Applications of a result of Aron, Hervés, and Valdivia to the homology of Banach algebras

Félix Cabello Sánchez and Ricardo García

Departamento de Matemáticas, Universidad de Extremadura
Avenida de Elvas, 06071-Badajoz, España
E-mail: fcabello@unex.es, rgarcia@unex.es

Dedicated to Professor Manuel Valdivia on his 70-th birthday.

Abstract
As an application of a celebrate result of Aron, Hervés, and Valdivia about weakly continuous multilinear maps, we obtain a sequence $(A_n)$ of finite dimensional (hence amenable) Lipschitz algebras for which the algebra $\ell_\infty(A_n)$ fails to be even weakly amenable.

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Introduction and main result

Let $A$ be an associative Banach algebra and $X$ a Banach bimodule over $A$. A derivation $D : A \to X$ is a (linear, continuous) operator satisfying Leibniz’s rule:

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

The simplest derivations have the form $D(a) = a \cdot x - x \cdot a$ for some fixed $x \in X$. They are called inner. A Banach algebra is said to be amenable if every derivation $D : A \to X$ is inner for all dual bimodules $X$. When this holds merely for $X = A^*$ we say that $A$ is weakly amenable.

Let us recall the trivial fact that if $B \to A$ is a bounded homomorphism with dense range and $B$ is amenable, then so is $A$. The same is true for weak amenability provided $B$ (hence $A$) is commutative [3] (see also [10] for counterexamples in the noncommutative case). We refer the reader to [11,12,4] for background on amenability and weak amenability.

Let $(A_n)$ be a sequence of associative Banach algebras. As usual, we write $\ell_\infty(A_n)$ for the Banach algebra of all sequences $f = (f_n)$, with $f_n \in A_n$ for all $n$, and $\|f\| = \sup_n \|f_n\|_{A_n}$ finite, equipped with the obvious norm and coordinatewise multiplication. If $A_n = A$ for some fixed algebra $A$, we simply write $\ell_\infty(A)$.

In this note, we exhibit sequences $(A_n)$ of finite dimensional amenable Banach algebras for which the algebra $\ell_\infty(A_n)$ fails to be (weakly) amenable.

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For basic information about the Arens product in the second dual of a Banach algebra the reader can consult [8,9,6]. Here we only recall that, given a bilinear operator \( B : X \times Y \to Z \) acting between Banach spaces, there is a bilinear extension \( B'' : X'' \times Y'' \to Z'' \) given by

\[
B''(x'', y'') = \lim_x\left( \lim_y B(x, y) \right)
\]

where the iterated limits are taken first for \( y \in Y \) converging to \( y'' \) in the weak* topology of \( Y'' \) and then for \( x \in X \) converging to \( x'' \) in the weak* topology of \( X'' \). The map \( B'' \) is often called the first Arens extension of \( B \); see [1]. In particular, if \( A \) is a Banach algebra, then the bidual space \( A'' \) is always a Banach algebra under the (first) Arens product

\[
a'' \cdot b'' = \lim_a\left( \lim_b (a \cdot b) \right)
\]

where the iterated limits are taken for \( a \) and \( b \) in \( A \) converging respectively to \( a'' \) and \( b'' \) in the weak* topology of \( A'' \).

Our main result is the following device that allows one to obtain \( A'' \) as a quotient algebra of \( \ell_\infty(A_n) \) if \( A_n \) are nicely placed linear subspaces of \( A \), even if they cannot be embedded as subalgebras in \( A \). We feel that the most remarkable feature of the paper is that we get homomorphisms on \( \ell_\infty(A_n) \) from linear operators on \( A_n \) which are not multiplicative.

**Theorem.** Let \( A_n \) and \( A \) be Banach algebras. Suppose there are linear embeddings \( T_n : A_n \to A \) satisfying:

(a) There is a constant \( M \) such that \( M^{-1} \|f\| \leq \|T_n f\| \leq M \|f\| \) for all \( n \) and every \( f_n \in A_n \).

(b) \( T_{n+1}(A_{n+1}) \) contains \( T_n(A_n) \) and \( \cup_n T_n(A_n) \) is (strongly) dense in \( A \).

(c) Given sequences \( (f_n) \) and \( (g_n) \) in \( \ell_\infty(A_n) \), the sequence \( T_n(f_n) \cdot T_n(g_n) - T_n(f_n \cdot g_n) \) is weakly null in \( A \).

Assume, moreover that

(d) the product \( A \times A \to A \) is jointly weakly continuous on bounded sets; and

(e) \( A' \) is a separable Banach space.

Then there exists a surjective homomorphism from \( \ell_\infty(A_n) \) onto \( A'' \).

So, if \( A'' \) fails to be amenable, then \( \ell_\infty(A_n) \) cannot be amenable, even if all \( A_n \) are. Also, if \( A_n \) are commutative and \( A'' \) is not weakly amenable, then neither is \( \ell_\infty(A_n) \).

Here, we are interested in the case in which \( A_n \) are finite dimensional, but note that if \( A \) satisfies (d) and (e), then the remaining conditions automatically hold for \( A_n = A \) and \( T_n = 1_A \) and we obtain \( A'' \) as a quotient of \( \ell_\infty(A) \).

The proof of the above Theorem uses in a critical way the following result of Aron, Hervés and Valdivia [2]. See [5] for a simpler proof.
Lemma. For a bilinear operator $B : X \times Y \to Z$ the following conditions are equivalent:

(a) $B$ is jointly weakly continuous on bounded sets.

(b) $B$ is jointly weakly uniformly continuous on bounded sets.

(c) $B^*$ is jointly weakly* (uniformly) continuous on bounded sets.

Proof of the Theorem. Let $U$ be an ultrafilter on $\mathbb{N}$. Define $\Psi : \ell_\infty(A_n) \to A''$ by $\Psi(f) = \text{weak}^* - \lim_{U(n)} T_n(f_n)$. This definition makes sense because of the weak* compactness of balls in $A''$. Clearly, $\Psi$ is linear and bounded, with $\|\Psi\| \leq \sup_n \|T_n\|$. We show that $\Psi$ is surjective. Take $f'' \in A''$. By (b) and (e) there is a sequence $(f_n)$, with $f_n \in A_n$ such that $T_n(f_n)$ is weakly* convergent to $f''$ in $A''$ and bounded in $A$. By (a) the sequence $(f_n)$ is itself bounded, and taking $f = (f_n)$, it is clear that $\Psi(f) = f''$.

It remains to prove that $\Psi$ is a homomorphism. Take $f, g \in \ell_\infty(A_n)$. Then,

$$
\Psi(f) \cdot \Psi(g) - \Psi(f \cdot g) = \left(\text{weak}^* - \lim_{U(n)} T_n(f_n) \cdot \text{weak}^* - \lim_{U(n)} T_n(g_n) \right) - \left(\text{weak}^* - \lim_{U(n)} T_n(f_n \cdot g_n) \right)
$$

$$
= \left(\text{weak}^* - \lim_{U(n)} (T_n f_n \cdot T_n g_n) \right) - \left(\text{weak}^* - \lim_{U(n)} T_n (f_n \cdot g_n) \right)
$$

$$
= \text{weak}^* - \lim_{U(n)} (T_n f_n \cdot T_n g_n - T_n (f_n g_n)) = 0.
$$

This completes the proof.

Construction of the example

Example. A sequence of finite dimensional (hence amenable) Lipschitz algebras $A_n$ such that $\ell_\infty(A_n)$ is not even weakly amenable.

Proof. Let $K$ be a compact metric space with metric $d(\cdot, \cdot)$ and let $0 < \alpha < 1$. Then Lip$_\alpha(K)$ is the algebra of all complex-valued functions on $K$ for which

$$
g_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}
$$

is finite and lip$_\alpha(K)$ is the subalgebra of those $f$ such that

$$
\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \to 0 \text{ as } d(x, y) \to 0.
$$

Both algebras are equipped with the norm $\|f\|_\alpha = \|f\|_\infty + g_\alpha(f)$. Bade, Curtis and Dales proved in [3] that the algebra lip$_\alpha(K)^\#$ is isometrically isomorphic to Lip$_\alpha(K)$ which has point derivations for every infinite $K$ (and, therefore, is not weakly amenable).

Take $A = \text{lip}_\alpha(I)$, where $I = [0, 1]$ has the usual metric. Then the Banach space $A$ turns out to be isomorphic (in the pure linear sense) to $c_0$, the space of all null sequences [7,15]. This implies that every bilinear operator from $A \times A$ into any Banach space is
jointly weakly continuous on bounded sets [2] and also that $A'$ is separable, which yields (d) and (e).

We now construct the required sequence $A_n$. For each $n$, let $I_n$ be the (finite) subset of $I$ consisting of all points of the form $k/2^n$, for $0 \leq k \leq 2^n$. Put $A_n = \text{lip}_\alpha(I_n)$. Clearly, $A_n$ is amenable for all $n$ since it is isomorphic to the algebra $C(I_n)$.

There is a natural quotient homomorphism $Q_n : A \rightarrow A_n$, given by plain restriction. Obviously, $\|Q_n\| = 1$ for all $n$ (this will be used later). Let $T_n : A_n \rightarrow A$ be defined as follows: for each $f \in A_n$, $T_n(f)$ is the polygonal interpolating $f$ on $I_n$. Clearly, $T_n$ is a linear operator, although it fails to be multiplicative. Since $Q_n \circ T_n$ is the identity on $A_n$ it is clear that $\|T_n f\| \geq \|f\|$ for all $f \in A_n$.

Moreover, $\|T_n\| = 1$ for all $n$. Clearly, $\|T_n(f)\|_{\infty} = \|f\|_{\infty}$, so the point is to show that $\varrho_\alpha(T_n f)$ equals $\varrho_\alpha(f)$. It obviously suffices to see that if $g$ is a polygonal with nodes in $I_n$ then

$$
\varrho_\alpha(g) = \sup_{x \neq y} \frac{|g(y) - g(x)|}{|y - x|^\alpha}
$$

is attained at some $(x, y) \in I_n \times I_n$. This is an amusing exercise in elementary calculus. The solution appears in [13, chapter III, lemma 3.2, p. 203]. Thus, $T_n$ is an into isometry and (a) holds.

Let us verify (b). Obviously, $T_{n+1}(A_{n+1})$ contains $T_n(A_n)$ for each $n$, so that $\cup_n T_n A_n$ is a (not closed) linear subspace of $\text{lip}_\alpha(I)$. We show that $\cup_n T_n A_n$ is (strongly) dense in $\text{lip}_\alpha(I)$. It clearly suffices to show weak density. We claim that for every $f \in \text{lip}_\alpha(I)$ the sequence $T_n Q_n(f)$ converges weakly to $f$ in $\text{lip}_\alpha(I)$. We need some information about weak convergent sequences in the small space of Lipschitz functions.

Consider the operator $\Phi : \text{lip}_\alpha(I) \rightarrow C_0(P^2 \setminus \Delta) \oplus_1 C(I)$ given by $\Phi(f) = (\tilde{f}, f)$, where

$$
\tilde{f}(x, y) = \frac{f(y) - f(x)}{|y - x|^\alpha}
$$

and $\Delta$ is the diagonal of $P^2$. Clearly, it is an isometric embedding, so that the weak topology in $\text{lip}_\alpha(I)$ is the relative weak topology as a subspace of $C_0(P^2 \setminus \Delta) \oplus_1 C(I)$. On the other hand, weakly null sequences in $C_0(\Omega)$ spaces are bounded sequences pointwise convergent to zero. Hence $f_n \rightarrow f$ weakly in $\text{lip}_\alpha(I)$ if and only if $(f_n)$ is bounded and $f_n(x) \rightarrow f(x)$ for all $x \in I$, and this happens if and only if $(f_n)$ is bounded and $f_n(x) \rightarrow f(x)$ for all $x$ in some dense subset of $I$.

But, for $f \in \text{lip}_\alpha(I)$ the sequence $(T_n Q_n(f))$ is bounded (by the norm of $f$) and converges pointwise to $f$ on $\cup_n I_n$. This proves our claim. So, (b) also holds.

It only remains to verify (c). Take $(f_n), (g_n) \in \ell_\infty(A_n)$. Then $T_n(f_n \cdot T_n(g_n) - T_n(f_n \cdot g_n)$ is weakly null in $A$ if and only if for every ultrafilter $V$ on $\mathbb{N}$ one has

$$
\lim_{V(n)} (T_n(f_n) \cdot T_n(g_n) - T_n(f_n \cdot g_n)) = 0
$$

in the weak* topology of $A'' = \text{Lip}_\alpha(I)$. Take $x \in \cup_n I_n$ and let $\delta_x$ be the associated
evaluation functional. Then,
\[
\left\langle \text{weak}^* \lim_{V(n)} T_n(f_n \cdot g_n), \delta_x \right\rangle = \lim_{V(n)} \left\langle T_n(f_n \cdot g_n), \delta_x \right\rangle \\
= \lim_{V(n)} T_n(f_n \cdot g_n)(x) \\
= \lim_{V(n)} (f_n \cdot g_n)(x) \\
= \lim_{V(n)} (f_n g_n)(x) \\
= \lim_{V(n)} f_n(x) \cdot \lim_{V(n)} g_n(x) \\
= \left\langle \text{weak}^* \lim_{V(n)} T_n(f_n), \delta_x \right\rangle \left\langle \text{weak}^* \lim_{V(n)} T_n(g_n), \delta_x \right\rangle,
\]
so that
\[
\text{weak}^* \lim_{V(n)} T_n(f_n \cdot g_n) = \left( \text{weak}^* \lim_{V(n)} T_n(f_n) \right) \cdot \left( \text{weak}^* \lim_{V(n)} T_n(g_n) \right).
\]
Since the product of \( \text{Lip}_\alpha(I) \) is jointly weakly* continuous on bounded sets, the right hand side of the preceding equation becomes
\[
\text{weak}^* \lim_{V(n)} (T_n(f_n) \cdot T_n(g_n)),
\]
which completes the proof of (c).

Thus, the Theorem yields a surjective homomorphism \( \ell_\infty(A_n) \to A'' \), which shows that \( \ell_\infty(A_n) \) is not weakly amenable and completes the proof.

**Concluding remarks**

As the referee pointed out, it is implicit in [14] that there are finite dimensional (hence amenable) \( C^* \)-algebras \( A_n \) for which \( \ell_\infty(A_n) \) fails to be amenable. To see this, let \( H \) be a separable Hilbert space with a fixed orthonormal basis and let \( H_n \) be the subspace spanned by the first \( n \) elements of the basis. Write \( i_n \) for the obvious inclusion of \( H_n \) into \( H \) and \( \pi_n \) for the obvious projection of \( H \) onto \( H_n \). Take \( A_n = L(H_n) \), the algebra of all operators on \( H_n \) and \( A = K(H) \), the algebra of all compact operators on \( H \). Then \( L(H_n) \) embeds isometrically as a subalgebra in \( A \) taking \( T_n(L) = i_n \circ L \circ \pi_n \). Although (d) fails, it is clear from the proof of the Theorem that \( \Psi \) is still an onto operator from \( \ell_\infty(L(H_n)) \) onto \( K(H)'' = L(H) \). Moreover the map \( \Phi : L(H) \to \ell_\infty(L(H_n)) \) given by \( \Phi(T) = (\pi_n \circ T \circ i_n) \) is a right inverse for \( \Psi \) and \( L(H) \) is thus a complemented subspace of \( \ell_\infty(L(H_n)) \). This implies that \( \ell_\infty(L(H_n)) \) lacks the approximation property and cannot be amenable (see [14] and references therein).

Needless to say, our example is far simpler since the existence of point derivations in \( \text{Lip}_\alpha(I) \) is a straightforward consequence of the Banach-Alaoglu theorem.

It follows from the remarks made after the Theorem that if \( A \) is \text{lip}_\alpha(I) \), then there is a surjective homomorphism from \( \ell_\infty(A) \) onto \( A'' \). Hence \( \ell_\infty(A) \) fails to be amenable and the same occurs with any ultrapower \( A_V \) (with respect to a non-trivial ultrafilter \( V \) on \( \mathbb{N} \)) since the quotient mapping constructed in the Theorem factorizes throughout the natural homomorphism \( \ell_\infty(A) \to A_V \).
It would be interesting to study Banach algebras which are “super-amenable” in the sense of having amenable ultrapowers. A reasonable conjecture appears to the that \( A \) is super-amenable if and only of \( A^\sigma \) is amenable. Note that, in view of [14, theorem 2.5], the conjecture is true for \( C^* \)-algebras. See [9] for some (loosely) related results.

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REFERENCES