ON A QUESTION OF PEŁCZYŃSKI ABOUT MULTILINEAR OPERATORS

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Abstract. We show that a separable Banach space X has the Schur property if and only if every separately compact bilinear application from X into c_0 is completely continuous, thus answering a question raised by Pełczyński.

1. Completely continuous multilinear maps

Let $T : X_1 \times \cdots \times X_k \to Y$ be a (continuous) multilinear operator acting between Banach spaces. As in the linear case, T is said to be weakly compact if it maps bounded sets into (relatively) weakly compact sets. Also, we say that T is completely continuous if it maps weakly Cauchy sequences of $X_1 \times \cdots \times X_k$ into norm convergent sequences.

In [10] (see also [9]) Pełczyński proved the following

Theorem 1. *Suppose X_i are (possibly different) $L_1(\mu)$ spaces for $1 \leq i \leq k$. Then every weakly compact multilinear operator from $X_1 \times \cdots \times X_k$ to an arbitrary Banach space is completely continuous.*

Later on, Ryan [11] showed that the same is true whenever all the X_i's above have the Dunford-Pettis property.

In the same article (see p. 385, Remark 2), Pełczyński suggests that "Theorem 1 above may be generalized to the case in which T is separately weakly compact" (that is, for every fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$, the linear operator

$$x \in X_i \longmapsto T(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k) \in Y$$

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is weakly compact). The purpose of this note is to prove that such a generalization is impossible. In fact, the following result shows that the conclusion of Theorem 1 may fail even if T is assumed to be separately compact.

Fact 1. *For a separable Banach space X the following are equivalent:*

(a) *X has the Schur property (weakly convergent sequences converge in norm).*

(b) *For any k and every Banach space Y, every k-linear operator from X^k into Y is completely continuous.*

(c) *Every symmetric bilinear application $S : X \times X \to c_0$ which is separately compact is completely continuous.*

Proof. We only need to prove that (c) implies (a): If X is not a Schur space, there is a weakly null sequence in the unit sphere of X. Using the Bessaga-Pełczyński selection principle [3] we can pick a basic subsequence of it, which we will call (x_n). Let Z be the closed subspace of X generated by (x_n), and let us define an operator $T : Z \to c_0$ by $T(x_n) = e_n$, where (e_n) is the traditional basis of c_0. We can now use Sobczyk's theorem [12] to extend T to the whole of X. Let us also call T to this extension. Now, consider the symmetric bilinear operator $S : X \times X \to c_0$ given by $S(x, y) = T(x) \cdot T(y)$, the product being that of c_0.

Fixing $y \in X$, we see that the operator $S(\cdot, y) : X \to c_0$ given by $x \mapsto S(x, y)$ is compact, since it can be decomposed as $S(\cdot, y) = D_{T(x)} \circ T$, where $D_{T(x)} : c_0 \to c_0$ denotes the (obviously compact) diagonal operator given by $z \mapsto T(x) \cdot z$. Proceeding analogously with the other variable, we infer that S is separately compact. On the other hand, S is not completely continuous, since it maps the sequence (x_n, x_n), which is weakly null in $X \times X$, into the basis of c_0, which is divergent. □

Let us give an explicit counterexample for $X = L_1$.

Example 1. *The bilinear operator $S : L_1(\mathbb{T}) \times L_1(\mathbb{T}) \to c_0(\mathbb{Z})$ given by $S(f, g) = \hat{f} \cdot \hat{g}$ is separately compact but not completely continuous.*

Proof. Here $\hat{f}(n)$ denotes the n-th Fourier coefficient of f. To see that S is not completely continuous, note that if k_n is a lacunary sequence of integers, for instance if $k_n = 3^n$, then the sequence given by $f_n(z) = z^{k_n}$ is equivalent to the unit basis of ℓ_2 (see [8, Theorem 3.4, p. 39] for a simple proof). Thus (f_n, f_n) is a weakly null sequence in $L_1(\mathbb{T}) \times L_1(\mathbb{T})$, while $S(f_n, f_n) = e_{k_n}$ cannot converge in norm. □

2. The range of Aron-Berner extensions

In a sense, Fact 1 is a characterization of separable Schur spaces via c_0-valued bilinear forms. We close with a similar characterization of the Grothendieck property (every linear operator to a separable Banach space, equivalently to c_0, is weakly compact) and reflexivity for separable Banach spaces.

Let $T : X_1 \times \cdots \times X_k \to Y$ be a multilinear operator. The Aron-Berner [1] extension $\alpha\beta(T)$ of T is the multilinear operator $X_1^{**} \times \cdots \times X_k^{**} \to Y^{**}$ given by

$$\alpha\beta(T)(x_1^{**}, \dots, x_k^{**}) = w^* - \lim_{x_1 \to x_1^{**}} \cdots \lim_{x_k \to x_k^{**}} T(x_1, \dots, x_k),$$

where the iterated limit is taken for $x_i \in X_i$ converging to x_i^{**} in the *weak topology of X_i^{**}. (We could have chosen any other order of the variables and we would have got another extension, in general different from the one given above).

In general, the Aron-Berner extensions are Y^{**}-valued. Sometimes, however, it may happen that they take values in Y. (The question of whether or not a given extension is Y-valued arise naturally when discussing "polynomial" properties of Banach spaces; see [6]).

Fact 2. *For a Banach space X the following are equivalent:*

(a) *X has the Grothendieck property.*
(b) *For any separable Banach space Y, every k-linear operator $X^k \to Y$ has Y-valued Aron-Berner extension.*
(c) *Every symmetric bilinear application $S : X \times X \to c_0$ which is separately compact has c_0-valued Aron-Berner extension.*

Moreover, if X is separable, these statements are equivalent to

(d) *X is reflexive.*

Proof. The "moreover" part is clear, since a separable space has the Grothendieck property if and only if is reflexive.

We now prove that (a) implies (b). Let $T : X^k \to Y$ be a multilinear map, where Y is separable and X has the Grothendieck property.

For $1 \leq i \leq k-1$, let us fix $x_i \in X$ and $x_k^{**} \in X^{**}$ and consider the first limit appearing in the Aron-Berner extension of T

$$\hat{T}(x_1, \ldots, x_{k-1}, x_k^{**}) = w^* - \lim_{x_k \to x_k^{**}} T(x_1, \ldots, x_{k-1}, x_k).$$

Then $\hat{T}(x_1, \ldots, x_{k-1}, x_k^{**})$ belongs to Y (instead of Y^{**}) since it is the value at x_k^{**} of the bitranspose of the weakly compact operator $T(x_1, \ldots, x_{k-1}, \cdot) : X \to Y$, by an old result of Gantmacher [4]. Now, for $1 \leq i \leq k-2$ let x_i be fixed in X and take $x_{k-1}^{**}, x_k^{**} \in X^{**}$. Then we have

$$\alpha\beta(T)(x_1, \ldots, x_{k-2}, x_{k-1}^{**}, x_k^{**}) = w^* - \lim_{x_{k-1} \to x_{k-1}^{**}} \hat{T}(x_1, \ldots, x_{k-2}, x_{k-1}, x_k^{**}),$$

which also belongs to Y since it is the value at x_{k-1}^{**} of the bitranspose of $\hat{T}(x_1, \ldots, x_{k-2}, \cdot, x_k^{**}) : X \to Y$ which is weakly compact by hypothesis. Continuing these reasonings we obtain that $\alpha\beta(T)$ is Y-valued.

It is trivial that (b) implies (c) . It remains to show that (c) implies (a).

As before, if $T : X \to c_0$ is a linear operator, we can consider the symmetric bilinear form $S : X \times X \to c_0$ given by $S(x, y) = T(x) \cdot T(y)$. It is easily seen that

$$\alpha\beta(B)(x^{**}, y^{**}) = T^{**}(x^{**}) \cdot T^{**}(y^{**}),$$

now the product being that of ℓ_∞. If X lacks the Grothendieck property, there is a linear operator $T : X \to c_0$ that is not weakly compact, which implies that T^{**} cannot fall into c_0 ([4]). Thus there exists $x^{**} \in X^{**}$ so that $T^{**}(x^{**}) \notin c_0$ and therefore $\alpha\beta(S)(x^{**}, x^{**}) \notin c_0$. This completes the proof. □

The equivalence between (a) and (b) above was already known for polynomials ([5]).

Let us discuss two extreme examples of the "growth" of the range of Aron-Berner extensions.

Example 2. *Suppose $S : \ell_1 \times \ell_1 \to c_0$ is obtained from some quotient mapping $T : \ell_1 \to c_0$ [2]. Then $T^{**} : (\ell_1)^{**} \to \ell_\infty$ is surjective as well, and therefore the range of $\alpha\beta(S)$ is the whole of ℓ_∞.*

Example 3. *To obtain an example of "minimal" growth, let J be James' quasireflexive space [7], set $T : J \to c_0$ to be the obvious inclusion and let $S : J \times J \to c_0$ be as before. It is easily seen that* *the range of $\alpha\beta(S)$ is contained in the subspace c of all convergent sequences of ℓ_∞.*

Finally, let us remark that, despite Fact 1, we do have the following result (note that $L_1(\mu)$ spaces have the Dunford-Pettis property):

Fact 3. *Suppose X_i are spaces with the Dunford-Pettis property. If T is a multilinear operator such that its Aron-Berner extension is separately weakly compact, then T is completely continuous.*

Proof. Suppose $T : X_1 \times \cdots \times X_k \to Y$ is a multilinear map whose Aron-Berner extension is separately weakly compact. Then, reasoning as in the proof that (a) implies (b) in Fact 2, one obtains that $\alpha\beta(T)$ takes values in Y. Now [6, Theorem 3.7] states that if all the X_i's have the Dunford-Pettis property and $\alpha\beta(T)$ is Y-valued then T is completely continuous. □

Thus, in view of Facts 2 and 3, every multilinear operator from a Banach space having the Dunford-Pettis and Grothendieck properties (say l_∞) into c_0 (or any separable space) is completely continuous. This shows that Fact 1 may fail is one allows nonseparable spaces.

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