Applications of a result of Aron, Hervés, and Valdivia to the homology of Banach algebras*

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Dedicated to Professor Manuel Valdivia on his 70-th birthday.

Abstract

As an application of a celebrate result of Aron, Hervés, and Valdivia about weakly continuous multilinear maps, we obtain a sequence (A_n) of finite dimensional (hence amenable) Lipschitz algebras for which the algebra $\ell_\infty(A_n)$ fails to be even weakly amenable.

MCS 2000 Primary 46H05; Secondary 46H25, 46G25

Introduction and main result

Let A be an associative Banach algebra and X a Banach bimodule over A. A derivation $D : A \to X$ is a (linear, continuous) operator satisfying Leibniz's rule:

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

The simplest derivations have the form $D(a) = a \cdot x - x \cdot a$ for some fixed $x \in X$. They are called inner. A Banach algebra is said to be amenable is every derivation $D : A \to X$ is inner for all dual bimodules X. When this holds merely for $X = A'$ we say that A is weakly amenable.

Let us recall the trivial fact that if $B \to A$ is a bounded homomorphism with dense range and B is amenable, then so is A. The same is true for weak amenability provided B (hence A) is commutative [3] (see also [10] for counterexamples in the noncommutative case). We refer the reader to [11,12,4] for background on amenability and weak amenability.

Let (A_n) be a sequence of associative Banach algebras. As usual, we write $\ell_\infty(A_n)$ for the Banach algebra of all sequences $f = (f_n)$, with $f_n \in A_n$ for all n, and $\|f\| = \sup_n \|f_n\|_{A_n}$ finite, equipped with the obvious norm and coordinatewise multiplication. If $A_n = A$ for some fixed algebra A, we simply write $\ell_\infty(A)$.

In this note, we exhibit sequences (A_n) of finite dimensional amenable Banach algebras for which the algebra $\ell_\infty(A_n)$ fails to be (weakly) amenable.

*Supported in part by DGICYT project PB97-0377.

For basic information about the Arens product in the second dual of a Banach algebra the reader can consult [8,9,6]. Here we only recall that, given a bilinear operator $B : X \times Y \to Z$ acting between Banach spaces, there is a bilinear extension $B'' : X'' \times Y'' \to Z''$ given by

$$B''(x'', y'') = \text{weak*--}\lim_x \left(\text{weak*--}\lim_y B(x, y) \right) \qquad (x'' \in X'', y'' \in Y'')$$

where the iterated limits are taken first for $y \in Y$ converging to y'' in the weak* topology of Y'' and then for $x \in X$ converging to x'' in the weak* topology of X''. The map B'' is often called the first Arens extension of B; see [1]. In particular, if A is a Banach algebra, then the bidual space A'' is always a Banach algebra under the (first) Arens product

$$a'' \cdot b'' = \text{weak*--}\lim_a \left(\text{weak*--}\lim_b (a \cdot b) \right) \qquad (a'', b'' \in A'')$$

where the iterated limits are taken for a and b in A converging respectively to a'' and b'' in the weak* topology of A''.

Our main result is the following device that allows one to obtain A'' as a quotient algebra of $\ell_\infty(A_n)$ if A_n are nicely placed linear subspaces of A, even if they cannot be embedded as subalgebras in A. We feel that the most remarkable feature of the paper is that we get homomorphisms on $\ell_\infty(A_n)$ from linear operators on A_n which are not multiplicative.

Theorem. *Let A_n and A be Banach algebras. Suppose there are linear embeddings $T_n : A_n \to A$ satisfying:*

(a) *There is a constant M such that $M^{-1}\|f\| \leq \|T_n f\| \leq M\|f\|$ for all n and every $f_n \in A_n$.*

(b) *$T_{n+1}(A_{n+1})$ contains $T_n(A_n)$ and $\cup_n T_n(A_n)$ is (strongly) dense in A.*

(c) *Given sequences (f_n) and (g_n) in $\ell_\infty(A_n)$, the sequence $T_n(f_n) \cdot T_n(g_n) - T_n(f_n \cdot g_n)$ is weakly null in A.*

Assume, moreover that

(d) *the product $A \times A \to A$ is jointly weakly continuous on bounded sets; and*

(e) *A' is a separable Banach space.*

Then there exists a surjective homomorphism from $\ell_\infty(A_n)$ onto A''.

So, if A'' fails to be amenable, then $\ell_\infty(A_n)$ cannot be amenable, even if all A_n are. Also, if A_n are commutative and A'' is not weakly amenable, then neither is $\ell_\infty(A_n)$.

Here, we are interested in the case in which A_n are finite dimensional, but note that if A satisfies (d) and (e), then the remaining conditions automatically hold for $A_n = A$ and $T_n = 1_A$ and we obtain A'' as a quotient of $\ell_\infty(A)$.

The proof of the above Theorem uses in a critical way the following result of Aron, Hervés and Valdivia [2]. See [5] for a simpler proof.

Lemma. *For a bilinear operator $B : X \times Y \to Z$ the following conditions are equivalent:*

(a) *B is jointly weakly continuous on bounded sets.*

(b) *B is jointly weakly uniformly continuous on bounded sets.*

(c) *B'' is jointly weakly* (uniformly) continuous on bounded sets.*

Proof of the Theorem. Let U be an ultrafilter on $\mathbf{N}$. Define $\Psi : \ell_\infty(A_n) \to A''$ by $\Psi(f) = \text{weak*}-\lim_{U(n)} T_n(f_n)$. This definition makes sense because of the weak* compactness of balls in A''. Clearly, Ψ is linear and bounded, with $\|\Psi\| \le \sup_n \|T_n\|$.

We show that Ψ is surjective. Take $f'' \in A''$. By (b) and (e) there is a sequence (f_n), with $f_n \in A_n$ such that $T_n(f_n)$ is weakly* convergent to f'' in A'' and bounded in A. By (a) the sequence (f_n) is itself bounded, and taking $f = (f_n)$, it is clear that $\Psi(f) = f''$.

It remains to prove that Ψ is a homomorphism. Take $f, g \in \ell_\infty(A_n)$. Then,

$$\begin{aligned}
&\Psi(f) \cdot \Psi(g) - \Psi(f \cdot g) \\
&\quad = \Big(\text{weak*}-\lim_{U(n)} T_n f_n\Big) \cdot \Big(\text{weak*}-\lim_{U(n)} T_n g_n\Big) - \Big(\text{weak*}-\lim_{U(n)} T_n(f_n \cdot g_n)\Big) \\
&\quad = \Big(\text{weak*}-\lim_{U(n)} (T_n f_n \cdot T_n g_n)\Big) - \Big(\text{weak*}-\lim_{U(n)} T_n(f_n \cdot g_n)\Big) \\
&\quad = \text{weak*}-\lim_{U(n)} \left(T_n f_n \cdot T_n g_n - T_n(f_n g_n)\right) = 0.
\end{aligned}$$

This completes the proof.

Construction of the example

Example. A sequence of finite dimensional (hence amenable) Lipschitz algebras A_n such that $\ell_\infty(A_n)$ is not even weakly amenable.

Proof. Let K be a compact metric space with metric $d(\cdot,\cdot)$ and let $0 < \alpha < 1$. Then $\operatorname{Lip}_\alpha(K)$ is the algebra of all complex-valued functions on K for which

$$\varrho_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

is finite and $\operatorname{lip}_\alpha(K)$ is the subalgebra of those f such that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as} \quad d(x,y) \to 0.$$

Both algebras are equipped with the norm $\|f\|_\alpha = \|f\|_\infty + \varrho_\alpha(f)$. Bade, Curtis and Dales proved in [3] that the algebra $\operatorname{lip}_\alpha(K)''$ is isometrically isomorphic to $\operatorname{Lip}_\alpha(K)$ which has point derivations for every infinite K (and, therefore, is not weakly amenable).

Take $A = \operatorname{lip}_\alpha(I)$, where $I = [0,1]$ has the usual metric. Then the Banach space A turns out to be isomorphic (in the pure linear sense) to c_0, the space of all null sequences [7,15]. This implies that every bilinear operator from $A \times A$ into any Banach space is jointly weakly continuous on bounded sets [2] and also that A' is separable, which yields (d) and (e).

We now construct the required sequence A_n. For each n, let I_n be the (finite) subset of I consisting of all points of the form $k/2^n$, for $0 \leq k \leq 2^n$. Put $A_n = \mathrm{lip}_\alpha(I_n)$. Clearly, A_n is amenable for all n since it is isomorphic to the algebra $C(I_n)$.

There is a natural quotient homomorphism $Q_n : A \to A_n$, given by plain restriction. Obviously, $\|Q_n\| = 1$ for all n (this will be used later). Let $T_n : A_n \to A$ be defined as follows: for each $f \in A_n$, $T_n(f)$ is the polygonal interpolating f on I_n. Clearly, T_n is a linear operator, although it fails to be multiplicative. Since $Q_n \circ T_n$ is the identity on A_n it is clear that $\|T_n f\| \geq \|f\|$ for all $f \in A_n$.

Moreover, $\|T_n\| = 1$ for all n. Clearly, $\|T_n(f)\|_\infty = \|f\|_\infty$, so the point is to show that $\varrho_\alpha(T_n f)$ equals $\varrho_\alpha(f)$. It obviously suffices to see that if g is a polygonal with nodes in I_n then

$$\varrho_\alpha(g) = \sup_{x \neq y} \frac{|g(y) - g(x)|}{|y - x|^\alpha}$$

is attained at some $(x, y) \in I_n \times I_n$. This is an amusing exercise in elementary calculus. The solution appears in [13, chapter III, lemma 3.2, p. 203]. Thus, T_n is an into isometry and (a) holds.

Let us verify (b). Obviously, $T_{n+1}(A_{n+1})$ contains $T_n(A_n)$ for each n, so that $\cup_n T_n A_n$ is a (not closed) linear subspace of $\mathrm{lip}_\alpha(I)$. We show that $\cup_n T_n A_n$ is (strongly) dense in $\mathrm{lip}_\alpha(I)$. It clearly suffices to show weak density. We claim that for every $f \in \mathrm{lip}_\alpha(I)$ the sequence $T_n Q_n(f)$ converges weakly to f in $\mathrm{lip}_\alpha(I)$. We need some information about weak convergent sequences in the small space of Lipschitz functions.

Consider the operator $\Phi : \mathrm{lip}_\alpha(I) \to C_0(I^2 \backslash \Delta) \oplus_1 C(I)$ given by $\Phi(f) = (\tilde{f}, f)$, where

$$\tilde{f}(x, y) = \frac{f(y) - f(x)}{|y - x|^\alpha}$$

and Δ is the diagonal of I^2. Clearly, it is an isometric embedding, so that the weak topology in $\mathrm{lip}_\alpha(I)$ is the relative weak topology as a subspace of $C_0(I^2 \backslash \Delta) \oplus_1 C(I)$. On the other hand, weakly null sequences in $C_0(\Omega)$ spaces are bounded sequences pointwise convergent to zero. Hence $f_n \to f$ weakly in $\mathrm{lip}_\alpha(I)$ if and only if (f_n) is bounded and $f_n(x) \to f(x)$ for all $x \in I$, and this happens if and only if (f_n) is bounded and $f_n(x) \to f(x)$ for all x in some dense subset of I.

But, for $f \in \mathrm{lip}_\alpha(I)$ the sequence $(T_n Q_n(f))$ is bounded (by the norm of f) and converges pointwise to f on $\cup_n I_n$. This proves our claim. So, (b) also holds.

It only remains to verify (c). Take $(f_n), (g_n) \in \ell_\infty(A_n)$. Then $T_n(f_n) \cdot T_n(g_n) - T_n(f_n \cdot g_n)$ is weakly null in A if and only if for every ultrafilter V on $\mathbf{N}$ one has

$$\lim_{V(n)} \left(T_n(f_n) \cdot T_n(g_n) - T_n(f_n \cdot g_n) \right) = 0$$

in the weak* topology of $A'' = \mathrm{Lip}_\alpha(I)$. Take $x \in \cup_n I_n$ and let δ_x be the associated evaluation functional. Then,

$$\begin{aligned}
\left\langle \text{weak*}\text{–}\lim\nolimits_{V(n)} T_n(f_n \cdot g_n), \delta_x \right\rangle &= \lim\nolimits_{V(n)} \langle T_n(f_n \cdot g_n), \delta_x \rangle \\
&= \lim\nolimits_{V(n)} T_n(f_n \cdot g_n)(x) \\
&= \lim\nolimits_{V(n)} (f_n \cdot g_n)(x) \\
&= \lim\nolimits_{V(n)} (f_n(x) g_n(x)) \\
&= \lim\nolimits_{V(n)} f_n(x) \cdot \lim\nolimits_{V(n)} g_n(x) \\
&= \left\langle \text{weak*}\text{–}\lim\nolimits_{V(n)} T_n(f_n), \delta_x \right\rangle \left\langle \text{weak*}\text{–}\lim\nolimits_{V(n)} T_n(g_n), \delta_x \right\rangle,
\end{aligned}$$

so that

$$\text{weak*}\text{–}\lim_{V(n)} T_n(f_n \cdot g_n) = \left(\text{weak*}\text{–}\lim_{V(n)} T_n(f_n)\right) \cdot \left(\text{weak*}\text{–}\lim_{V(n)} T_n(g_n)\right).$$

Since the product of $\mathrm{Lip}_\alpha(I)$ is jointly weakly* continuous on bounded sets, the right hand side of the preceding equation becomes

$$\text{weak*}\text{–}\lim_{V(n)} (T_n(f_n) \cdot T_n(g_n)),$$

which completes the proof of (c).

Thus, the Theorem yields a surjective homomorphism $\ell_\infty(A_n) \to A''$, which shows that $\ell_\infty(A_n)$ is not weakly amenable and completes the proof.

Concluding remarks

As the referee pointed out, it is implicit in [14] that there are finite dimensional (hence amenable) C^*-algebras A_n for which $\ell_\infty(A_n)$ fails to be amenable. To see this, let H be a separable Hilbert space with a fixed orthonormal basis and let H_n be the subspace spanned by the first n elements of the basis. Write i_n for the obvious inclusion of H_n into H and π_n for the obvious projection of H onto H_n. Take $A_n = L(H_n)$, the algebra of all operators on H_n and $A = K(H)$, the algebra of all compact operators on H. Then $L(H_n)$ embeds isometrically as a subalgebra in A taking $T_n(L) = i_n \circ L \circ \pi_n$. Although (d) fails, it is clear from the proof of the Theorem that Ψ is still an onto operator from $\ell_\infty(L(H_n))$ onto $K(H)'' = L(H)$. Moreover the map $\Phi : L(H) \to \ell_\infty(L(H_n))$ given by $\Phi(T) = (\pi_n \circ T \circ i_n)$ is a right inverse for Ψ and $L(H)$ is thus a complemented subspace of $\ell_\infty(L(H_n))$. This implies that $\ell_\infty(L(H_n))$ lacks the approximation property and cannot be amenable (see [14] and references therein).

Needless to say, our example is far simpler since the existence of point derivations in $\mathrm{Lip}_\alpha(I)$ is a straightforward consequence of the Banach-Alaoglu theorem.

It follows from the remarks made after the Theorem that if A is $\mathrm{lip}_\alpha(I)$, then there is a surjective homomorphism from $\ell_\infty(A)$ onto A''. Hence $\ell_\infty(A)$ fails to be amenable and the same occurs with any ultrapower A_V (with respect to a non-trivial ultrafilter V on $\mathbf{N}$) since the quotient mapping constructed in the Theorem factorizes throughout the natural homomorphism $\ell_\infty(A) \to A_V$.

It would be interesting to study Banach algebras which are "super-amenable" in the sense of having amenable ultrapowers. A reasonable conjecture appears to the that A is super-amenable if and only of A'' is amenable. Note that, in view of [14, theorem 2.5], the conjecture is true for C^*-algebras. See [9] for some (loosely) related results.

Acknowledgements

It is a pleasure to thank the anonymous referee for the correction of a serious mistake in a previous version of the paper, for helpful comments, and for calling our attention to some useful references. We also thank Jesús Jaramillo for being so amenable.

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