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NOTE

Complemented Subspaces of Spaces of Multilinear Forms and Tensor Products[1]

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Among other things, we show that L_q is isomorphic to a complemented subspace of the space of multilinear forms on $L_{p_1} \times \cdots \times L_{p_n}$, where $q \geq 1$ is given by $1/p_1 + \cdots + 1/p_n + 1/q = 1$. The proof strongly depends on the L_∞-module structure of the spaces L_p. © 2001 Academic Press

Key Words: Banach space; multilinear form; tensor product; complemented subspace; Banach module; Banach algebra; Köthe space.

1. INTRODUCTION AND SAMPLE RESULT

This note stems from a misreading of [2, 3] although I hope this is not entirely obvious. Our main "concrete" result is the following.

THEOREM 1. *Let $p_1, \ldots, p_n$ and q be numbers such that $1/p_1 + \cdots + 1/p_n + 1/q = 1$, with $1 \leq p_i, q \leq \infty$. Then $L_q(\mu)$ is a complemented subspace of the space of multilinear forms on $L_{p_1}(\mu) \times \cdots \times L_{p_n}(\mu)$, for every σ-finite measure μ.*

That $L_q = L_q(\Omega, \mu)$ is a subspace of $\mathscr{L}(L_{p_1}, \ldots, L_{p_n})$ is surely well known. Indeed, let f be fixed in L_q. Then we can define an n-linear form on $L_{p_1} \times \cdots \times L_{p_n}$ by

$$(x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n} \mapsto \int_\Omega f(t) x_1(t) \ldots x_n(t)\, d\mu(t).$$

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According to Hölder inequality, one has $\|f \cdot x_1 \cdots x_n\|_1 \leq \|f\|_q \|x_1\|_{p_1} \cdots \|x_n\|_{p_n}$, which shows that the norm of f acting as a form on $L_{p_1} \times \cdots \times L_{p_n}$ is at most $\|f\|_q$. As for the reverse inequality, we may and do assume that f is nonnegative, with $\|f\|_q = 1$. Taking $x_i = f^{q/p_i}$ for $1 \leq i \leq n$, we see that $\|x_i\|_{p_i} = 1$ and

$$\int_\Omega f(t) x_1(t) \cdots x_n(t)\, d\mu(t) = \int_\Omega f^q\, d\mu = 1,$$

so that $\|f\| = 1$, as a multilinear form. This shows that L_q is isometrically isomorphic to a closed subspace of $\mathscr{L}(L_{p_1}, \dots, L_{p_n})$, provided $1/p_1 + \cdots + 1/p_n + 1/q = 1$.

The proof that L_q is a complemented subspace of $\mathscr{L}(L_{p_1}, \dots, L_{p_n})$ will require an intrinsic description of the multilinear forms induced by L_q-functions in terms of certain structural properties of the spaces L_p viewed as L_∞-modules. Then the desired projection is obtained from a standard averaging technique.

2. MULTILINEAR FORMS ON BANACH MODULES

Let $X_1, \dots, X_n$ be (Banach) modules over the same (commutative, Banach) algebra A. (The typical situation will be $A = L_\infty(\mu)$ and each X_i a Köthe function space on μ. See [5, 7] for information on Banach modules and Köthe spaces, respectively.) We are interested in those multilinear forms $f : X_1 \times \cdots \times X_n \to \mathbb{K}$ that are balanced (with respect to the module structure of the spaces X_i) in the sense of satisfying

$$f(x_1, \dots, ax_i, \dots, x_j, \dots, x_n) = f(x_1, \dots, x_i, \dots, ax_j, \dots, x_n)$$

for each $1 \leq i, j \leq n$ and all $a \in A$, $x_k \in X_k$. The set of all these f is obviously a closed subspace of $\mathscr{L}(X_1, \dots, X_n)$ which we denote by $\mathscr{L}_A(X_1, \dots, X_n)$. The following result yields a useful characterization of these forms $\mathscr{L}_A(X_1, \dots, X_n)$.

LEMMA 1. *Let X_i be A-modules. For a multilinear form $f : X_1 \times \cdots \times X_n \to \mathbb{K}$, the following are equivalent*:

(a) $f \in \mathscr{L}_A(X_1, \dots, X_n)$.

(b) *The associated $(n-1)$-linear operator $X_1 \times \cdots \times X_{n-1} \to X_n^*$ is an A-module homomorphism in each variable.*

(c) *The associated linear operator $X_1 \hat{\otimes} \cdots \hat{\otimes} X_{n-1} \to X_n^*$ is a homomorphism of $(A \hat{\otimes} \cdots \hat{\otimes} A)$-modules.*

Proof. This is a straightforward verification which is left to the reader. (The A-module structure of X_n^* is given by $\langle ax^*, x\rangle = \langle x^*, ax\rangle$ for $a \in A, x \in X_n, x^* \in X_n^*$. The structure of a module over $A \hat{\otimes} \cdots \hat{\otimes} A$ in $X_1 \hat{\otimes} \cdots \hat{\otimes} X_{n-1}$ and X_n^* is given by $(a_1 \otimes \cdots \otimes a_{n-1}) \cdot (x_1 \otimes \cdots \otimes x_{n-1}) = a_1 x_1 \otimes \cdots \otimes a_{n-1} x_{n-1}$ and $(a_1 \otimes \cdots \otimes a_{n-1}) \cdot x_n^* = (a_1 \cdots a_{n-1}) \cdot x_n^*$, respectively.) ∎

An element z of an A-module X will be called cyclic if the set $A \cdot z = \{az : a \in A\}$ is dense in X. (Clearly, if X is a minimal Köthe function space on a σ-finite measure μ, then every nonvanishing function in X is a cyclic element for the natural $L_\infty(\mu)$-module structure of X.) The presence of cyclic elements greatly simplifies the determination of balanced forms. Suppose $z_i \in X_i$, $1 \leq i \leq n-1$, are cyclic elements over A and let $f \in \mathscr{L}_A(X_1, \dots, X_n)$. If $T : X_1 \hat{\otimes} \cdots \hat{\otimes} X_{n-1} \to X_n^*$ is the associated operator, then we can recover f from the functional $T(z_1 \otimes \cdots \otimes z_{n-1}) \in X_n^*$. Indeed, if $x_n^* = T(z_1 \otimes \cdots \otimes z_{n-1})$, then

$$f(a_1 z_1, \dots, a_{n-1} z_{n-1} x_n) = \langle x_n^*, a_1 \dots a_{n-1} x_n \rangle.$$

So, at least in the pure linear sense, $\mathscr{L}_A(X_1, \dots, X_n)$ can be seen as a linear subspace of X_n^*.

COROLLARY 1. *Let X_i be minimal Köthe function spaces on a σ-finite measure space (Ω, μ). Suppose either that μ is finite and X_n is σ-order continuous or that each X_i is σ-order continuous. Then, for each $f \in \mathscr{L}_{L_\infty(\mu)}(X_1, \dots, X_n)$, there exists a (essentially) unique measurable, locally integrable function $g : \Omega \to \mathbb{K}$ such that*

$$f(x_1, \dots, x_n) = \int_\Omega g(t) x_1(t) \dots x_n(t) \, d\mu(t).$$

Moreover, if μ is finite, then g belongs to X_n'.

Proof. Let us recall here that a Köthe function space X is σ-order continuous if every order bounded increasing sequence converges in X. (Thus, L_p is σ-order continuous if and only if p is finite.) It is well known that X is σ-order continuous if and only if $X^* = X'$; that is, every bounded linear functional x^* on X can be written as $\langle x^*, x \rangle = \int_\Omega g(t) x(t) \, d\mu(t)$, for some (obviously locally integrable) measurable g.

Suppose μ is finite. Then $L_\infty(\mu) \subset X_i$ and f must be given by

$$f(a_1, \dots, a_n) = \langle x_n^* a_1 \cdots a_n \rangle,$$

for $a_i \in L_\infty(\mu)$. On the other hand, x_n^* is representable as an integral, so there is $g \in X_n^*$ such that

$$f(x_1, \dots, x_n) = \int_\Omega g(t) x_1(t) \cdots x_n(t)\, d\mu(t),$$

for $x_i \in L_\infty(\mu)$. Since $L_\infty(\mu)$ is dense in each X_i the same representation holds for all $x_i \in X_i$.

As for the σ-finite case, write Ω as an increasing union of measurable subsets Ω_k of finite measure. Since each X_i is σ-order continuous $\bigcup_k X_i(\Omega_k)$ is dense in X_i. Now, consider $X_i(\Omega_k)$ as an $L_\infty(\Omega_k)$-module in the obvious way. Taking into account that $L_\infty(\Omega_k)$ is an ideal in $L_\infty(\Omega)$ we infer the existence of measurable functions $g_k : \Omega_k \to \mathbb{K}$ such that

$$f(x_1, \dots, x_n) = \int_{\Omega_k} g_k(t) x_1(t) \dots x_n(t)\, d\mu(t),$$

provided $x_i \in X_i(\Omega_k)$. Since all these g_k coincide on the common domain they define a locally integrable function $g : \Omega \to \mathbb{K}$ in such a way that

$$f(x_1, \dots, x_n) = \int_\Omega g(t) x_1(t) \dots x_n(t)\, d\mu(t)$$

holds for every $x_i \in \bigcup_k X_i(\Omega_k)$ and, therefore, for every $x_i \in X_i$. ∎

COROLLARY 2. *Let $X_i = L_{p_i}(\mu)$ and $A = L_\infty(\mu)$, with μ a σ-finite measure and $p_n < \infty$. Then $\mathscr{L}_A(X_1, \dots, X_n)$ is the space of multilinear forms induced by L_q-functions, where q is given by $1/p_1 + \cdots + 1/p_n + 1/q = 1$. In particular $\mathscr{L}_A(X_1, \dots, X_n)$ is isometrically isomorphic to $L_q(\mu)$.*

Proof. If p_i is finite for all $1 \le i \le n$, this is a straightforward consequence of Corollary 1 and the computations of Section 1. The general case then follows from the obvious fact that $\mathscr{L}_A(A, X_1, \dots, X_n)$ can be identified with $\mathscr{L}_A(X_1, \dots, X_n)$ if A is unital. The isomorphism is given as follows: for $f \in \mathscr{L}_A(A, X_1, \dots, X_n)$ define a balanced form on $X_1 \times \cdots \times X_n$ by $(x_1, \dots, x_n) \mapsto f(1, x_1, \dots, x_n)$. The inverse map transforms $g \in \mathscr{L}_A(X_1, \dots, X_n)$ into $(a, x_1, \dots, x_n) \mapsto g(ax_1, \dots, x_n)$. ∎

Remark 1. Of course, if $X_i = L_\infty$ for all i, then $\mathscr{L}_A(X_1, \dots, X_n) = L_\infty^*$, which contains a complemented copy of L_1.

Corollary 2 remains true even if μ fails to be σ-finite. This follows from a classical result of Maharam about Boolean algebras and the obvious fact that our arguments still work for strictly localizable measures. Since the next Section 3 is independent on measure theory, the assumption on μ made in Theorem 1 turns out to be superfluous. We do not give the details here.

3. AVERAGING

In this section we shall see that, under rather mild assumptions on the ground algebra, $\mathscr{L}_A(X_1, \dots, X_n)$ is a complemented subspace of $\mathscr{L}(X_1, \dots, X_n)$. Recall that an element u of a Banach algebra A is said to be unitary if $\|u\| = \|u^{-1}\| = 1$. The set of all unitary elements of A is a group under multiplication and will be denoted by U.

THEOREM 2. *Let X_i be Banach modules over the commutative Banach algebra A. If U spans a dense subspace of A, then $\mathscr{L}_A(X_1, \dots, X_n)$ is the range of a contractive projection on $\mathscr{L}(X_1, \dots, X_n)$.*

Proof. Let $d\mu(u_1, \dots, u_{n-1})$ be an invariant mean for the locally compact group U^{n-1} viewed as a discrete space. (We refer the reader to [6, Chap. IV] or to the Greenleaf booklet [4] for information on invariant means.) For $f \in \mathscr{L}(X_1, \dots, X_n)$, put

$$Pf(x_1, \dots, x_n) = \int_{U^{n-1}} f\big(u_1 x_1, \dots, u_{n-1} x_{n-1}, u_1^{-1} \cdots u_{n-1}^{-1} x_n\big)\, d\mu(u_1, \dots, u_{n-1}).$$

Clearly, Pf is a multilinear form on $X_1 \times \cdots \times X_n$, with $\|Pf\| \le \|f\|$. On the other hand, it is clear that $Pf = f$ for all $f \in \mathscr{L}_A(X_1, \dots, X_n)$ and also that Pf depends linearly on f. Thus, the proof will be complete if we see that Pf is balanced for all f. A moment of reflection shows that it suffices to verify the identity

$$Pf(x_1, \dots, ux_i, \dots, x_n) = Pf(x_1, \dots, x_i, \dots, ux_n)$$

for each $1 \le i < n$ and all $x_k \in X_k, u \in U$. Using the invariance of the mean $d\mu(u_1, \dots, u_{n-1})$ and letting $v_i = u_i u$, we get

$$\begin{aligned}
&Pf(x_1, \dots, ux_i, \dots, x_n)\\
&\quad = \int f\big(u_1 x_1, \dots, u_i u x_i, \dots, u_{n-1} x_{n-1}, u_1^{-1} \cdots u_{n-1}^{-1} x_n\big)\\
&\qquad \times d\mu(u_1, \dots, u_{n-1})\\
&\quad = \int f\big(u_1 x_1, \dots, v_i x_i, \dots, u_{n-1} x_{n-1}, u_1^{-1} \cdots u v_i^{-1} u_{n-1}^{-1} x_n\big)\\
&\qquad \times d\mu(u_1, \dots, v_i, \dots, u_{n-1})\\
&\quad = \int f\big(u_1 x_1, \dots, u_i x_i, \dots, u_{n-1} x_{n-1}, u_1^{-1} \cdots u_{n-1}^{-1} u x_n\big)\\
&\qquad \times d\mu(u_1, \dots, u_i, \dots, u_{n-1})\\
&\quad = Pf(x_1, \dots, x_i, \dots, ux_n),
\end{aligned}$$

as desired. ∎

Theorem 1 now follows from Corollary 2 (and its Remark) and Theorem 2. Taking into account that balanced forms on L_p spaces are automatically symmetric, we can adhere the following corollary to Theorem 1.

COROLLARY 3. *Suppose $n/p + 1/q = 1$, with $1 \leq p, q \leq \infty$. Then L_q is a complemented subspace of the space of n-homogeneous polynomials on L_p.*

Remark 2. The hypothesis about A appearing in Theorem 2 can be understood as a stronger form of amenability. (Amenability is a central theme in the homology of Banach algebras; see [5].) It seems very likely that $\mathscr{L}_A(X_1, \ldots, X_n)$ is complemented in $\mathscr{L}(X_1, \ldots, X_n)$ provided A is an amenable algebra. This is true, for instance, for group algebras. It is worth noting that if $\mathscr{L}_A(X_1, \ldots, X_n)$ is complemented in $\mathscr{L}(X_1, \ldots, X_n)$ by any bounded projection and A is amenable, then there is an A-module projection from $\mathscr{L}(X_1, \ldots, X_n)$ onto $\mathscr{L}_A(X_1, \ldots, X_n)$. This is so because $\mathscr{L}_A(X_1, \ldots, X_n)$ is a dual module over A and all dual modules over an amenable algebra are injective; see [5, Theorem VII.1.6.I].

4. DIAGONAL SUBSPACES OF TENSOR PRODUCTS

We close the paper by showing that Theorem 1 can be predualized for sequence spaces.

Let X_k be Banach spaces with bases $(e_i^k)_{i=1}^{\infty}$ for $1 \leq k \leq n$. The main diagonal of $X = X_1 \hat{\otimes} \cdots \hat{\otimes} X_n$ is the closed subspace Δ spanned in X by the diagonal vectors $e_i \otimes \cdots \otimes e_i$. Let us show that Δ is complemented in X provided each factor X_k has an unconditional basis.

LEMMA 2. *Let $X_1, \ldots, X_n$ be m-dimensional Banach spaces with 1-unconditional bases $(e_i^k)_{i=1}^m$ for $1 \leq k \leq n$. Then the "diagonal" projection Q given on $X_1 \hat{\otimes} \cdots \hat{\otimes} X_n$ by*

$$Q\left(e_{i(1)}^1 \otimes \cdots \otimes e_{i(n)}^n\right) = \begin{cases} e_{i(1)}^1 \otimes \cdots \otimes e_{i(n)}^n & \text{if } i(1) = \cdots = i(n); \\ 0 & \text{otherwise} \end{cases}$$

is contractive. Consequently, if $X_1, \ldots, X_n$ are (infinite-dimensional) Banach spaces with unconditional bases $(e_i^k)_{i=1}^{\infty}$, then the diagonal projection is bounded on the space $X_1 \hat{\otimes} \cdots \hat{\otimes} X_n$.

Proof. We only prove the first part. Consider each X_k as an l_∞^m-module in the obvious way and let U be the unitary group of the Banach algebra l_∞^m, endowed with the norm-topology. Obviously U is compact and so there is a unique normalized Haar measure du on U. define a linear operator R on $X_1 \hat{\otimes} \cdots \hat{\otimes} X_n$ by

$$R(x_1 \otimes \cdots \otimes x_n) = \int \cdots \int u_1 x_1 \otimes \cdots \otimes u_{n-1} x_{n-1} \otimes \left(u_1^{-1} \ldots u_{n-1}^{-1}\right) x_n \, du_1 \ldots du_{n-1}.$$

Clearly, R is linear and bounded, with $\|R\| \leq 1$. Routine computations now show that $R = Q$. Hence Q is contractive too and the proof is complete. ∎

This lemma, in combination with Corollary 2, provides us with a very simple proof of the following nice result of Arias and Farmer [1]. From now on, we make the convention that l_p means c_0 if $p = \infty$.

PROPOSITION 1 (Arias and Farmer). *The main diagonal of $l_{p_1} \hat{\otimes} \cdots \hat{\otimes} l_{p_n}$ is a complemented subspace isomorphic to l_p, where p is given by $1/p = \min\{1, \sum_{i=1}^n 1/p_i\}$.*

Proof. We already know that Δ is complemented in $X = l_{p_1} \hat{\otimes} \cdots \hat{\otimes} l_{p_n}$ by a contractive projection. We prove that $(e_i \otimes \cdots \otimes e_i)_{i=1}^\infty$ is an l_p-basis. Take

$$x = \sum_{i=1}^m x_i (e_i \otimes \cdots \otimes e_i).$$

We shall consider two cases. First, suppose $1/p_1 + \cdots + 1/p_n \leq 1$. Let $P : \mathscr{L}(1_{p_1}, \ldots, l_{p_n}) = X^* \to \mathscr{L}_{l_\infty}(l_{p_1}, \ldots, l_{p_n})$ be the averaging projection described in the proof of Theorem 2. Since $\mathscr{L}_{l_\infty}(l_{p_1}, \ldots, l_{p_n}) = l_p^* = l_q$ and, taking into account that $\langle Pf, x \rangle = \langle f, x \rangle$ for all f and $x \in \Delta$ (this is obvious), we have

$$\begin{aligned}
\|x\|_X &= \sup_{\|f\| \leq 1} \langle f, x \rangle = \sup_{\|f\| \leq 1} \langle Pf, x \rangle \\
&= \sup\left\{\langle f, x \rangle : f \text{ is induced by a norm-one } l_q\text{-sequence}\right\} \\
&= \sup\left\{\sum_i f_i x_i : \left(\sum_i |f_i|^q\right)^{1/q} = 1\right\} \\
&= \left(\sum_{i=1}^m |x_i|^p\right)^{1/p},
\end{aligned}$$

so that Δ is isometric to l_p.

Now suppose $1/p_1 + \cdots + 1/p_n > 1$. Choose $s_k \geq p_k$ in such a way that $1/s_1 + \cdots + 1/s_n = 1$. Since $\|y\|_{s_k} \leq \|y\|_{p_k}$ for each $1 \leq k \leq n$ and all $y \in l_{p_k}$ the formal identity $l_{p_1} \hat{\otimes} \cdots \hat{\otimes} l_{p_n} \to l_{s_1} \hat{\otimes} \cdots \hat{\otimes} l_{s_n}$ has norm one. Hence,

$$\sum_i |x_i| = \left\| \sum_i x_i (e_i \otimes \cdots \otimes e_i) \right\|_{l_{s_1} \hat{\otimes} \cdots \hat{\otimes} l_{s_n}} \leq \left\| \sum_i x_i (e_i \otimes \cdots \otimes e_i) \right\|_X \leq \sum_i |x_i|$$

and $(e_i \otimes \cdots \otimes e_i)$ is an l_1-basis. This completes the proof. ∎

5. CONCLUDING REMARKS

Perhaps a few remarks about the module-theoretic content of this note are in order. First, there is a classical construction which linearizes balanced forms. Indeed, if the X_i are modules over the commutative Banach algebra A, then $\mathscr{L}_A(X_1, \ldots, X_n) = (X_1 \hat{\otimes}_A \cdots \hat{\otimes}_A X_n)^*$ in a canonical way. In fact, this identity could be taken as the definition of $X_1 \hat{\otimes}_A \cdots \hat{\otimes}_A X_n$ in [5]. To obtain a suitable description of the tensor product of modules on an algebra, consider the usual (projective, Banach) tensor product $X_1 \hat{\otimes} \cdots \hat{\otimes} X_n$ and the closed subspace N spanned by the elements of the form

$$\left(x_1 \otimes \cdots \otimes a x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n\right) - \left(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes a x_j \otimes \cdots \otimes x_n\right).$$

Then $X_1 \hat{\otimes}_A \cdots \hat{\otimes}_A X_n$ equals $(X_1 \hat{\otimes} \cdots \hat{\otimes} X_n)/N$.

Thus, Corollary 2 and the comments made after Theorem 1 imply that if $1/p_1 + \cdots + 1/p_n \leq 1$, then $L_{p_1} \hat{\otimes}_{L_\infty} \cdots \hat{\otimes}_{L_\infty} L_{p_n} = L_p$ (as L_∞-modules), where p is given by $1/p = 1/p_1 + \cdots + 1/p_n$. Surprisingly enough, if μ is nonatomic and $1/p_1 + \cdots + 1/p_n > 1$, then $L_{p_1} \hat{\otimes}_{L_\infty} \cdots \hat{\otimes}_{L_\infty} L_{p_n} = 0$ since in this case the space of balanced forms reduces to zero. We remark, however, that this situation is not too strange in the pure algebraic setting: it is easily checked that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

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