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Sobczyk's Theorems from A to B

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1. Sobczyk's theorem and how to prove it

Sobczyk's theorem is usually stated as: *Every copy of c_0 inside a separable Banach space is complemented by a projection with norm at most* 2. Nevertheless, our understanding is not complete until we also recall: *and c_0 is not complemented in ℓ_∞.* Now the limits of the phenomenon are set: although c_0 is complemented in separable superspaces, it is not necessarily complemented in a nonseparable superspace, such as ℓ_∞.

The history of complemented and uncomplemented subspaces of Banach spaces is traced back in another article of this volume [48]. It is probably worth mentioning that it starts with two propositions: *Every closed subspace of a Hilbert space is complemented by a norm one projection* and *ℓ_1 contains uncomplemented subspaces.* The first result easily follows by proving that the metric projection onto a closed subspace acts linearly; the second result holds since the kernels of quotient maps $\ell_1 \to X$ are necessarily uncomplemented when X has not been previously chosen a subspace of ℓ_1 (and recalling that all separable Banach spaces are quotients of ℓ_1).

A more interesting question for us is: why should one suspect that c_0 is complemented inside separable superspaces? A previous result in this direction had been proved by Phillips [47]: *Every copy of ℓ_∞ inside a Banach space is complemented by a norm one projection.* In other words, the spaces $\ell_\infty(\Gamma)$ are *injective.* Since it was well known (and can be easily proved) that every Banach space is isometric to a subspace of some $\ell_\infty(\Gamma)$ it is clear that the injective spaces are precisely the $\ell_\infty(\Gamma)$ spaces and their complemented subspaces. In order to determine all the injective spaces the story starts with

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Lindenstrauss's [38] proof that a complemented subspace of ℓ_∞ is again isomorphic to ℓ_∞. Hence the question arises whether a complemented subspace of $\ell_\infty(\Gamma)$ has to be isomorphic to some $\ell_\infty(I)$. It took much work by Rosenthal [50], beyond the scope of this article, to show that there exist injective spaces that are not isomorphic to any $\ell_\infty(\Gamma)$.

A Banach space injective among separable spaces is called *separably injective*. Well might one guess that the separable version of ℓ_∞, namely c_0, could be as injective among separable spaces justas ℓ_∞ is among all spaces. Sobczyk's theorem substantiates this: c_0 is separably injective. In this case, moreover, the story has a happy ending since Zippin [63] was able to prove that a separaby injective space is isomorphic to c_0. Again, Zippin's theorem is out of reach for us.

Another point to be careful about is the difference between working with isometric copies of c_0 and with isomorphic copies of c_0. It is an easy exercise to show that if a Banach space X contains a K-isomorphic copy Y_0 of some Banach space Y then X can be renormed so that one obtains a K-isomorphic copy of X containing an isometric copy of Y. Pełczyński [45, Proposition 1] was probably the first to prove this. Taking $Y = c_0$, it follows that if Sobczyk's theorem holds for a Banach space X, i.e. if every isomorphic copy of c_0 inside X is 2-complemented, then every K-isomorphic copy of c_0 in X is $2K$-complemented. Thus, from now on a *copy* of c_0 means an isometric copy; otherwise we will say explicitly isomorphic or K-isomorphic copy.

How to prove Sobczyk's theorems? There are several apparently different ways to tackle the proof that c_0 is complemented in a separable X or is uncomplemented in l_∞. We shall assign to each "method" one of the suits of playing cards; so, whenever we present a proof the closing suit means which type of approach was (mainly) used.

(♠) The first method is to appeal to the plain definition: if $j : c_0 \to X$ is an isomorphic embedding, one needs to obtain an operator $P : X \to c_0$ such that $Pj = id$ (or show that such operator cannot exist). The full force of Sobczyk's theorem is that if j is an isometric embedding and X is separable then P can be chosen with $\|P\| \leq 2$.

(♣) The second approach introduces duality. Operators $X \to c_0$ are no different from weak* null sequences of X^*; in particular, the identity operator on c_0 is the sequence of coordinate functionals (δ_n). Thus, what one needs is a weak* null sequence of X^* formed by extensions (D_n) of the (δ_n); alternatively, to show that such extensions do not exist. Again, the full force of Sobczyk's theorem is to obtain extensions with $\sup_n \|D_n\| \leq 2$.

Playing harder on duality, what one needs is a weak*-continuous section s^* for the transpose j^* of the embedding j. It is elementary that $j^* : X^* \to \ell_1$ admits a norm-continuous section since ℓ_1 is projective; but what one is looking for is the transpose of a projection $P : X \to c_0$; i.e. $s = P^*$, or $s^* = P$, a section whose transpose yields an operator $X \to c_0$. The weak* null sequence described at (♣) is precisely $(s\delta_n)_n$ (since $(\delta_n)_n$ can be identified with the canonical basis of ℓ_1). Again, the result is optimal when one obtains $\|s\| \leq 2$.

Zippin [64, 65] introduced a more topological-oriented view: one needs to show that there exists some constant $C > 0$ and a continuous (in the weak*-topology) map $\phi : \alpha\mathbb{N} \to CBall(X^*)$ such that $\phi(n)(e_m) = \delta_{nm}$ (here, $\alpha\mathbb{N}$ is the one point compactification of $\mathbb{N}$).

(♡) This method means taking as a whole the subspace c_0 and the quotient space X/c_0. A precise formulation requires some machinery from the theory of exact sequences of Banach spaces, which is too much of a pleasure for (some of) us to present. An exact sequence of Banach spaces is a diagram $0 \to Y \to X \to Z \to 0$ in which the points are Banach spaces and the arrows are operators, with the property that the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem guarantees that Y is a subspace of X such that the corresponding quotient X/Y is Z. An exact sequence is said to split if the arrow $j : Y \to X$ admits a left-inverse, i.e. some arrow $p : X \to Y$ exists such that $pj = id_Y$. The space X is also called a *twisted sum* of Y and Z. So, the approach is to try to decide when an exact sequence

$$0 \to c_0 \to X \to Z \to 0$$

splits.

(♢) This approach is only good for showing that c_0 is not complemented in X. The idea is to detect properties of $Z = X/c_0$ which prevent it from being a subspace of X. It willll mainly be used in 3.7.

2. c_0 IS COMPLEMENTED IN ANY SEPARABLE SUPERSPACE

2.1. SOBCZYK'S PROOF, 1944. Probably not many people have struggled through Sobczyk's original proof in [54]. There are good reasons for that, such as the existence of Veech's proof, but also the fact that Sobczyk's paper is written in an old-fashioned style. The reader may prefer to postpone the reading of this section until after the "Understanding Sobczyk" section 2.7.

Sobczyk's theorem is never stated as such in [54], it is just a comment at page 946, lines 21-23; while its proof occupies theorems 1, 2 and 5, and the comments on pages 942 and 945. We can, however, deconstruct Sobczyk's arguments.

Let c_0 be the natural copy inside ℓ_∞. Assume that we have proved that if W is a separable subspace of ℓ_∞ containing c_0 then there exists a norm 2 projection of W onto c_0. Then the way is paved to prove that every isometric copy Y_0 of c_0 inside every separable subspace W of ℓ_∞ is 2-complemented: for if $T : Y_0 \to c_0$ is the isometry, it can be extended to a norm 1 operator $T_1 : W \to \ell_\infty$. Putting $W' = T_1(W)$ the existence of a norm 2 projection $P : W' \to c_0$ guarantees that $T^{-1}PT_1 : W \to Y_0$ is a norm 2 projection of W onto Y_0.

Thus, let W be a separable subspace of ℓ_∞ containing c_0. The separability assumption is used to express W as the closure of $c_0 + [u_j]$, where $\{x_j\}$ is a finite or countable quantity of elements of ℓ_∞. Sobczyk's good idea and hard work are then to assume that the projection one is searching for has the form $P(x_0 + \sum t_j x_j) = x_0$ on $c_0 + [u_j]$ (and then extend it to the closure) with a proper, clever, but also awkward, choice of the points x_j so that $c_0 + [x_j] = c_0 + [u_j]$.

Which points x_j work? Well, the case easiest to handle –and actually the core of Sobczyk's proof– is that of points such that $x_j(n) = \pm 1$. We state that as a separate lemma:

LEMMA 2.1. *Let $\{x_j\}$ be a sequence of points of ℓ_∞ such that*

(1) $[x_j] \cap c_0 = 0$.

(2) $x_j(n) = \pm 1$ *for all* $n \in \mathbb{N}$.

(3) *For every index n there exists an infinite set $A_n \subset \mathbb{N}$ such that for all $i \in A_n$ one has $x_j(n) = x_j(i)$ for all j.*

Then $P(x_0 + \sum_{j=1}^k t_j x_j) = x_0$ defines a projection $P : c_0 + [x_n] \to c_0$ with $\|P\| \leq 2$.

Proof. Fix scalars $t_1, \dots, t_k$ and $x_0 \in c_0$. Everything is based on the observation that the hypotheses yield that there exists $n_0 \in \mathbb{N}$ such that for all $i \in A_{n_0}$ one has

$$\sup_n \left| \sum_{j=1}^k t_j x_j(n) \right| = \left| \sum_{j=1}^k t_j x_j(n_0) \right| = \left| \sum_{j=1}^k t_j x_j(i) \right|.$$

Since A_{n_0} is infinite and $x_0 \in c_0$ then, for $i \in A_{n_0}$

$$\Big\|\sum_{j=1}^{k} t_j x_j\Big\| \leq \Big|\sum_{j=1}^{k} t_j x_j(i) + x_0(i)\Big| + |x_0(i)|$$

and thus

$$\Big\|\sum_{j=1}^{k} t_j x_j\Big\| \leq \Big\|\sum_{j=1}^{k} t_j x_j + x_0\Big\|.$$

This means that c_0 is the kernel of a norm one projection, namely: $x_0 + \sum t_j x_j \to \sum t_j x_j$. Does Sobczyk's argument for the general situation still give us this conclusion? Yes; see also § 2.8. Back to the proof,

$$\|Px\| = \|x_0\| \leq \Big\|x_0 + \sum_{j=1}^{k} t_j x_j\Big\| + \Big\|\sum_{j=1}^{k} t_j x_j(n)\Big\| \leq 2\|x\|.$$

∎

What remains of the proof is to show how the conditions on the x_j can be relaxed and reduced to just (1) without affecting the norm of the projection. For instance, assume that (3) does not hold. Consider the set $x_1, \cdots, x_n$. Since there is only a finite quantity of elements of $\{1,-1\}^n$ that are not infinitely repeated in the sequences $(x_1(k), \cdots, x_n(k))_k$, replacing (actually, adding some ± 1) a finite quantity of coordinates of of x_1, then of x_2, and so on until x_n one obtains new elements $x'_1, \cdots, x'_n$ verifying (3). The rest of the conditions remains unaltered since $x_j - x'_j \in c_0$; which, in particular, gives $c_0 + [x_j] = c_0 + [x'_j]$.

The situation when it is (2) which fails is tough. Nevertheless, the door opens when one observes that if all the elements x_j take only a finite number of values then one can reproduce the preceding argument without great difficulties.

So, let x_j be elements that only verify condition (1). Assume, as can be done without loss of generality, that $\|x_j\| \leq 1$. Let $N \in \mathbb{N}$ and divide the interval $[0,1]$ into N subintervals of equal length. Let $\pi(x_j(n))$ denote the number of the interval in which $|x_j(n)|$ lies. Let

$$s_j(n) = \frac{\pi(x_j(n))}{N}\operatorname{sign}(x_j(n)).$$

The elements $S_j = (s_j(n))_k$ take only the values $\{\pm k/N,\ 1 \leq k \leq N\}$. Therefore, a finite number of alterations (consistent with adding some $\pm k/N$)

to each S_j produces a new sequence S_j^N satisfying (3) (and so that $S_j - S_j^N \in c_0$). Making the same alteration to the x_j one obtains new elements x_j^N so that $x_j - x_j^N \in c_0$.

We shall study what will happen when $N \to \infty$. It is easier to compare x_j^N and x_j^M when $M = 2^l N$. So, we shall assume from now on that that $N = 2^l$ for $l = 1, 2, \cdots$ In that case, it is not difficult to realize two things

- $x_j^N - x_j^M \in c_0$.
- $\lim_{N,M\to\infty} \|x_j^N - x_j^M\| = 0$.

The first line implies that if $x = x_{0,N} + \sum_{j=1}^k t_j x_j^N$ and also $x = x_{0,M} + \sum_{j=1}^k r_j x_j^M$ then it is possible to choose $r_j = t_j$. The second line says that, given the above, $(x_{0,N})_N$ is a Cauchy sequence in c_0. If we set $P^N x = x_{0,N}$ then he projection we are looking for is

$$Px = \lim_{N\to\infty} P^N(x) = \lim_{N\to\infty} x_{0,N};$$

as we prove next. Since

$$\begin{aligned}
\|P^N x\| &= \Big\| x - \sum_{j=1}^k t_j x_j^N \Big\| \\
&\le \|x\| + \Big\| \sum_{j=1}^k t_j S_j^N \Big\| + \Big\| \sum_{j=1}^k t_j \big(S_j^N - x_j^N\big) \Big\| \\
&\le \|x\| + \Big\| x_{0,N} - \sum_{j=1}^k t_j S_j^N \Big\| + \Big\| \sum_{j=1}^k t_j \big(S_j^N - x_j^N\big) \Big\| \\
&\le \|x\| + \Big\| x_{0,N} + \sum_{j=1}^k t_j x_j^N \Big\| + 2\Big\| \sum_{j=1}^k t_j \big(S_j^N - x_j^N\big) \Big\| \\
&\le 2\|x\| + 2\Big\| \sum_{j=1}^k t_j \big(S_j^N - x_j^N\big) \Big\|,
\end{aligned}$$

taking limits as $N \to \infty$ we obtain $\|Px\| \le 2\|x\|$.

♠

2.2. Pełczyński's proof, 1960. This proof appeared first in [45, Theorem 4]. The idea this time is that since every separable Banach space is isometric to a closed subspace of $C[0,1]$, it is enough to prove that isometric copies of c_0 inside $C[0,1]$ are 2-complemented. To this end, let f_n be the images of the canonical basis e_n of c_0. Let $p_n \in [0,1]$ be such that $|f_n(p_n)| = \|f_n\| = 1$ and let Δ be the set of accumulation points of $\{p_n\}_{n\in N}$ in $[0,1]$. Since $\|f_n \pm f_m\| = 1$ it follows that $|f_n(p_m)| = \delta_{nm}$; and thus that for every $t \in \Delta$ one has $f_n(t) = 0$. Let us verify two assertions:

- *c_0 is 1-complemented in the subspace*

$$C = \{f \in C[0,1] : \forall\, t \in \Delta,\ f(t) = 0\}.$$

Indeed,

$$P(f) = \sum_{n=1}^{\infty} f(p_n)\mathrm{sign} f_n(p_n) f_n$$

is a well-defined (note that $\lim f(p_n) = 0$) norm-one projection.

- *C is 2-complemented in $C[0,1]$.*

Clearly $[0,1]\backslash\Delta$ is a countable union of open intervals. Thus by affine interpolation, each continuous function $g \in C(\Delta)$ can be extended to a continuous function in $C[0,1]$ with the same norm. This gives us a linear operator that we shall call E.

Although we don't need, we can't resist mentioning the following generalization of this argument, the Borsuk-Dugundji theorem ([5, 18]; or else [27]).

Theorem 2.2. *Let D be a closed subspace of a metric space M, and let F be a locally convex space. Each continuous map $f : D \to F$ has a continuous extension $E(f) : M \to F$ such that $E(f)(M) \subset \mathrm{conv} f(D)$.*

The map E is actually linear and thus it defines an extension operator $E : C(D,F) \to C(M,F)$ which is continuous in the compact-open topology. Hence, one has

Theorem 2.3. *Let K be a compact metric space and let $D \subset K$ a closed subset. There exists a linear extension operator $E : C(D) \to C(K)$; i.e., for each $f \in C(D)$ one has $E(f)|D = f$. Moreover, $\|E\| = 1$.*

Then

$$Q(f) = f - E(f|\Delta)$$

defines a linear projection $C[0,1] \to C$ with norm at most 2, which finishes the proof.

♠

2.3. MARTINEAU'S PROOF, 1964. We have not had access to Martineau's paper [39] and thus we had to reconstruct his arguments out from the comments in [52] and the Mathematical Reviews (MR 35#3418) and Zentralblatt reviews. Let c_0 be an isometric copy inside a separable Banach space X. It seems that the difference with Pełczyński's proof is that Martineau embeds X into ℓ_∞ and then considers the algebra it generates, actually a $C(K)$ space with K a metrizable compactification of N. The method of constructing a norm 2 projection $C(K) \to c_0$ proceeds on as before.

♠

2.4. KÖTHE'S PROOF, 1966. As early as 1954 Grothendieck stated in [25, Part 4, 3, Exercise 1] the following lifting result.

LEMMA 2.4. *Let E be a separable locally convex space, F a vector subspace and (f_n) an equicontinuous and weakly convergent sequence in F^*; show that we can find extensions e_n of the f_n to E such that (e_n) is an equicontinuous and weakly convergent sequence in E^*.*

This result is in fact a generalization of the analogous lifting result obtained by Köthe in [32] for a particular class of locally convex spaces, now called Köthe spaces.

This line of thought was reconsidered by Köthe in [33] to derive a proof of Sobczyk's theorem in its isomorphic form. There is also a fairly complete description of its contents in [35, § 33, 5]. The surprising and surprisingly simple lifting result is restated as follows.

LEMMA 2.5. (KÖTHE'S LIFTING) *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of Banach spaces in which X is separable. Let $0 \to Y^\perp \to X^* \to X^*/Y^\perp \to 0$ be its dual sequence. Then every weak*-null sequence in the ball of radius r in $X^*/Y^\perp$ admits a weak*-null lifting sequence in the ball of radius $2r$ in X^*. More precisely, if $(u_n^* + Y^\perp)$ is a weak*-null sequence in $X^*/Y^\perp$*

with $\|u_n^* + Y^\perp\| \leq r$ *then there exists a weak*-null sequence* (x_n^*) *in* X^* *such that, for all* n, $\|x_n^*\| \leq 2r$ *and* $x_n^*|Y = u_n^*|Y$.

Proof. There is no loss of generality in assuming that $\|u_n^*\| \leq r$. We observe that the set of accumulation points of the sequence $(u_n^*)_n$ in B_{X^*} is contained in $Y^\perp$: indeed, since B_{X^*} in the weak* topology is a metrizable compact, every subsequence of $(u_n^*)_n$ admits a weak*-convergent subsequence; thus, since $(u_n^* + Y^\perp)$ is weak*-null if $u = \text{weak}^* \lim u_{n(k)}^*$ then $u \in Y^\perp$. Since the norm is weak*-lower semicontinuous, $\|u\| \leq r$. This can be spelled out as:

- For each $\varepsilon > 0$ and each finite set $F \subset Y$ there exists some $N(\varepsilon)$ so way that whenever $n > N(\varepsilon)$ there exists some $w_n \in Y^\perp$ vrifying $\|w_n\| \leq r$ and $|(u_n^* - w_n)(f)| \leq \varepsilon$ for all $f \in F$.

We only have to be careful with the induction now. Since X is separable, let $\{y_n\}_n$ be a dense subset. Applying the preceding tool to $F_1 = \{y_1\}$ and $\varepsilon_1 = 2^{-1}$ we know that for $n \geq N(1)$ one has

$$|(u_n^* - w_n^1)(y_1)| \leq 2^{-1}.$$

And, in general, if $F_k = \{y_1, \cdots, y_k\}$ and $\varepsilon_k = 2^{-k}$ we know that for $n \geq N(k)$ and $1 \leq j \leq k$

$$|(u_n^* - w_n^k)(y_j)| \leq 2^{-k}.$$

The sequence $w_n = w_n^k$ for $N(k) < n \leq N(k+1)$ (completed with some elements for $n \leq N(1)$) is such that

$$\lim_{n\to\infty}(u_n^* - w_n)(y_j) = 0$$

for all j. Since $\{y_n\}_n$ is dense in X, the sequence $(u_n^* - w_n)_n$ is weak* null. Quite clearly $\|u_n^* - w_n\| \leq 2r$. ∎

From that Köthe obtains:

PROPOSITION 2.6. *Let* $A : c_0 \to X$ *be an isomorphism from* c_0 *into a separable Banach space* X*. Then there exists a linear and continuous left inverse* $B : X \to c_0$ *for* A *such that* $\|B\| \leq 2\|A^{-1}\|$.

Proof. Let $H = A(c_0)$. Since $A^* : X^*/H^\perp \to \ell_1$ is a weak* isomorphism, the elements $(A^*)^{-1}(e_n)$ have norm $\|(A^*)^{-1}(e_n)\| \leq \|A^{-1}\|$ and form a weak* null sequence. Let x_n^* be a lifting to X^* forming a weak* null sequence with norm at most $2\|A^{-1}\|$. Then we define $Bx = (x_n^*(x))_n$. It is clear that B is linear, continuous and with

$$\|Bx\| \leq \sup_n |(x_n^*(x)| \leq 2\|A^{-1}\|\|x\|.$$

Finally $BA = id_{c_0}$ since

$$BAe_k = (x_n^*(Ae_k))_n = (A^*(x_n^* + H^\perp(e_k))_n = (A^*(A^{*-1}e_n)(e_k))_n = e_k.$$

∎

It follows that there exists a projection onto $A(c_0)$, namely AB, with norm at most $2\|A\|\|A^{-1}\|$.

♣

2.5. Goldberg's simplification, 1969. Goldberg's short note [22] has the declared purpose of presenting a simpler proof of Köthe's result. Everything is stated as:

Theorem 2.7. *Let A be an into isomorphism from c_0 into a separable Banach space X. Then there exists a weak* closed subspace M of X^* such that*

$$X^* = M \oplus A(c_0)^\perp; \qquad X = {}^\perp M \oplus A(c_0).$$

Furthermore, the projection P from X onto $A(c_0)$ with kernel ${}^\perp M$ has norm at most $2\|A\|\|A^{-1}\|$ and $B = A^{-1}P$ is a left inverse of A with norm at most $2\|A^{-1}\|$.

2.6. Veech's proof, 1971. This proof [59] is one of the masterpieces in "the book" (which Erdös must be reading now). It follows the strategy ♣, and so it tries to find a weak*-null extension of the coordinate functionals (δ_n) in $2B_{X^*}$. Let D_n be a Hahn-Banach extension of δ_n. Since X is separable, its dual ball is weak*-metrizable by a translation invariant metric, say d. Let Δ be the set of accumulation points of $(D_n)_n$ in B_{X^*}. The following ridiculously simple observation is the key: a sequence such that every subsequence contains a further subsequence converging to zero is itself convergent to zero. It is then clear that

$$\lim_{n\to\infty} \text{dist}(D_n, \Delta) = 0.$$

Choosing $f_n \in \Delta$ such that $d(D_n, f_n) \leq \text{dist}(D_n, \Delta) + 1/n$ one has that the sequence $(D_n - f_n)_n \subset 2B_{X^*}$ is weak*-null. Moreover $D_n - f_n$ extends δ_n since $f_n(e_m)$ is an accumulation point of $(D_n(e_m))_n$, i.e. 0.

♣

2.7. Understanding Sobczyk. We can now translate what Sobczyk did. Assume that one is trying to show that a weak* null sequence of norm one functionals defined on a subspace Y of a separable Banach space X can be extended to a weak*-null sequence of functionals on X with norm (at most) two. The simplest situation that can be thought of is to perform such an extension from Y to a superspace $Y + [u_1, \ldots, u_n]$. Perhaps one would even daydream about being able to exactly determine the extensions. Let us show that such hope has a price: we can exactly determine the extensions at the cost of letting them have norm $2 + \varepsilon$!

Let $(F_n)_n$ be a Hahn-Banach extension of $(f_n)_n$ to X. Let Δ be the set of accumulation points of $\{(F_n(u_1), \ldots, F_n(u_k))\}_n$ in R^k. Choose for each n a point $(p_1^n, \ldots, p_k^n) \in \Delta$ such that $\|(F_n(u_1), \ldots, F_n(u_k) - (p_1^n, \ldots, p_k^n)\| = \operatorname{dist}((F_n(u_1), \ldots, F_n(u_k)), \Delta)$. We define the functionals h_n as a Hahn-Banach extension of the functional taking the value p_j^n on u_j and 0 on Y. It is clear that $F_n - h_n$ is pointwise convergent to zero on $Y + [u_1, \ldots, u_k]$. We only have to calculate the norm of $F_n - h_n$.

$\limsup \|h_n\| \leq 1$: taking w an accumulation point of $(F_m)_m$ in B_{X^*} (observe that $\|w\| \leq 1$ and $w|Y = 0$) and since $\lim \|(h_m - w)_{|[u_1,\ldots,u_k]}\| = 0$ one gets

$$\begin{aligned}\left\|h_{m|Y+[u_1,\ldots,u_k]}\right\| &\leq \left\|(h_m - w)_{|Y+[u_1,\ldots,u_k]}\right\| + \|w\| \\ &\leq \left\|(h_m - w)_{|[u_1,\ldots,u_k]}\right\| + 1 \\ &\leq 1.\end{aligned}$$

Therefore, $(F_n - h_n)_n$ is a pointwise null sequence of extensions of (F_n) with norm $\|F_n - h_n\| \leq 2 + \varepsilon$ for large n. Repeating the process increasing the number of points (u_k) and with a diagonalization one gets a sequence $(g_n)_n$ of extensions of $(f_n)_n$ with norms $\|g_n\| \leq 2 + \varepsilon$.

When $Y = c_0$ what one has obtained is a projection $P_\varepsilon : X \to c_0$ with norm $\|P_\varepsilon\| \leq 2 + \varepsilon$.

(By the way, observe that there is no way of pasting together all those projections: attempts of diagonalization when $\varepsilon \to 0$, such as considering a free ultrafilter $\mathcal{U}$ refining the Fréchet filter and setting

$$P(x)(k) = \lim_{\mathcal{U}(\varepsilon)} P_\varepsilon(x)(k)$$

typically produce nothing different from $(F_n)_n$.) This erratic behaviour has its roots in the choice of the values of the extended functional at the points

u_j. Observe that even for a single point our choice is wrong. Consider the extension from Y to $Y + [u]$. At first glance our choice for $h_n(u)$ is logical and an examination of Veech's proof should convince us that we are setting for $h_n(u)$ the only possible value: if w is a weak*-accumulation point for (F_n) then ... is not, we wonder, $w(u)$ an accumulation point of $F_n(u)$? Well ... yes, it is, but maybe not the *right* accumulation point we have chosen for $h_n(u)$; maybe it is not the closest accumulation point to $F_n(u)$; after all, w only accumulates a certain subsequence of the F_n, not all. The values at Y are correct: 0, since in that case that is the only possible accumulation point.)

After that it only remains one way: to choose carefully the points $u_1, \dots,$ u_n. For instance, in the case of a single point, we can save the proof and obtain a wonderful 2 with a special choice of u: use Riesz's lemma to get some norm one point u such that $\mathrm{dist}(u, Y) = 1$. In fact this is, in some sense, what Sobczyk did: the several detours of his proof have the objective of choosing the right points x_j that make the projection appear.

2.8. Hasanov's proof, 1980 The following extension of Sobczyk's theorem appeared in [28]. Let $\mathcal{F}$ be a filter on a set S. Let τ be a cardinal. Then $\mathcal{F}$ is called a τ-filter if whenever $A_i \in \mathcal{F}$ for all $i \in I$ and $\mathrm{card} I < \tau$ then $\cap_{i\in I} A_i \in \mathcal{F}$. The space $m_0(S, \mathcal{F})$ is the closed span in $l_\infty(S)$ of the set $\{x \in l_\infty(S) : \lim_{\mathcal{F}} x = 0\}$. With this notation Hasanov shows:

Theorem 2.8. *The space $m_0(S, \mathcal{F})$ is at most 2-complemented in any Banach superspace E such that $E/m_0(S, \mathcal{F})$ has density character at most τ.*

2.9. Werner's proof, 1989. This one, originally in [60], but which can also be found in [27], was described by its author as "probably the most complicated proof of Sobczyk's theorem that has appeared in the literature". (We think –and deplore– that the next proof 2.10 beats that record.) It is similar to Pełczyński's proof, inasmuch as it uses a (more abstract) version of Borsuk's theorem 2.3. Let K be a compact space and $D \subset K$ a closed subspace. Then $J_D = \{f \in C(K) : f|D = 0\}$ is not only an ideal in $C(K)$. It is even an *M-ideal*, which means that there is a subspace $V \subset C(K)^*$ for which the decomposition

$$C(K)^* = V^* \oplus_1 J_D^\perp$$

holds. Here the subscript 1 indicates that if $\mu = (\nu, \phi)$ then $\|\mu\| = \|\nu\| + \|\phi\|$. The following result of Ando [3] Theorem 5, Choi and Effros [15] can be thought of as an abstract Borsuk's theorem.

THEOREM 2.9. *Let J be an M-ideal in a Banach space X. Let Y be a separable Banach space and let $T : Y \to X/J$ be a norm one operator. If J is an L_1-predual then there is a linear continuous lifting operator $L : Y \to X$ for T; i.e. if $q : X \to X/J$ is the quotient map then $qL = T$. Moreover, with $\|L\| = 1$.*

Now let c_0 be an isometric subspace of a separable Banach space X. Sobczyk's Theorem will follow if we apply the *ABCDE* result with $J = c_0$ and $Y = X/J$. But one first has to work to get c_0 as an M-ideal of X, something that need not be true without renorming X. This is what Werner does, but we prefer to simplify the proof of [62, Thm. 8].

Let $f_n \in X^*$ be a norm preserving extension of the n^{th}-evaluation functional on c_0. An easy calculation then shows that $Y = \overline{span(f_n)}$ is isometric to ℓ_1, and that $X^* = Y \oplus (c_0)^\perp$. Define a new norm on X^* by $\sharp y+z\sharp = \|y\|+\|z\|$. We will show that this is a dual norm. So, let $y_\alpha + z_\alpha$ be a bounded net weak*-convergent to $y + z$; we need to show that $\sharp y + z\sharp \leq \liminf \sharp y_\alpha + z_\alpha\sharp$.

Since $(c_0)^\perp$ is weak*-closed, we may pass to a subnet and assume that the weak* limit z_1 of z_α belongs to $(c_0)^\perp$; then $\|z_1\| \leq \liminf \|z_\alpha\|$. Also $(y_\alpha - y)$ is weak* convergent to $z - z_1$ and $\|z - z_1\| \leq \liminf \|y_\alpha - y\|$.

Although y_α is not weak* convergent to y, it is pointwise convergent on c_0. A simple calculation with the ℓ_1 norm then yields $\lim(\|y_\alpha\| - \|y_\alpha - y\|) = \|y\|$. Now we just add everything up:

$$\begin{aligned}
\sharp y + z\sharp &\leq \|y\| + \|z - z_1\| + \|z_1\| \\
&\leq \|y\| + \liminf \|y_\alpha - y\| + \liminf \|z_\alpha\| \\
&= \liminf \|y_\alpha\| + \liminf \|z_\alpha\| \\
&\leq \liminf(\|y_\alpha\| + \|z_\alpha\|) \\
&= \liminf(\sharp y_\alpha + z_\alpha\sharp).
\end{aligned}$$

♠

The *ABCDE* Theorem then gives a linear lifting $L : X/J \to X$, with $\sharp L\sharp = 1$, and so J is complemented in X. Clearly $\sharp \cdot \sharp \geq \| \cdot \|$ on X^*, whence $\sharp \cdot \sharp \leq \| \cdot \|$ on X. Since $(X/J)^* \cong J^\perp$, we see that $\sharp \cdot \sharp = \| \cdot \|$ on X/J, and thus $\|L\| = 1$. Norm one projections are better than norm two projections, aren't they? This is what we have finally achieved: in any separable Banach space, any copy of c_0 is the kernel of a norm one projection. For Lq is a norm one projection on X whose kernel is just J. Moreover, we can replace c_0 by $c_0(\Gamma)$ throughout this argument.

Proposition 2.10. *If X is any Banach space, J is a subspace isometric to $c_0(\Gamma)$, and X/J is separable, then there is a norm one projection P on X with* $\ker P = J$.

It seems that this can also be deduced from Sobczyk's original proof, but not from any of the other proofs we know. This topic will be pursued further in § 4.2.

2.10. Cabello and Castillo's proof, 1998. This proof can be found in [8] in a rather eccentric form, with the purpose of extending Sobczyk's theorem to the domain of topological semigroups. We shall not go that far here. The theory of Kalton and Peck [30, 31], see also [12], describes twisted sums of quasi-Banach spaces in terms of the so-called quasi-linear maps. A map $F : Z \to Y$ acting between quasi-normed spaces is said to be quasi-linear if it is homogeneous and there exists a constant K such that for all points $x, y \in Z$ one has

$$\|F(x+y) - F(x) - F(y)\| \leq Q(\|x\| + \|y\|).$$

The infimum of those constants Q satisfying this inequality shall be called the quasi-linearity constant of F and denoted $Q(F)$. We shall say that a quasi-linear map is trivial if it can be written as the sum of a bounded (homogeneous) and a linear (not necessarily continuous) map. A quasi-linear map $F : Z \to Y$ gives rise to a twisted sum of Y and Z, denoted $Y \oplus_F Z$, by endowing the product space $Y \times Z$ with the quasi-norm $\|(y,z)\|_F = \|y - F(z)\| + \|z\|$. Clearly, the map $Y \to Y \oplus_F Z$ sending y to $(y, 0)$ is an into isometry while the map $Y \oplus_F Z \to Z$ sending (y, z) to z is surjective and continuous. In this way Y can be thought of as a subspace of $Y \oplus_F Z$ for which the corresponding quotient space is Z. Conversely, given an exact sequence $0 \to Y \to X \to Z \to 0$, if one takes a bounded homogeneous selection B and a linear selection L for the quotient map, then their difference $B - L$ is a quasi-linear map $Z \to Y$. The two processes are inverse to one another in a functorial sense.

Other basic results of Kalton [30] are
(1) that the exact sequence constructed with a quasi-linear map $F : Z \to Y$ splits if and only if F can be written as a sum $F = B + L$ of a bounded homogeneous map $B : Z \to Y$ and a linear one $L : Z \to Y$. Equivalently, if we measure the distance between two homogeneous maps F and G as

$$\operatorname{dist}(F, G) = \sup_{\|x\| \leq 1} \{\|F(x) - G(x)\|\},$$

then F is at finite distance from a linear map $L : Z \to Y$. And
(2) quasi-linear maps defined on a dense subspace can be extended to the whole space (see [31]).

A rather delicate point in the theory is that a twisted sum of Banach spaces might not be locally convex: this is shown by Ribe's example [49] of an exact sequence $0 \to R \to E \to \ell_1 \to 0$ that does not split. A twisted sum $Y \oplus_F Z$ of Banach spaces is a Banach space if and only if the quasi-linear map $F : Z \to Y$ has the property, called 0-linearity (see [8, 10, 12]), that there exists a constant K such that for all choices of finite sets $\{x_1, \ldots, x_n\}$ of points in Z one has

$$\left\| \sum_{i=1}^{n} F(x_i) - F\left(\sum_{i=1}^{n} x_i\right) \right\| \leq K \sum_{i=1}^{n} \|x_i\|.$$

The infimum of those constants K satisfying the preceding inequality will be called the 0-linearity constant of F and denoted by $Z(F)$.

So, Sobczyk's theorem means that every exact sequence $0 \to c_0 \to X \to Z \to 0$ of Banach spaces with Z separable must split. Hence, that every 0-linear map $F : Z \to c_0$ from a separable Banach space into c_0 must be at finite distance from some linear map $L : Z \to c_0$. We prove that.

THEOREM 2.11. *Let $F : Z \to c_0(I)$ be a 0-linear map with Z separable. Then there exists a linear map $L : Z \to c_0(I)$ at a finite distance from F.*

We shall define the linear map at finite distance of F over a dense subspace of Z and then, apply the extension result (2). Let $F : Z \to c_0(I)$ be a 0-linear map with constant $Z(F)$, and assume that Z is separable. If F is written as $(f_\alpha)_{\alpha \in I}$ then each $f_\alpha : Z \to R$ is again 0-linear with constant at most $Z(F)$. The Hahn-Banach theorem makes the sequence

$$0 \to \mathbb{R} \to \mathbb{R} \oplus_{f_\alpha} Z \to Z \to 0$$

split, and thus there exists some $l_\alpha \in Z'$ at distance at most $Z(F)$ from f_α. This, and the fact that for each $z \in Z$ the family $(f_\alpha(z)) \in c_0(I)$ imply that $(l_\alpha(z)) \in \ell_\infty(I)$, so that we have a linear map $L : Z \to \ell_\infty(I)$ at distance at most $Z(F)$ from F.

Let (z_k) be a countable subset of Z spanning a dense subspace D. We set

$$E = \{\Lambda \in D' : \sup_k \Lambda(z_k)(1 + \|L(z_k)\|)^{-1} < \infty\}$$

endowed with the distance function

$$\operatorname{dist}(R,T) = \sum_k \frac{|R(z_k) - T(z_k)|}{2^k(1 + \|L(z_k)\|)}.$$

Bounded sets are relatively compact in (E,d) (very much as in the standard proof of the Banach-Alaoglu Theorem), and so is the closure of $(l_\alpha)_\alpha$. If A denotes the set of its accumulation points and $p_\alpha \in A$ is such that $\operatorname{dist}(l_\alpha, A) = \operatorname{dist}(l_\alpha, p_\alpha)$ then it is easy to see that $(\operatorname{dist}(l_\alpha, p_\alpha))_{\alpha\in I} \in c_0(I)$ and thus $(|l_\alpha(z) - p_\alpha(z)|)_{\alpha\in I} \in c_0(I)$ for every $z \in D$. The key point is that actually $p_\alpha \in D^*$. This follows from

$$\begin{aligned} |p_\alpha(z)| &= \limsup_{n\to\infty} |l_\alpha(n)(z)| \\ &\leq \limsup_{n\to\infty} (|l_\alpha(n)(z) - f_\alpha(n)(z)| + |f_\alpha(n)(z)|) \\ &\leq Z(F)\|z\|. \end{aligned}$$

This is enough since $P = (p_\alpha)_\alpha$ is a linear continuous map $D \to \ell_\infty(I)$ such that, when restricted to D, the map $L - P$ is at distance at most $2Z(F)$ from F. Applying the extension result there must be some linear map at finite distance from F on Z.

♡

3. c_0 IS NOT COMPLEMENTED IN ℓ_∞

We pass now to the negative counterpart of Sobczyk's theorem: there exist spaces, such as ℓ_∞, in which no copy of c_0 is complemented. Many people proved this fact, some without realizing it, and we hope it will be instructive to review the proofs.

3.1. PHILLIPS'S PROOF, 1940. To be pedantic, what Phillips proved [47, 7.5] is that c, the space of convergent sequences is not complemented in ℓ_∞. To do that, Phillips observes that if a projection $P : \ell_\infty \to c$ existed then to each weak*-convergent sequence of functionals on c would correspond a weak*-convergent sequence of functionals on ℓ_∞. In modern notation, P^* would transform weak*-convergent sequences on c into weak*-convergent sequences on ℓ_∞.

Consider the sequence $(f_n)_n$ of functionals on c given by

$$f_n(x) = x(n+1) - x(n).$$

This sequence is weak*-convergent to zero on c. Let $(F_n)_n$ be a weak*-convergent sequence of extensions of $(f_n)_n$ to all ℓ_∞. We recall a couple of results of that same paper: first Phillips's lemma [47, Lemma 3.3].

LEMMA 3.1. *Let μ_n be a bounded sequence of finitely additive set functions on $\mathbb{N}$. If for every set $A \subset \mathbb{N}$ one has $\lim \mu_n(A) = 0$ then*

$$\lim \sum_{k\in\mathbb{N}} |\mu_n(k)| = 0.$$

Secondly, the observation [47, p. 526] that each linear continuous functional $F \in \ell_\infty^*$ can be represented by a measure μ on $P(\mathbb{N})$ as

$$F(x) = \int_{\mathbb{N}} x(n) d\mu.$$

If $(F_n)_n$ were weak*-convergent to zero then for each $A \subset \mathbb{N}$ one would obviously have $\lim \mu_n(A) = 0$. Hence, Phillips's lemma gives that if $(F_n)_n$ is weak*-convergent to zero on ℓ_∞ then $\lim_{n\to\infty} \sum_{k\in\mathbb{N}} |\mu_n(k)| = 0$.

Returning to the proof, if $F_n(x) = \int_N x(n) d\mu_n$ then one would have

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}} |\mu_n(k)| = 0.$$

But the equalities $\mu_n(n+1) = F_n(e_{n+1}) = f_n(e_{n+1}) = 1$ and $\mu_n(n) = F_n(e_n) = f_n(e_n) = -1$ make that impossible.

Thus no continuous projection $P : \ell_\infty \to c$ exists.

♠

3.2. SOBCZYK'S PROOF, 1941. Sobczyk observed in [54, p. 945]: "By an argument identical with that used by Phillips (to prove the nonexistence of a projection of ℓ_∞ on c) it may also be shown directly that there is no projection of ℓ_∞ on c_0." Instead, he preferred to use Phillips's statement in combination with the following simple lemma, which we refuse to prove.

LEMMA 3.2. *Suppose that $X = A \oplus B$ and that $B = B_1 \oplus B_2$. Then $X_1 = A + B_1$ is closed in X, and $X = X_1 \oplus B_2$.*

Now, were c_0 complemented in ℓ_∞, the choices $X = \ell_\infty$, $A = c_0$ and B its complement, together with $B_1 = [(1, 1, \ldots)]$, would imply $\ell_\infty = c \oplus B_2$, contrary to what one knows.

♠

3.3. Nakamura and Kakutani's proof, 1941. In [43, Thm. 5], Nakamura and Kakutani observed the existence of an uncountable family (M_γ) of infinite subsets of N with the property that for different indices γ, μ the intersection $M_\gamma \cap M_\mu$ is finite. The way in which they obtain such families is: after numbering the nodes of a dyadic tree, let M_γ the set of numbers falling in a given branch γ. They were unaware of Sierpinski's [53] earlier (more complicated) proof.

From that they derive [43, Thm. 6] the existence in $\beta\mathbb{N} \setminus \mathbb{N}$ of an uncountable family $(E_\gamma)_{\gamma\in\Gamma}$ of mutually disjoint clopen (simultaneously open and closed) sets, namely

$$E_\gamma = \overline{M}_\gamma \setminus \mathbb{N}.$$

The closure of M_γ in $\beta\mathbb{N}$ is obviously clopen, since it coincides with the support of the (unique) continuous extension of 1_{M_γ} to $\beta\mathbb{N}$. If two subsets $A, B \subset \mathbb{N}$ have finite intersection $A \cap B = F$ then it is easy to see that $\overline{A} \cap \overline{B} = F$. Since $M_\gamma \cap M_\mu$ is finite, $E_\gamma \cap E_\mu$ is empty.

Let us remark that the existence of such family is impossible in $\beta\mathbb{N}$. And that is not the worst possible case: Szpilrajn [56] (aka Marczewski) observed that it is possible for a compact space K to admit a family of size d of mutually disjoint clopen sets while a compact superspace of K does not. This follows from the fact that every compact Hausdorff space is homeomorphic to a closed subset of a compact topological group. The existence of invariant measures on a compact topological group G makes mutually disjoint families of clopen sets of G countable. It is thus enough to take as K the one point compactification of a discrete set of cardinal d.

Nakamura and Kakutani [43, § 7] derive from the existence of such family that the coordinate functionals on c_0 cannot be extended to the whole of ℓ_∞ maintaining the pointwise null character of the sequence.

To show this, first recall that each element $F \in \ell_\infty^*$ can be decomposed in such a way that for each $x \in \ell_\infty$ one has

$$F(x) = \sum_n \lambda_n x(n) + \int_{\beta\mathbb{N}\setminus\mathbb{N}} x(\omega)\, d\mu(\omega)$$

for some regular countably additive measure μ on $\beta\mathbb{N}$. Moreover,

$$\|F\| = \sum_n |\lambda_n| + \text{total variation of } \mu \text{ on } \beta\mathbb{N} \setminus \mathbb{N}.$$

Let now $F_n \in \ell_\infty^*$ be an extension of the n^{th}-coordinate functional δ_n on

c_0. One has

$$F_n(x) = \delta_n(x) + \int_{\beta\mathbb{N}\setminus\mathbb{N}} x(\omega) d\mu_n(\omega)$$

for some countably additive measures μ_n on $\beta\mathbb{N}$. Since the elements of the family $\{E_\gamma\}_\gamma$ are disjoint, the total variation of a measure μ on $\beta\mathbb{N} \setminus \mathbb{N}$ has to be zero on all except countably many of them.

Applying that to each measure μ_n it follows that there exists some set, say E_0, on which all the measures μ_n have total variation 0. Since the sets E_0 and $\mathbb{N} \setminus E_0$ are both infinite, it follows that the sequence $(F_n(1_{E_0}))_n$ contains both 0 and 1 infinitely often, so it cannot be convergent.

♣

3.4. GROTHENDIECK'S PROOF, 1953. A formidable improvement of Phillip's lemma was obtained by Grothendieck in [26]: If K is a compact extremally disconnected space, e.g. $\beta\mathbb{N}$, then the weak* and weak convergent sequences in $C(K)^*$ coincide. Banach spaces with that property are now called Grothendieck spaces. It is almost obvious that quotients of Grothendieck spaces are Grothendieck spaces and that separable Grothendieck spaces are reflexive. Thus infinite dimensional nonreflexive separable spaces cannot be at all complemented in Grothendieck spaces. In particular, c_0 cannot be even a quotient of ℓ_∞.

♡

3.5. GROTHENDIECK'S PROOF, 1954. Again as an exercise in [25], precisely ex. 2 in 3.7, Grothendieck freely considers on his own the fact that the coordinate functionals of $c_0(I)$ cannot be extended in a pointwise null fashion to the whole of $\ell_\infty(I)$. There are four steps, the first two well worth stepping into. Given a functional f on $c_0(I)$ we can consider it, as an element of $l_1(I)$, as a functional on $\ell_\infty(I)$.

- Let μ be a continuous linear functional on $\ell_\infty(I)$. If $J_1, \dots, J_n$ are disjoint subsets of I then

$$\|\mu_{|\ell_\infty(J_1)}\| + \cdots + \|\mu_{|\ell_\infty(J_n)}\| \leq \|\mu\|.$$

- For each sequence $(\mu_n)_n$ of functionals on $\ell_\infty(I)$ there exists an infinite set $J \subset I$ such that, for all n,

$$\mu_{n|c_0(I)}{}_{|l_\infty(J)} = \mu_{n|l_\infty(J)}.$$

Maybe we could say some words about this point. For each n there must be some infinite set J_n such that $\|\mu_{n|\ell_\infty(J_n)}\| \leq 1/n$ (since I can be partitioned into an infinitely countable quantity of infinite sets). But, proceeding inductively, it is possible to obtain a decreasing sequence J_n of infinite sets such that

$$\|\mu_{k|\ell_\infty(J_n)}\| \leq \frac{1}{n}$$

for $1 \leq k \leq n$. It is enough to consider an infinite set J such that, for every n, $J \setminus J_n$ is finite.

Intermission.

Maybe it is worthwhile to mention here that Drewnowski and Roberts unpublished manuscript [17] (see also [12]) contains a more general version of this; precisely:

LEMMA 3.3. *Given a weakly compact operator $T : \ell_\infty \to Z$ there exists an infinite subset $M \subset \mathbb{N}$ such that the restriction $T|\ell_\infty(M)$ is weak*-to-weak continuous.*

In particular, a continuous functional $\mu : \ell_\infty \to \mathbb{R}$ (quite weakly compact), has to have an infinite set $M \subset \mathbb{N}$ such that the restriction $\mu|\ell_\infty(M)$ is weak* continuous. This means that $\mu_{|\ell_\infty(M)} \in \ell_1(M)$, and thus that $\mu_{|\ell_\infty(M)} = \mu_{|c_0(M)}$.

End of the intermission.

Going ahead, Grothendieck's argument claims now that if (μ_n) is a weakly* null sequence in $\ell_\infty(I)^*$ then $(\mu_{n|c_0(I)})$ is norm null. *New intermission.* There is, however, a mistifying point here: it seems that one had proved that without using that $l_1(I)$ has the Schur property; but at the end of the proof (p.131) the Schur property of l_1 appears; on the other hand, it is far simpler to realize that $l_1(I)$ must also have the Schur property once l_1 has it! *End of the new intermission.* From which it clearly follows that when $I = \mathbb{N}$ the weak* null sequence of coordinate functionals on c_0 cannot be lifted to a weak* null sequence of functionals on ℓ_∞.

♣

3.6. BOURBAKI'S PROOF, 1955 A clever insight on Phillips (or Grothendieck's) proof was presented by Bourbaki in [6, EVT IV, 55. Ex. 16]. It can also be found, no more as an exercise, cleanly in [34, § 31, 2, (3) and (5)]:

A pointwise zero sequence of extensions of the coordinate functionals to the whole ℓ_∞ is a weak*-null sequence in ℓ_∞^*, whose restrictions to c_0 have to form a weakly null sequence in ℓ_1; hence, by Schur's lemma, a norm null sequence. In conclusion, that the coordinate functionals of c_0 do not admit extensions to the whole ℓ_∞ forming a pointwise null sequence.

♣

3.7. CORSON'S PROOF, 1961; BOURGAIN'S PROOF 1980; AND OTHERS. Let us recall the line of reasoning $\diamondsuit$: to prove that ℓ_∞/c_0 is not a subspace of ℓ_∞ would show that c_0 is not complemented in ℓ_∞ (since any complement would have to be isomorphic to ℓ_∞/c_0). It only remains to choose which isomorphic hereditary property $\mathcal{P}$ the space ℓ_∞ has and ℓ_∞/c_0 has not. In general, this may not be the most efficient way to achieve our aim. Anyway, here are several possibilities:

1943 (Nakamura and Kakutani, implicitly) $\mathcal{P}=$ *to admit, in the weak topology, an uncountable discrete set*. That ℓ_∞ cannot admit such a subset follows from the separability of its weakly compact sets. That ℓ_∞/c_0 does was already shown with the family (E_γ).

1961 (Corson) $\mathcal{P}$ = *to be weakly realcompact*, whatever that means. Corson [16] proved that if X^* is weak* separable, then X is weakly realcompact. This obviously includes ℓ_∞. His proof that ℓ_∞/c_0 is not weakly realcompact is a modification of the proof that $c_0(\Gamma)^*$ is not weak* separable. Given both things, he derived again that c_0 is not complemented in ℓ_∞.

1972 $\mathcal{P}$ =*weakly compact sets are separable*. That this is so in ℓ_∞ follows from the obvious fact that its dual is weak*-separable. It was already observed in [37, p. 240] that the canonical injection $l_2(\Gamma) \to c_0(\Gamma)$ yields a non-separable weakly compact subset of $c_0(\Gamma)$; that $c_0(\Gamma)$ is a subspace of ℓ_∞/c_0 appears as a footnote in Rosenthal, but he avoided claiming priority.

1980 (Bourgain) $\mathcal{P}$ = *to admit a strictly convex renorming*. That ℓ_∞/c_0 has no such renorming was proved by Bourgain [7]. That l_∞ admits a strictly convex renorming follows from the well-known facts that l_2

admits a strictly convex renorming and that if $T : X \to l_2$ is an injective operator then X also admits a strictly convex renorming.

♢

Further information about the space ℓ_∞/c_0 can be found in [36]; and about its generalizations ℓ_ϕ/h_ϕ in [24].

3.8. Whitley's proof, 1966. Anecdotal evidence suggests this is the best known proof, perhaps because it appears in the eminently readable textbook [29]. The proof of Whitley [61] is a simplification of Nakamura and Kakutani's proof, although discovered independently. It makes the measures disappear, replacing them by functionals. In other words, it requires no representation theorem for ℓ_∞^*.

Recall the existence of an uncountable family (M_γ) of infinite subsets of $\mathbb{N}$ with the property that for different indices γ, μ the intersection $M_\gamma \cap M_\mu$ is finite. Whitley credits Arthur Kruse for the following ingenious method to obtain such families, but it was probably first discovered by Alexandrov in 1922 [1, Espace A_6]: ordering the rational numbers into a sequence, assign to each irrational number γ a set M_γ of (indices of) rationals converging to γ.

The argument now is that given any linear continuous functional $f \in c_0^\perp$, its kernel contains all, except perhaps countably many elements of the family (M_γ). To prove this, observe that the set $A_n = \{M_\gamma : f(M_\gamma) \geq 1/n\}$ cannot have more than $n\|f\|$ elements since given k elements $M_\gamma \in A_n$ then $\|\sum \mathrm{sign} f(M_\gamma) M_\gamma\| \leq 1$ and

$$f\left(\sum_{i=1}^{k} \mathrm{sign} f(M_{\gamma_i}) M_{\gamma_i}\right) \geq \frac{k}{n}.$$

That being true, the existence of an operator $T : \ell_\infty \to \ell_\infty$ with kernel c_0 would imply the existence of a bounded sequence (f_n) of functionals in $c_0^\perp$ such that $\cap_n \mathrm{Ker} f_n = c_0$, something impossible since uncountably many members of (M_γ) would be in that intersection.

♣

3.9. Amir's proof, 1962; Ülger's proof, 1999 More proofs? Yes, why not. Since complemented subspaces of injective spaces are injective, it is enough to show that c_0 is not injective. Or, using an argument worthy of Bertrand Russell: if we prove that c_0 is not complemented in some Banach space then it is not complemented in ℓ_∞. Of course, relatively recently, Rosenthal [51] showed us that injective spaces contain ℓ_∞.

Amir [2] proved that if a $C(K)$-space is injective then every convergent sequence in K must be eventually constant. Amir's argument is as follows: let (x_n) be a convergent sequence of distinct points. Set the elements $\mu_n = \delta_{x_{2n+1}} - \delta_{x_{2n}} \in C(K)^*$. It is clear that (μ_n) is weak*-convergent to 0. Now choose distinct neighborhoods U_n of x_n and define an element $F \in C(K)^{**}$ by $F(\mu) = \sum \mu(U_{2n}$. One has $F(\mu_n) = -1$, and thus (μ_n) is not weakly convergent to 0. This obviously implies that $C(K)$ is not a Grothendieck space. But every Banach space is isometric to a subspace of $\ell_\infty(\Gamma)$ for some Γ, and the latter is a Grothendieck space. Thus every injective space is a Grothendieck space. 3.4).

A. Ülger mentioned to the second author during the 1999 Spring School at Paseky the following approach (see [19]). A Banach space X is said to have the Phillips property (see [19] if the canonical projection $p : X^{***} \to X^*$ is sequentially weak*-to-norm continuous. The name comes from Phillips who proved that c_0 has that property (it is Phillips's lemma we've already seen in section 3.1). In [19] they also define the weak-Phillips property of X when p is sequentially weak*-to-weak continuous; and then they prove in theorem 2.4 that a Banach space X has the (weak) Phillips property if and only if for every operator $T : X^{**} \to c_0$ the restriction $T_{|X}$ is (weakly) compact. Hence, no operator $T : \ell_\infty \to c_0$ can exist such that $T(c_0) = c_0$.

4. Which is the statement of Sobczyk's theorem?

Good question. We have a couple of possibilties to explore.

4.1. In which spaces is every copy of c_0 complemented? This line of thought starts with Rosenthal's observation that Veech's proof also shows that copies of c_0 inside WCG spaces are complemented. In other words, copies of c_0 inside $C(K)$ spaces with K an Eberlein compact are complemented. Thus, let us consider the problem shifting the situation from the space X to $C(B_{X^*})$; and then formulating the properties of X in terms of topological properties of the compact space (B_{X^*}, w^*). Following this line one is asking in which $C(K)$ spaces are the copies of c_0 complemented. Let us call, momentarily, K-Sobczyk any Banach space in which every isometric copy of c_0 is K-complemented. The classical result asserts that separable spaces are 2-Sobczyk.

In [42] Molto shows that if the compact space (B_{X^*}, w^*) is (so-called) *cofinitely sequential* then X is 2-Sobczyk. Since Corson compact spaces are cofinitely sequential, it follows (corollary 6) that Banach spaces such that (B_{X^*}, w^*) is a Corson compact are 2-Sobczyk. Of course, this is not the last word since Valdivia introduced in [58] a more general type of compact space, called nowadays Valdivia compact, and proved that if K is a Valdivia compact then every separable subspace of $C(K)$ is contained in a separable 1-complemented subspace of $C(K)$. In this also, $C(K)$-spaces are 2-Sobczyk, as well as spaces X such that (B_{X^*}, w^*) is a Valdivia compact.

Molto shows in [42] that the well known compact space Δ with $\Delta^3 = \emptyset$ (see [1, Espace A_6]) is a Rosenthal compact, whatever that means; as we should know $C(\Delta)$ contains an uncomplemented copy of c_0. This suggests that maybe Valdivia compact are the last word. Or maybe not: Patterson [44] shows that if K denotes the two-arrows space (which is a Rosenthal compact, see [20]) then $C(K)$ is 2-Sobczyk.

4.2. The statement of Sobczyk's theorem A more general line of thought appears when one observes that Sobczyk's classical proof can be amended (see also the proof 2.9 or Hasanov's argument) to prove that copies of $c_0(I)$ inside spaces X such that $X/c_0(I)$ is separable are complemented. Thus, one may ask if there exists a general version of Sobczyk's containing the two results, namely:

Question. Are copies of $c_0(\Gamma)$ inside WCG spaces complemented?

This problem was considered in [4]. Let us first observe that the definition of K-Sobczyk spaces given in that paper is more general than the previous one: a Banach space X is said to be K-Sobczyk if every M-isomorphic copy of $c_0(I)$ is KM-complemented in X. The following observations might help to clarify this point.

When one has an isometric copy of $c_0(I)$ inside some $C(K)$-space, if p_i is some point of K where $\widehat{e_i}$ attains its norm 1 then the family (p_i) is a copy of I inside K, and moreover $\widehat{e_i}(p_j) = \delta_{ij}$. After some lemma or other this means the existence of a map $\phi : I \to B(C(K)^*)$ such that $\phi(i)(\widehat{e_j}) = \delta_{ij}$ for all $i, j \in I$. Of course, this is not enough to get a complemented copy, since it remains to verify if the condition weak*-lim$\phi(i) = 0$ can be obtained.

When one has instead a k-isomorphic copy of $c_0(I)$ inside $C(K)$ then one only knows that $k^{-1} \leq \|\widehat{e_i}\| \leq k$; so, if p_i are points where $\widehat{e_i}(p_i) = \|\widehat{e_i}\|$ then each point p_i can be "shared" by at most k functions (i.e., it is possible that for some i one has $\widehat{e_j}(p_i) = \|\widehat{e_j}\|$ for at most k indices j). Therefore one cannot guarantee that $\widehat{e_i}(p_j) = \delta_{ij}$.

In any case, if $\hat{I}$ denotes the closure of I in K, while it is clear that if $c_0(I)$ is complemented in $C(\hat{I})$ then it is complemented in $C(K)$. We do not know if the converse is true.

QUESTION. If $c_0(I)$ is complemented in $C(K)$, is it complemented in $C(\hat{I})$?

It is therefore worth mentioning the following partial answer regarding the topological nature of the compactification.

PROPOSITION 4.1. *Let I be a fixed set. Then the assertions (E1) and (E2) are equivalent; so are as assertions (RN1) and (RN2); and asssertions (G1) and (G2).*

- (E1) *Every copy of $c_0(I)$ inside a WCG Banach space is complemented.*
- (E2) *If $Eb(I)$ is an Eberlein compactification of I then $c_0(I)$ in complemented in $C(Eb(I))$.*
- (RN1) *Every copy of $c_0(I)$ inside an Asplund generated Banach space is complemented.*
- (RN2) *If $RN(I)$ is a Radon-Nikodym compactification of I then $c_0(I)$ in complemented in $C(RN(I))$.*
- (G1) *Every copy of $c_0(I)$ inside a WCD Banach space is complemented.*
- (G2) *If $G(I)$ is a Gulko compactification of I then $c_0(I)$ in complemented in $C(G(I))$.*

Proof. The proofs follows the same schema. $(*1)$ implies $(*2)$ since if K is an Eberlein (resp. Gulko, Radon-Nikodym) compact then $C(K)$ is WCG (resp. WCD, Asplund generated). Conversely, $(*2)$ implies $(*1)$ since when X is WCG (resp. WCD, Asplund generated) then $(B(X^*), w^*)$ is an Eberlein (resp. Gulko, Radon-Nikodym) compact; and because closed subspaces of an Eberlein (resp. Gulko, Radon-Nikodym) compact is a compact of the same type. ∎

We left out the case of Valdivia compacta because it is not true that subspaces of Valdivia compact are Valdivia compact. Nonetheless, some of the best results in [4] have been obtained for Valdivia compacts. Precisely:

THEOREM 4.2. *Let K be a Valdivia compact. The space $C(K)$ is 2^{m+1}-Sobczyk for copies of $c_0(I)$ with $cardI \leq \aleph_m$.*

This result is optimal since [4] also exhibits a scattered Eberlein compact K with density character $\aleph_\omega$ containing uncomplemented copies of some $c_0(\Gamma)$ (see also [11]). Further, in [4] it is also shown that the spaces of continuous functions on ordinal spaces are "quite" Sobczyk; precisely:

Theorem 4.3. *Let κ be an ordinal. Let X be a $(1+\varepsilon)$-isomorphic copy of $c_0(I)$ inside $C[1,\kappa]$ for $\varepsilon < \sqrt{\frac{3}{2}-1}$. Then X is ocomplemented in $C[1,\kappa]$.*

Let us close the paper with some related information. The following "necessity version" of Sobczyk's theorem can be found in [23]:

Theorem 4.4. *A closed subspace of $c_0(I)$ is complemented if and only if isomorphic to some $c_0(J)$.*

Several oblique readings of Sobczyk's theorem can be followed in [11]: the existence of retractions onto the derived space and their connection with Sobczyk's theorem, the definition of an ordinal uncomplementation index, and the relationships between the nature of a Boolean algebra $\mathcal{A}$ on $\mathbb{N}$, the Stone compactification, $\mathcal{A}\mathbb{N}$, of $\mathbb{N}$ it defines and complemented copies of c_0 inside $C(\mathcal{A}\mathbb{N})$.

The paper [4] contains many more results with a Sobczyk's like flavour; as a token, let us mention:

Theorem 4.5. *Let K be a Valdivia compact. Let X be an isomorphic copy of $c_0(I)$ inside $C(K)$. There exists a subset $J \subset I$ with $cardJ = cardI$ such that $c_0(J)$ is complemented.*

Further variations on non-Sobczyk's theorems can be followed through [13] and [9]. For instance, in [13] it is shown:

Theorem 4.6. *Let Z be any non-separable Banach space. Then there exists a nontrivial exact sequence $0 \to c_0 \to X \to Z \to 0$ in which X is not WCG.*

(this complements the fact that every exact sequence $0 \to c_0 \to X \to Z \to 0$ with X WCG splits); of course the same result is valid for $c_0(I)$.

Some of the results of [9] can be considered as explorations of the limits of Sobczyk's theorem. For instance, could $C(\omega^\omega)$ replace $c_0 = C(\omega)$ in some sense? Could the previous Z be a given nonseparable $C(K)$? The answer to the first question seems to be a resounding no since

THEOREM 4.7. *There exists a nontrivial exact sequence* $0 \to C(\omega^\omega) \to X \to c_0 \to 0$.

while we left open the second one:

PROBLEM. Let K be a nonmetrizable compact space. Does there exist a nontrivial sequence $0 \to c_0 \to X \to C(K) \to 0$?

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