2000

We give a negative answer to the three-space problem for the Banach space properties *to be complemented in a dual space* and *to be isomorphic to a dual space* (solving a problem of Vogt [Lectures held in the Functional Analysis Seminar, Dusseldorf/Wuppertal, Jan–Feb. 1987] and another posed by Díaz *et al.* in [*Bull. Polish Acad. Sci. Math.* **40** (1992), 221–224]). Precisely, we construct an exact sequence $0 \to \ell_2 \to D \to W^* \to 0$ in which W^* is a separable dual and D is not isomorphic to a dual space. We also show the existence of an exact sequence $0 \to Y \to X \to Z \to 0$ where both Y and Z are dual spaces and X is not even complemented in its bidual. To do that we perform a study of the basic questions on duality from the point of view of exact sequences of Banach spaces.

1. INTRODUCTION

In this paper we give a negative answer to the three-space problem for the Banach space properties *to be* (*isomorphic to*) *a dual space* and *to be complemented in a dual space.*

Let us briefly explain the contents and organization of the paper. Section 2 is preliminary and contains background on exact sequences and the pull-back construction. We present in Section 3 easy counterexamples to the stated three-space problems. The key point is that every Banach space B with B^{**}/B reflexive embeds as a complemented subspace in a twisted sum of two dual spaces. This shows the existence of an exact sequence $0 \to Y \to X \to Z \to 0$ where both Y and Z are dual spaces and X is not even complemented in its bidual since it has a complemented subspace isomorphic to the standard predual of the James–Tree space. This example answers a question of Vogt [19] and solves another problem posed by Díaz *et al.* in [5].

[1] Supported in part by DGICYT Project PB-97-0377.

0022-1236/00 $35.00

Nevertheless, these counterexamples are not very nice: the kernel is ℓ_1-saturated and the quotient space is nonseparable. As we shall see, they can be considerably sharpened, at the cost of using more sophisticated tools. Section 4 is in some sense elementary, but contains some never recorded folklore which we will use in an essential way, mainly that locally convex twisted sums of Banach spaces come defined by a special type of quasi-linear maps that we call 0-linear, and a nonlinear version of the Hahn–Banach theorem. In Section 5 we construct, given a 0-linear map F, the dual 0-linear map F^* corresponding to the dual exact sequence. This allows us to work from now on with the 0-linear map $F\colon Z \to Y$ that defines an exact sequence $0 \to Y \to X \to Z \to 0$ paying no attention to the middle space X. Section 6 contains the basic criteria to determine when an exact sequence made with dual spaces is itself a dual sequence. It is most important to our purposes that a 0-linear map $G\colon Y^* \to Z^*$ induces a dual sequence if and only if G^* transforms a predual of Z^* into a predual of Y^*. This is the central section of the paper together with Section 7, in which we construct a twisted sum of a separable Hilbert space and a separable dual which is not isomorphic to a dual space. An outline of the counterexamples, without proofs, appeared in [4, 3.7].

2. PRELIMINARIES AND NOTATIONS

Exact Sequences of (Quasi)-Banach Spaces

For general information about exact sequences the reader can consult [9]. Information about categorical constructions in the quasi-Banach space setting can be found in the monograph [4]. Throughout the paper, the word *operator* means linear continuous map. A diagram $0 \to Y \to X \to Z \to 0$ of quasi-Banach spaces and operators is said to be an exact sequence if the kernel of each arrow coincides with the image of the preceding. This means, by the open mapping theorem, that Y is (isomorphic to) a closed subspace of X and the corresponding quotient is (isomorphic to) Z. We shall also say that X is a twisted sum of Y and Z. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there is an operator T making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

commutative. The three-lemma (see [9, p. 14, Lemma 1.1]) and the open mapping theorem imply that T must be an isomorphism. The exact sequence $0 \to Y \to X \to Z \to 0$ is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This already implies that the twisted sum X is isomorphic to the direct sum $Y \oplus Z$ (the converse is not true).

The Pull-Back Square

Let $A\colon U \to Z$ and $B\colon V \to Z$ be two operators. The pull-back of $\{A, B\}$ is the space $\Xi = \{(u, v) : Au = Bv\} \subset U \times V$ endowed with the relative product topology, together with the restrictions of the canonical projections of $U \times V$ onto, respectively, U and V. If $0 \to Y \to X \to Z \to 0$ is an exact sequence with quotient map q, $Q\colon V \to Z$ is a surjective operator, and Ξ denotes the pull-back of the couple $\{q, Q\}$, then the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle Q} & & \\
0 & \longrightarrow & Y & \longrightarrow & \Xi & \longrightarrow & V & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & \ker Q & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

is commutative with exact rows and columns. It is straightforward from the definition that if Ξ denotes the pull-back of two operators $\{A, B\}$ and the operator $A - B\colon U \times V \to Z$ defined by $(A - B)(u, v) = Au - Bv$ is surjective, then the sequence $0 \to \Xi \to U \times V \to Z \to 0$ is also exact.

3. FIRST COUNTEREXAMPLES

The tools exhibited so far are enough to prove the following result, which provides our first counterexample to the three-space problem for the two properties.

PROPOSITION 1. *Let Z be a Banach space such that Z^{**}/Z has the Radon–Nikodym property and is complemented in its bidual. Then Z is a complemented subspace of a twisted sum of two dual spaces.*

Proof. Let $i: Z \to Z^{**}$ be the canonical inclusion. Let I be some index set such that there exists a quotient map $q: l_1(I) \to Z^{**}$. Let Ξ be the pull-back of the operators $\{q, i\}$. One has the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Z^{**}/Z & = & Z^{**}/Z & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & l_1(I) & \xrightarrow{q} & Z^{**} & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & K & \longrightarrow & \Xi & \longrightarrow & Z & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The pull-back space Ξ is complemented in its bidual since it is the kernel of a quotient map $q: l_1(I) \to X$ in which X has the RNP and is itself complemented in its bidual (see [12, Proposition 2.3]). Moreover, since q is surjective one has the exact sequence

$$0 \to \Xi \to l_1(I) \oplus Z \to Z^{**} \to 0,$$

and the assertion follows. ∎

From here, we obtain

THEOREM 1. *To be complemented in the bidual is not a three-space property. To be isomorphic to a dual space is not a three-space property.*

Proof. Let JT denote the James–Tree space and B its standard predual. The reference [4] contains all the information about those spaces that is needed for our construction, which essentially is: the space B is uncomplemented in its bidual JT^* and JT^*/B is isomorphic to a (non-separable) Hilbert space. Applying the previous construction with $Z = B$ one gets the exact sequence

$$0 \to \Xi \to l_1(I) \oplus B \to JT^* \to 0$$

in which the middle space is not complemented in its bidual.

Whether or not the space Ξ is itself a dual space is open for discussion. However, multiplying by the complement of Ξ into Ξ^{**} one obtains the exact sequence

$$0 \to \Xi^{**} \to (\Xi^{**}/\Xi) \oplus l_1(I) \oplus B \to JT^* \to 0$$

in which the middle space is not complemented in its bidual (hence, it is not a dual space). ▌

4. LOCALLY CONVEX TWISTED SUMS AND ZERO-LINEAR MAPS

The remainder of the paper is devoted to construct a much more extreme counterexample for the property of being a dual space.

It will be convenient to introduce the following distance between maps. Given two homogeneous maps A and B acting between the same spaces their (eventually infinite) distance is defined as

$$\operatorname{dist}(A, B) = \sup_{\|x\| \leqslant 1} \|A(x) - B(x)\|.$$

In this setting, bounded maps are those maps at finite distance from the zero map. Also, the reader should keep in mind that linear maps are not assumed to be bounded (i.e., operators) in what follows.

Quasi-linear Maps

The classical theory of Kalton and Peck [11] describes short exact sequences of quasi-Banach spaces in terms of the so-called quasi-linear maps. A map $F\colon Z \to Y$ acting between quasi-normed spaces is said to be quasi-linear if it is homogeneous and satisfies that for some constant K and all points x, y in Z one has

$$\|F(x+y) - F(x) - F(y)\| \leqslant K(\|x\| + \|y\|).$$

Quasi-linear maps give rise to twisted sums: given a quasi-linear map $F\colon Z \to Y$, it is possible to construct a twisted sum, which we shall denote by $Y \oplus_F Z$, endowing the product space $Y \times Z$ with the quasi-norm

$$\|(y, z)\|_F = \|y - Fz\| + \|z\|.$$

Clearly, the subspace $\{(y, 0) : y \in Y\}$ is isometric to Y and the corresponding quotient is isomorphic to Z. One has the following fundamental result.

PROPOSITION 2. [11, p. 6, Theorem 2.5].

(a) *Let F and G be quasi-linear maps from Z to Y. Then the induced sequences* $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *and* $0 \to Y \to Y \oplus_G Z \to Z \to 0$ *are equivalent if and only if the difference* $G - F$ *is at finite distance from some linear map* $Z \to Y$.

(b) *Consequently* $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *splits if and only if F is at finite distance from some linear map* $Z \to Y$.

Accordingly, we shall say that two quasi-linear maps F and G (defined between the same spaces) are equivalent if their difference $G - F$ is at finite distance from linear maps. (Sometimes we say that G is a version of F, or vice versa.) Also, we say that F is trivial if it is at finite distance from some linear map (that is, if F is a version of the 0 map). This means that F can be written as the sum of a linear map and a bounded map.

And conversely, quasi-linear maps arise from exact sequences: given a short exact sequence $0 \to Y \to X \to Z \to 0$ of quasi-Banach spaces a quasi-linear map $F: Z \to Y$ can be obtained taking a linear (possibly non-continuous) selection $L: Z \to X$ for the quotient map $q: X \to Z$ and a bounded homogeneous (possibly non-linear nor continuous) selection $B: Z \to X$ for q. The difference $F = B - L$ is quasi-linear and takes values in Y since $q \circ (B - L) = 0$. It is clear that the sequences $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y \to X \to Z \to 0$ are equivalent and that any quasi-linear map defined as before by a sequence $0 \to Y \to Y \oplus_F Z \to Z \to 0$ is equivalent to F.

Zero-Linear Maps

The quasi-Banach space $Y \oplus_F Z$ constructed via a quasi-linear map $F: Z \to Y$ need not be locally convex, even when Y and Z are. In [18], Ribe constructed a twisted sum of $\mathbb{R}$ and ℓ_1 that is not locally convex. Nevertheless, some positive results are available. In [10], Kalton proved that when Y and Z are B-convex Banach spaces then every twisted sum $Y \oplus_F Z$ is locally convex. In [13], Kalton and Roberts proved that every twisted sum of a Banach space and an $\mathscr{L}_\infty$-space is locally convex.

It is, however, possible to obtain a simple characterization of locally convex twisted sums in terms of the quasi-linear map F (see [3, 4]).

DEFINITION 1. A homogeneous map $F: Z \to Y$ acting between normed spaces is said to be zero-linear if there is a constant K such that

$$\left\| \sum_{i=1}^{n} F(z_i) \right\| \leqslant K \left(\sum_{i=1}^{n} \|z_i\| \right)$$

whenever $\{z_i\}_{i=1}^n$ is a finite subset of Z with $\sum_{i=1}^n z_i = 0$. The smallest constant satisfying the inequality above is denoted $Z(F)$ and referred to as the 0-linearity constant of the map F.

It is clear that 0-linear maps are quasi-linear. It is not true, however, that quasi-linear maps are 0-linear after Ribe's example and the following characterization (perhaps implicit in [7]):

THEOREM 2. *A twisted sum of Banach spaces $Y \oplus_F Z$ is locally convex (being thus isomorphic to a Banach space) if and only if F is 0-linear.*

Proof. Assume that $F: Z \to Y$ is 0-linear with constant K. It is straightforward that

$$\left\| \sum_{i=1}^n (y_i, z_i) \right\|_F \leqslant 2K \left(\sum_{i=1}^n \|(y_i, x_i)\|_F \right),$$

and therefore $\|\cdot\|_F$ is equivalent to a norm. Conversely, if $Y \oplus_F Z$ is locally convex, $\|\cdot\|_F$ is equivalent to a norm $\|\cdot\|$ in the sense that $\|x\| \leqslant \|x\|_F \leqslant K \|x\|$ for some K and all x. Hence if $\sum_{i=1}^n z_i = 0$, then

$$\left\| \sum_{i=1}^n F(z_i) \right\|_Y = \left\| \sum_{i=1}^n (Fz_i, z_i) \right\|_F \leqslant K \sum_{i=1}^n \|(Fz_i, z_i)\| \leqslant K \sum_{i=1}^n \|z_i\|_Z.$$

It follows that F is 0-linear with constant at most K. ∎

The first step in duality theory is the Hahn–Banach theorem. From the point of view of exact sequences, the Hahn–Banach theorem asserts that every exact sequence of locally convex spaces $0 \to \mathbb{K} \to X \to Z$ splits. Thus, given a 0-linear map $F: Z \to \mathbb{K}$ there must be some linear map $L: Z \to \mathbb{K}$ at finite distance from F. The following more precise result gives exactly the distance between F and Z', the algebraic dual of Z, and is the key of the nonlinear theory of duality to be developed in the next section.

LEMMA 1. *Let $F: Z \to \mathbb{K}$ be a 0-linear map with constant K. Then there exists a linear map $L: Z \to \mathbb{K}$ such that* $\operatorname{dist}(F, L) \leqslant K$.

Proof. We write the proof only for the real case. Suppose that $F: Z \to \mathbb{R}$ is 0-linear with constant K. Consider the functional $\varrho: Z \to \mathbb{R}$ given by

$$\varrho(z) = \inf \left\{ \sum_{i=1}^n F(z_i) + K \sum_{i=1}^n \|z_i\| : z = \sum_{i=1}^n z_i \right\}.$$

(Note that ϱ takes only finite values and, in fact, one has $\varrho(z) \geqslant F(z) - K\|z\|$ for all $z \in Z$.) Clearly, $\varrho(tz) = t\varrho(z)$ for every $t \geqslant 0$ and all z. Moreover

ϱ is subadditive. By an old theorem of Banach [1, p. 226, Théorème 2] ϱ dominates some $\mathbb{R}$-linear map $L\colon X \to \mathbb{R}$. Thus, $L(z) \leqslant \varrho(z) \leqslant F(z) + K\,\|z\|$ for all $z \in Z$ and $L(z) - F(z) \leqslant K\,\|z\|$. Taking into account that $\|\cdot\|$ is an even function and that both L and F are homogeneous (hence, odd) maps, we arrive at

$$|L(z) - F(z)| \leqslant K\,\|z\|. \quad \blacksquare$$

5. NONLINEAR DUALITY

From the point of view of exact sequences the existence of a duality theory means that if $0 \to Y \to X \to X \to 0$ is an exact sequence then the dual sequence $0 \to Z^* \to X^* \to Y^* \to 0$ is well defined and exact. Thus, if $F\colon Z \to Y$ is a 0-linear map defining the starting sequence then there must be a 0-linear map $Y^* \to Z^*$ inducing the dual sequence $0 \to Z^* \to (Y \oplus_F Z)^* \to Y^* \to 0$. This map could properly be called the transpose or dual map F^* of F. Let us show how to construct F^*.

Let $y^* \in Y^*$. The composition $y^* \circ F\colon Z \to \mathbb{K}$ is 0-linear with $Z(y^* \circ F) \leqslant \|y^*\|\, Z(F)$. By Lemma 1 there must be some linear map $H(y^*)\colon Z \to \mathbb{K}$ at a distance of at most $\|y^*\|\, Z(F)$ from $y^* \circ F$. The map $H\colon Y^* \to Z'$ so defined is not linear, although assuming that it is homogeneous, there is no loss of generality. Take now a Hamel basis (f_α) for Y^* and define a linear map $L_H\colon Y^* \to Z'$ by $L_H(f_\alpha) = H(f_\alpha)$ (and linearly on the rest).

THEOREM 3. *The map $F^* = L_H - H$ is a 0-linear map from Y^* to Z^* with $Z(F^*) \leqslant Z(F)$ and the sequences $0 \to Z^* \to Z^* \oplus_{F^*} Y^* \to Y^* \to 0$ and $0 \to Z^* \to (Y \oplus_F Z)^* \to Y^* \to 0$ are equivalent.*

Proof. First of all one has to show that $H(y^*) - L_H(y^*)$ belongs to Z^* (and not merely to Z'). If we write y^* as a finite linear combination $y^* = \sum_\alpha t_\alpha f_\alpha$, then

$$\begin{aligned}
&\|H(y^*) - L_H(y^*)\| \\
&\quad = \operatorname{dist}(H(y^*), L_H(y^*)) \\
&\quad = \operatorname{dist}(L_H(y^*), y^* \circ F) + \operatorname{dist}(y^* \circ F, H(y^*)) \\
&\quad \leqslant \operatorname{dist}\left(L_H\left(\sum t_\alpha f_\alpha\right), \left(\sum t_\alpha f_\alpha\right) \circ F\right) + \operatorname{dist}(y^* \circ F, H(y^*)) \\
&\quad \leqslant \sum |t_\alpha| \operatorname{dist}(L_H(f_\alpha), f_\alpha \circ F) + \operatorname{dist}(y^* \circ F, H(y^*)) \\
&\quad \leqslant K(F)\left(\sum |t_\alpha|\, \|f_\alpha\| + \|y^*\|\right) < \infty.
\end{aligned}$$

That F^* is 0-linear is straightforward: If $\{y_i^*\}$ is a finite subset of Y^* such that $\sum_i y_i^* = 0$, then

$$\begin{aligned}\left\|\sum_i F^* y_i^*\right\| &= \left\|\sum_i (L_H y_i^* - H y_i^*)\right\| \\ &= \left\|\sum_i (H y_i^* - y_i^* \circ F)\right\| \leqslant Z(F) \sum_i \|y_i^*\|.\end{aligned}$$

It remains to obtain an operator T making commutative the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^* & \xrightarrow{j} & Z^* \oplus_{F^*} Y^* & \xrightarrow{p} & Y^* & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & Z^* & \xrightarrow[q^*]{} & (Y \oplus_F Z)^* & \xrightarrow[i^*]{} & Y^* & \longrightarrow & 0 \end{array}$$

where $j(z^*) = (z^*, 0)$ and $p(z^*, y^*) = y^*$ are the canonical embedding and quotient map, respectively. This operator gives the action of the elements of the twisted sum $Z^* \oplus_{F^*} Y^*$ as functionals on $Y \oplus_F Z$. We define:

$$T(z^*, y^*)(y, z) = z^*(z) + y^*(y) - L_H(y^*)(z).$$

The linearity of T and the property of making the diagram commutative are clear, while the fact that T is well defined and continuous is a consequence of

$$\begin{aligned}&|T(z^*, y^*)(y, z)| \\ &\quad = |z^*(z) + y^*(y) - L_H(y^*)(z)| \\ &\quad = |z^*(z) + y^*(y) + F^*(y^*)(z) - F^*(y^*)(z) - L_H(y^*)(z)| \\ &\quad = |z^*(z) + y^*(y) - H(y^*)(z) - F^*(y^*)(z)| \\ &\quad \leqslant |z^*(z) - F^*(y^*)(z)| + |y^*(y) - y^*(Fz) + y^*(Fz) - H(y^*)(z)| \\ &\quad \leqslant \|z^* - F^* y^*\| \, \|z\| + \|y^*\| \, \|y - Fz\| + K(F) \, \|y^*\| \, \|z\| \\ &\quad \leqslant \max\{1, K(F)\} \, \|(z^*, y^*)\|_{F^*} \, \|(y, z)\|_F. \quad \blacksquare\end{aligned}$$

Remark 1. The dual 0-linear map F^* can be written as $L_H - H$, where $H\colon Y^* \to Z'$ satisfies $y^* \circ F - H(y^*) \leqslant Z(F) \|y^*\|$ and $L_H\colon Y^* \to Z'$ is linear. Actually, every version of F^* can be written as $L' - H'$ with L' linear and H' satisfying an estimate $\|y^* \circ F - H'(y^*)\| \leqslant M \|y^*\|$ for some constant M, because if $G = F^* + B + L$ is a version of $F^* = L_H - H$ then $G = (L_H + L) - (H - B)$ is the desired decomposition.

Remark 2. Given a 0-linear map $F: Z \to Y$ the meaning to be given to $F^{**}: Z^{**} \to Y^{**}$ is now clear. Analogously to the linear case one has:

LEMMA 2. *There is a version of F^{**} that restricted to Z coincides with F.*

Proof. Let B and L be homogeneous bounded and linear selections for the quotient map $q: X \to Z$ such that $F = B - L$. One only needs to obtain homogeneous bounded and linear selections B^{**} and L^{**} for $q^{**}: X^{**} \to Z^{**}$ such that $B^{**}_{|Z} = B$ and $L^{**}_{|Z} = L$. It is not necessary to say more. ∎

6. CHARACTERIZATIONS OF DUAL SEQUENCES

In this section we give criteria for a sequence $0 \to Z^* \to X \to Y^* \to 0$ to be a dual sequence. It is necessary to give a precise meaning to expressions such as being a dual sequence.

Preduals

By a predual of a Banach space X we understand a Banach space M and an isomorphism $T: X \to M^*$. Therefore $Y = T^*(M)$ is a subspace of X^* such that Y^* is X (with the obvious duality), and for this reason it is possible to admit that preduals of X are subspaces of X^*. (Dixmier [6] characterized precisely which subspaces of X^* are preduals of X.) Two preduals of X^*, say M_1 and M_2, have to be considered differently if they occupy different positions at X^*, even if they are isomorphic or isometric. Observe that they correspond to different isomorphisms T.

Dual Sequences

Given a sequence $0 \to Y \to X \to Z \to 0$ the sequence $0 \to Z^* \to X^* \to Y^* \to 0$ shall be referred to as *the* transpose sequence of the preceding. We will say that

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

is a dual sequence if there exists some exact sequence

$$0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \longrightarrow 0$$

such that $0 \to A \to B \to C \to 0$ and $0 \to Z^* \to X^* \to Y^* \to 0$ are isomorphically equivalent in the sense that there are isomorphisms α, β, and γ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Z^* & \xrightarrow[q^*]{} & X^* & \xrightarrow[i^*]{} & Y^* & \longrightarrow & 0 \end{array} \qquad (\dagger)$$

commutative. This already implies that Z, X, and Y are preduals of A, B, and C under $\alpha: A \to Z^*$, $\beta: B \to X^*$ and $\gamma: C \to Y^*$ respectively. To clarify the exposition as much as possible, let us show that every dual sequence is itself a transpose sequence with respect to the appropriate choice of preduals. So, assume that $0 \to A \to B \to C \to 0$ is a dual sequence. Taking adjoints in $(\dagger)$, one obtains the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longleftarrow & A^* & \xleftarrow{j^*} & B^* & \xleftarrow{p^*} & C^* & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle\alpha^*} & & \uparrow{\scriptstyle\beta^*} & & \uparrow{\scriptstyle\gamma^*} & & \\ 0 & \longleftarrow & Z^{**} & \xleftarrow{q^{**}} & X^{**} & \xleftarrow{i^{**}} & Y^{**} & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longleftarrow & Z & \xleftarrow{q} & X & \xleftarrow{i} & Y & \longleftarrow & 0 \end{array}$$

Set ${}_*A = \alpha^*(Z)$, ${}_*B = \beta^*(X)$, and ${}_*C = \gamma^*(Y)$. Then $({}_*A)^* = A$, $({}_*B)^* = B$, and $({}_*C)^* = C$ (with the obvious dualities). On the other hand ${}_*B$ contains the image of ${}_*C$ under p^* and $j^*({}_*B) = {}_*A$, so that there is an exact sequence $0 \leftarrow {}_*A \leftarrow {}_*B \leftarrow {}_*C \leftarrow 0$ whose transpose is $0 \to A \to B \to C \to 0$.

Remark 3. One essential point to be noted is that since a Banach space may have many different preduals, it may happen that a given sequence $0 \to A \to B \to C \to 0$ is a transpose sequence for a certain choice of the preduals Y, X, and Z while it is not for a different choice of preduals. A simple example is given by the sequence

$$0 \longrightarrow \ell_1 \xrightarrow{r} \ell_\infty \longrightarrow \ell_\infty / r(\ell_1) \longrightarrow 0,$$

where the inclusion map r consists of embedding the canonical basis of ℓ_1 as Rademacher-like sequences of ± 1 in ℓ_∞: this is not a weak*-to-weak* continuous (with respect to the weak* topologies induced by the usual preduals c_0 and ℓ_1, respectively) embedding and thus it cannot be the transpose of a sequence formed with precisely *those* preduals. Consider now a sequence $0 \to \ker q \to \ell_1 \to c_0$, where $q: \ell_1 \to c_0$ is some quotient map. Then one has the transpose sequence

$$0 \longrightarrow \ell_1 \xrightarrow{q^*} \ell_\infty \longrightarrow (\ker q)^* \longrightarrow 0.$$

The two sequences are isomorphically equivalent since when a separable Banach space (in our case ℓ_1) is embedded in two different forms r and q^* into ℓ_∞ there exists an automorphism T of ℓ_∞ such that $T \circ q^* = r$ (see [17, p. 110, Theorem 2.f.12]).

We enter now into the problem of determining when an exact sequence with dual spaces is a dual sequence.

THEOREM 4. (a) *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence defined by a* 0*-linear map* $F: C \to A$. *Fix preduals* $Y \subset C^*$ *and* $Z \subset A^*$ *for, respectively,* C *and* A. *The sequence is the transpose of a sequence* $0 \to Y \to {}_*X \to Z \to 0$ *if and only if some version of* F^* *maps* Z *into* Y.

(b) *Consequently, an exact sequence* $0 \to Z^* \to X \to Y^* \to 0$ *defined by a* 0*-linear map* $F: Y^* \to Z^*$ *is a dual sequence if and only if some version of* F^* *maps a predual of* Z^* *into a predual of* Y^*.

Proof. The necessity clearly follows from Lemma 2. Let us prove the sufficiency. Let us see that if a version G of F^* sends Z into Y then F is a version of $(G_{|Z})^*$. Actually, the hypothesis only appears to ensure that both maps are defined between the same spaces. Thus, write $G = L - H$ as in Section 5, that is, assuming that $H: A = Y^* \to Z'$ satisfies an estimate $\|y^* \circ F - H(y^*)\| \leqslant K \|y^*\|$ and $L: Y^* \to Z'$ is linear. By Remark 1, such a decomposition is possible for every version of F^* (although the constant might vary). Also, write $(G_{|Z})^* = l - h$ assuming that $h: Y^* \to Z'$ satisfies an estimate $\|y^* \circ (G_{|Z}) - h(y^*)\| \leqslant M \|y^*\|$ and $l: Y^* \to Z'$ is linear (same observation about the constant).

If $\Lambda: Y^* \to Z'$ is the linear map defined by

$$\Lambda(y^*)(z) = Lz(y^*) + ly^*(z),$$

then $\operatorname{dist}(F - (G_{|Z})^*, \Lambda) < \infty$ as can be easily checked:

$$\begin{aligned}
&|Fy^*(z) - (hy^*(z) - ly^*(z)) - (Lz(y^*) + ly^*(z))|\\
&\quad = |zF(y^*) - hy^*(z) - Lz(y^*)|\\
&\quad \leqslant |zF(y^*) - Hz(y^*)| + |Hz(y^*) - Lz(y^*) - hy^*(z)|\\
&\quad \leqslant K \|y^*\| \|z\| + |Gz(y^*) - zh(y^*)|\\
&\quad = K \|y^*\| \|z\| + |(y^* \circ (G_{|Z}) - h(y^*))(z)|\\
&\quad \leqslant (K + M) \|y^*\| \|z\|.
\end{aligned}$$

It also follows that Λ actually takes values in Z^* instead of in Z'. ∎

Next we show two interesting cases of dual sequences.

PROPOSITION 3. *If Y is a Banach space such that $Y^{**} = Y \oplus I$ with I injective (for instance, Y reflexive or quasi-reflexive) then every exact sequence $0 \to Z^* \to X \to Y^* \to 0$ is a dual sequence (and, in fact, the transpose of a sequence $0 \to Y \to {}_*X \to Z \to 0$).*

Proof. Let F: $Y^* \to Z^*$ be a 0-linear map defining the sequence. Since $Y^{**} = Y \oplus I$, then $F^* = \pi_Y \circ F^* + \pi_I \circ F^*$. Since I is an injective Banach space, $\pi_I \circ F^*$ can be decomposed as the sum of a bounded and a linear map. This means that $\pi_Y \circ F^*$ is a version of F^* with its range contained in Y. By Theorem 4, the sequence $0 \to Z^* \to X \to Y^* \to 0$ is a dual sequence. ▮

This result can be considered as inspired by [5, Proposition 3], in which the authors consider the situation when Y is reflexive.

A dual, in some sense, version of this result is

PROPOSITION 4. *The exact sequence $0 \to Y \to X \to Z \to 0$ is a dual sequence if and only if there is a predual ${}_*X$ of X such that Y is weak*-closed (with respect to ${}_*X$) in X^*.*

Proof. If Y is weak*-closed then, by the bipolar theorem (see [14]), $Y = ({}_*X/Y_\perp)^*$ and the sequence $0 \to Y \to X \to Z \to 0$ is the transpose of $0 \leftarrow X/Y_\perp \leftarrow {}_*X \leftarrow Y_\perp \leftarrow 0$. The other implication is obvious. ▮

The next two lemmata have obvious proofs (just by drawing the diagrams).

LEMMA 3. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence with Y complemented in its bidual such that the bidual sequence $0 \to Y^{**} \to X^{**} \to Z^{**} \to 0$ splits. Then the starting sequence splits.*

LEMMA 4. *Given Banach spaces Y, Z, and W, let i_Y: $Y \to W \oplus Y$ be the canonical injection and π_Z: $W \times Z \to Z$ the canonical projection. The exact sequence defined by F: $Z \to Y$ splits if and only if the sequence defined by $i \circ F$: $Z \to Y \oplus W$ splits; and if and only if the sequence defined by $F \circ \pi_Z$: $W \times Z \to Y$ splits.*

This result admits a more interesting nonlinear reformulation that eases the way to Theorem 5.

LEMMA 5. *Let $0 \to Y \to X \to Z \to 0$ be a nontrivial exact sequence defined by a quasi-linear map F: $Z \to Y$. If W is a Banach space, i_Y: $Y \to W \oplus Y$ is the canonical embedding and $G = i_Y \circ F$ then no version of G transforms Z into W.*

Proof. The quasi-linear map that defines $0 \to W \oplus Y \to W \oplus X \to Z \to 0$ is $G = 0 \oplus F$. It is clear that G can be written as $G = \pi_W \circ G + \pi_Y \circ G$. If

B and L are a bounded and a linear map respectively then also $B=\pi_W\circ B+\pi_Y\circ B$ and $L=\pi_W\circ L+\pi_Y\circ L$. Assume that some version $G+B+L$ of G has its range in W. Then $\pi_Y\circ G+\pi_Y\circ B+\pi_Y\circ L=0$ and $\pi_Y\circ G=F$ is trivial. ∎

7. A NONDUAL TWISTED SUM OF ℓ_2 AND A SEPARABLE DUAL

With all this machinery, we have the sought-after example.

THEOREM 5. *There exists a non-dual exact sequence* $0\to\mathscr{H}\to X\to W^*\to 0$ *in which* $\mathscr{H}$ *is a separable Hilbert space and* W^* *is a separable dual space.*

Proof. In [12, Corollary 4.5], is shown the existence of a nontrivial exact sequence

$$0\longrightarrow\mathscr{H}\xrightarrow{j}X\xrightarrow{q}c_0\longrightarrow 0.$$

By the preceding lemma the bidual sequence

$$0\longrightarrow\mathscr{H}\xrightarrow{j^{**}}X^{**}\xrightarrow{q^{**}}\ell_\infty\longrightarrow 0$$

does not split and thus the dual (of the starting) sequence

$$0\to\ell_1\to X^*\to\mathscr{H}^*\to 0$$

does not split. Assume that this sequence is defined by the 0-linear map F. Let W be a separable Banach space such that $W^{**}=W\oplus\ell_1$. Concrete examples of spaces W with the required properties appear in [8] or [16]. By Lemma 6.3 the sequence (defined by $i_{\ell_1}\circ F$)

$$0\to W\oplus\ell_1\to W\oplus X^*\to\mathscr{H}^*\to 0$$

does not split. By Proposition 3, this sequence is the transpose of some sequence

$$0\to\mathscr{H}\to{}_*(W\oplus X^*)\to W^*\to 0$$

(defined by some 0-linear map G). This sequence cannot be the transpose of a sequence

$$0\to W\to{}_{**}(W\oplus X^*)\to\mathscr{H}^*\to 0$$

since, otherwise, Theorem 4 would imply that some version of G^* should fall into W, something that Lemma 5 prevents.

To complete the proof, two things have to be fixed: one, that W^* has W as its unique isomorphic predual: this was proved by Brown and Ito [2] for separable spaces satisfying $W^{**} = W \oplus \ell_1$. Two, that once one knows that W is the unique predual of W^*, there still remains the question of the existence of other preduals of W^* (all of them copies of W inside W^{**} in different positions) able to contain the image of G^*. Fortunately, Brown and Ito prove more than announced. Recall that $W^{**} = W \oplus A$ where A is a subspace of W^{**} isomorphic to ℓ_1: Brown and Ito show that if W_1 is another predual of W^*, then there is a decomposition $W^{**} = W_1 \oplus C$ with C a subspace of W^{**} isomorphic to ℓ_1 and, in fact, one can choose C in such a way that $C \cap A$ is a finite codimensional part of A. Therefore since the image of G^* lies in A then the image of a version of G^* cannot fall into W_1.

THEOREM 6. *There is a twisted sum of ℓ_2 and a separable dual that is not isomorphic to a dual space.*

Proof. The middle space in the sequence $0 \to \mathscr{H} \to {}_*(W \oplus X^*) \to W^* \to 0$ is not a dual space: otherwise, $\mathscr{H}$ would be weak*-closed and Proposition 4 would imply that $0 \to \mathscr{H} \to {}_*(W \oplus X^*) \to W^* \to 0$ is a dual sequence. Needless to say, $\mathscr{H} = \ell_2$. ∎

Remark 4. The final subtleties of the proof of Theorem 5 cannot be avoided. Consider [7, 11] a nontrivial sequence $0 \to \ell_2 \to Z_2 \to \ell_2 \to 0$. If J denotes James quasi-reflexive space satisfying $J^{**} = J \oplus \mathbb{K}$, then $\ell_2(J)^{**} = \ell_2(J) \oplus \ell_2$. The sequence

$$0 \to \ell_2(J)^{**} = \ell_2(J) \oplus \ell_2 \to \ell_2(J) \oplus\ Z_2 \to \ell_2 \to 0$$

does not split and is the transpose of some sequence

$$\ell_2 \to {}_*(\ell_2(J) \oplus Z_2) \to \ell_2(J)^* \to 0.$$

Let F be a 0-linear map defining this sequence. When one chooses the obvious copy of $\ell_2(J)$ into $\ell_2(J)^{**}$ as predual for $\ell_2(J)^*$ then certainly no version of F^* sends ℓ_2 into that copy of $\ell_2(J)$ (roughly the same reasoning as in the proof of Theorem 5). However, it is possible to choose as predual of $\ell_2(J)^*$ a different copy of $\ell_2(J)$ inside $\ell_2(J)^{**}$ containing the image of ℓ_2 by F^*.

Remark 5. The construction of a sequence $0 \to R \to X \to Z^{**} \to 0$ that is not a bidual sequence can be performed as that of Theorem 5 (this was observed by Yost): start with a separable Banach space V such that

$V^{**}/V = c_0$ and set $W = V^*$. The sequence $0 \to \ell_2 \to D_V \to V^{**} \to 0$ is not a dual sequence (and the space D_V is not isomorphic to any dual space).

Remark 6. The sequence $0 \to \ell_2 \to D \to W^* \to 0$ shows:

THEOREM 7. *To have a boundedly complete basis is not a three-space property.*

REFERENCES

1. S. Banach, Sur les fonctionnelles linéaires, II, *Studia Math.* **1** (1929), 223–239.
2. L. Brown and T. Ito, Some non-quasi-reflexive spaces having unique isomorphic preduals, *Israel J. Math.* **20** (1975), 321–325.
3. J. M. F. Castillo, Snarked sums of Banach spaces, *in* "Proceedings of the II Congress in Banach spaces, Badajoz 1996," *Extracta Math.* **12** (1997) 117–128.
4. J. M. F. Castillo and M. González, "Three-Space Problems in Banach Space Theory," Lecture Notes in Mathematics, Vol. 1667, Springer-Verlag, Berlin, 1997.
5. J. C. Díaz, S. Dierolf, P. Domański, and C. Fernández, On the three space problem for dual Fréchet spaces, *Bull. Polish Acad. Sci. Math.* **40** (1992), 221–224.
6. J. Dixmier, Sur un théorème de Banach, *Duke Math. J.* **15** (1948), 1057–1071.
7. P. Enflo, J. Lindenstrauss, and G. Pisier, On the three space problem, *Math. Scand.* **36** (1975), 199–210.
8. R. C. James, Separable conjugate spaces, *Pacific J. Math.* **10** (1960), 563–571.
9. P. J. Hilton and U. Stammbach, "A Course in Homological Algebra," Graduate Texts in Mathematics, Vol. 4, Springer-Verlag, New York, 1970.
10. N. J. Kalton, The three-space problem for locally bounded F-spaces, *Compositio Math.* **37** (1978), 243–276.
11. N. J. Kalton and N. T. Peck, Twisted sums of sequence spaces and the three space problem, *Trans. Amer. Math. Soc.* **255** (1979), 1–30.
12. N. J. Kalton and A. Pelczyński, Kernels of surjections from $\mathscr{L}_1$-spaces with an application to Sidon sets, *Math. Ann.* **309** (1997), 135–158.
13. N. J. Kalton and J. W. Roberts, Uniformly exhaustive submeasures and nearly additive set functions, *Trans. Amer. Math. Soc.* **278** (1983), 803–816.
14. G. Köthe, "Topological Vector Spaces, I," Springer-Verlag, Berlin/New York, 1969.
15. Å. Lima and D. Yost, Absolutely Chebyshev subspaces, *in* "Workshop/Miniconference Funct. Analysis/Optimization, Canberra 1988" (S. Fitzpatrick and J. Giles, Eds.), *Proc. Centre Math. Appl. Austral. Nat. Univ.* **20** (1988), 116–127.
16. J. Lindenstrauss, On James' paper "Separable Conjugate Spaces," *Israel J. Math.* **9** (1971), 279–284.
17. J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces. I. Sequence Spaces," Springer-Verlag, Berlin/New York, 1977.
18. M. Ribe, Examples for the nonlocally convex three space problem, *Proc. Amer. Math. Soc.* **237** (1979), 351–355.
19. D. Vogt, Lectures on projective spectra of (DF) spaces, *in* "Lectures Held in the Functional Analysis Seminar, Dusseldorf/Wuppertal, Jan–Feb 1987," unpublished manuscript.