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The Singular Case in the Stability of Additive Functions

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0. INTRODUCTION

This paper deals with the "singular case" in the stability of additive functions. We are interested in mappings ω: $Z \to Y$ acting between (quasi) Banach spaces which are "quasi-additive" in the sense of satisfying an estimate

$$(1) \qquad \|\omega(x+y) - \omega(x) - \omega(y)\|_Y \le \varepsilon(\|x\|_Z + \|y\|_Z)$$

for some $\varepsilon > 0$ and all $x, y \in Z$.

Those who do not know why this is called the singular case are invited to peruse the book [7]. Suffice it to recall here that, putting together theorems 1.1, 2.1, and 2.2 therein, one has the following starting result.

THEOREM 1 (Hyers–Rassias–Gajda). *Let $p \neq 1$ be a fixed real number. Suppose ω: $Z \to Y$ is a mapping (acting between Banach spaces) satisfying an estimate*

$$\|\omega(x+y) - \omega(x) - \omega(y)\|_Y \le \varepsilon(\|x\|_Z^p + \|y\|_Z^p) \qquad (x, y \in Z)$$

for some $\varepsilon > 0$. Then there is a unique additive mapping a: $Z \to Y$ such that

$$\|\omega(x) - a(x)\|_Y \le |2^{p-1} - 1|^{-1}\varepsilon\|x\|_Z^p \quad (x \in Z).$$

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Thus, one may wonder if, given a quasi-additive mapping, there exists an additive map a: $Z \to Y$ such that $\|\omega(x) - a(x)\|_Y \le K\|x\|_Z$ for some K and all $x \in Z$.

Simple examples show that this is not the case. Actually, if θ: $\mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then the Kalton–Peck map ω_θ: $\mathbb{R} \to \mathbb{R}$ given by $\omega_\theta(t) = t\theta(\log_2 |t|)$ (and $\omega_\theta(0) = 0$) satisfies (1) for $\varepsilon = \mathrm{Lip}(\theta)$, the Lipschitz constant of θ. However, it is easily seen that an additive function a: $\mathbb{R} \to \mathbb{R}$ fulfilling $|\omega(t) - a(t)| \le K|t|$ for all $t \in \mathbb{R}$ exists if and only if θ is bounded on $\mathbb{R}$. Other counterexamples were found by Bourgin, Johnson, Gajda, and Rassias and Šemrl (see [7, p. 24] and the references therein).

Nevertheless, some results on the behavior of quasi-additive maps have been obtained under regularity hypotheses. For instance, Kalton and Peck proved in [11] that if ω: $\mathbb{R} \to \mathbb{R}$ satisfies (1) and is continuous at zero, then there is a Lipschitz function θ: $\mathbb{R} \to \mathbb{R}$, with $\mathrm{Lip}(\theta) \le \varepsilon$, such that $|\omega(t) - t\theta(\log_2 |t|)| \le K\varepsilon|t|$, where K is an absolute constant.

Also, Rassias and Šemrl proved in [15] that if ω is a quasi-additive map acting between finite-dimensional spaces which is bounded on the unit ball of Z (which is always the case if ω is continuous at zero), then, for each $p > 0$, there exists a constant M_p such that $\|\omega(x)\| \le M_p\|x\|^{1+p}$ for $\|x\| \ge 1$ and $\|\omega(x)\| \le M_p\|x\|^{1-p}$ for $\|x\| \le 1$.

Thus the relevant question seems to be whether or not quasi-additive maps can be approximated by additive ones in the (weaker) sense that $\omega - a$ is continuous at zero. If so, we will say that ω is approximable.

We now explain the organization of the paper and summarize our main results.

Section 1 contains the main reduction result. It is proved that if a quasi-additive map ω: $Z \to Y$ is approximable on some dense subgroup of Z, then it is approximable on the whole of Z (provided Y is complete).

Section 2 deals with quasi-additive maps on the line. We prove that each quasi-additive map from the line into a quasi-Banach space is approximable on the dyadic numbers, that is, the subgroup of all real numbers having a finite expansion of the form $t = \sum_k \epsilon_k 2^k$, where $\epsilon_k \in \{-1, 0, 1\}$ for all $k \in \mathbb{Z}$. We conclude that each quasi-additive map from the line (or a finite-dimensional space) into a quasi-Banach space is approximable. Actually, we obtain that a vector-valued quasi-additive map on the line cannot be too far from the additive ones: suppose ω: $\mathbb{R} \to Y$ satisfies (1), where Y is a Banach space. Then there exists an additive map a: $\mathbb{R} \to Y$ such that

$$\|\omega(t) - a(t)\| \le 2\varepsilon|t|(|\log_2 t| + 4).$$

A similar result is proved for quasi-Banach spaces.

Also, we classify all quasi-additive maps from the line into a given Banach space, up to an asymptotic term. If ω: $\mathbb{R} \to Y$ is quasi-additive, then there are an additive map a and a Lipschitz function $\theta\colon \mathbb{R} \to Y$, with $\mathrm{Lip}(\theta) \le \varepsilon$, such that

$$\|\omega(t) - a(t) - t\theta(\log_2 |t|)\| \le 19\varepsilon|t|.$$

In Section 3 we study quasi-additive maps on infinite-dimensional (quasi) Banach spaces. After establishing the key connection between quasi-additive maps and the theory of extensions of (quasi) Banach spaces, we prove that a (quasi) Banach space is a K-space if and only if it has only approximable quasi-additive (scalar-valued) functions. From this, it follows that quasi-additive functions on most classical (quasi) Banach spaces are approximable.

1. THE RESTRICTION LEMMA

The main result in this section is that a quasi-additive mapping $\omega\colon Z \to Y$ is approximable if and only if it is approximable on some dense subgroup of Z. This is somewhat surprising since, at first sight, quasi-additive maps have nothing to do with continuity. From now on, we shall work with quasi-Banach spaces. Let us recall the corresponding definitions. Let X be a linear space. A quasi-norm is a real-valued function on X satisfying

- $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$.
- $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in X$.
- There is a constant Δ_X such that $\|x + y\| \le \Delta_X(\|x\| + \|y\|)$ for all $x, y \in X$.

A quasi-normed space is a linear space together with a specified quasi-norm. On such a space there is a unique linear topology for which the balls

$$B(\varepsilon) = \{x \in X\colon \|x\| \le \varepsilon\} \quad (\varepsilon > 0)$$

are a neighborhood base at zero. This topology is locally bounded (that is, it has a bounded neighborhood of zero) and, conversely, every locally bounded linear topology comes from a quasi-norm. A quasi-Banach space is a complete, quasi-normed space. It should be noted that a quasi-norm need not be continuous with respect to its own topology. For instance, the map $\mathbb{R}^2 \to \mathbb{R}$ given by

$$\|(x, y)\| = \begin{cases} |x| + |y| & \text{if } \ y \ne 0, \\ 2|x| & \text{if } \ y = 0, \end{cases}$$

is a quasi-norm inducing the usual topology on $\mathbb{R}^2$, but it is discontinuous at every point of the form $(x, 0)$, with $x \ne 0$. The so-called p-norms

($0 < p \leq 1$) are quasi-norms satisfying the inequality $\|x+y\|^p \leq \|x\|^p + \|y\|^p$. Clearly, p-norms are continuous, and in fact, if $\|\cdot\|$ is a p-norm on X, then the formula $d(x, y) = \|y - x\|^p$ defines an invariant metric for X and $\|\cdot\|^p$ is a p-homogeneous F-norm. The Aoki–Rolewicz theorem [19, 12] guarantees that each quasi-norm is equivalent to some p-norm, for some $0 < p \leq 1$.

DEFINITION 1. A mapping $\omega: Z \to Y$ acting between quasi-normed spaces is quasi-additive if it satisfies an estimate

$$\|\omega(x+y) - \omega(x) - \omega(y)\|_Y \leq \varepsilon(\|x\|_Z + \|y\|_Z) \tag{2}$$

for some $\varepsilon > 0$ and all $x, y \in Z$. The least possible constant ε in the above inequality shall be denoted $Q(\omega)$ and referred to as the quasi-additivity constant of ω.

Similarly, if G is a subgroup of Z, we say that a map $\omega: G \to Y$ is quasi-additive if (2) holds true for all $x, y \in G$. Also, we say that ω is approximable on G if there is a (not necessarily continuous) additive map $a: G \to Y$ such that $w - a$ is continuous at the origin of G.

LEMMA 1. *Let $\omega: Z \to Y$ be a quasi-additive map. If (the restriction of) ω is approximable on some dense subgroup of Z and Y is complete, then it is approximable on the whole of Z.*

Proof. Suppose that, for some dense subgroup $G \subset Z$, there is an additive map $a: G \to Y$ such that $\omega(z) - a(z) \to 0$ as $z \to 0$ in G. We prove that the formula

$$a^*(z^*) = \lim_{z \to z^*} (\omega(z^* - z) + a(z)) \qquad (z^* \in Z, z \in G)$$

defines an additive map approximating ω on Z.

The definition makes sense since, for all $z^* \in Z$ and $z, z' \in G$, one has

$$\begin{aligned}
&\lim_{z, z' \to z^*} (\omega(z^* - z) + a(z) - \omega(z^* - z') - a(z')) \\
&\quad = \lim_{z, z' \to z^*} (\omega(z^* - z) - \omega(z^* - z') - \omega(z' - z) + \omega(z' - z) - a(z' - z)) \\
&\quad = \lim_{z, z' \to z^*} (\omega(z^* - z) - \omega(z^* - z') - \omega(z' - z)) \\
&\qquad + \lim_{z, z' \to z^*} (\omega(z' - z) - a(z' - z)) \\
&\quad = \lim_{x, y \to 0} (\omega(x+y) - \omega(x) - \omega(y)) + \lim_{z \to 0} (\omega(z) - a(z)) \\
&\hspace{22em} (x, y \in Z, z \in G) \\
&\quad = 0 \qquad \text{(by quasi-additivity on } Z \text{ and approximability on } G\text{).}
\end{aligned}$$

That a^* extends a is obvious: if $z^* \in G$, one has

$$\begin{aligned} a^*(z^*) &= \lim_{z\to z^*}(\omega(z^*-z)+a(z)) \\ &= a(z^*) + \lim_{z\to z^*}(\omega(z^*-z)-a(z^*-z)) \\ &= a(z^*). \end{aligned}$$

We now prove that a^* is additive. Take $x^*, y^* \in Z$. Then

$$\begin{aligned} &a^*(x^*+y^*)-a^*(x^*)-a^*(y^*) \\ &\quad= \lim_{(x,y)\to(x^*,y^*)}(\omega(x^*+y^*-x-y)-\omega(x^*-x)-\omega(y^*-y)) \\ &\quad= 0, \end{aligned}$$

by quasi-additivity of ω on Z.

Finally, let us verify that $\omega - a^*$ is continuous at the origin of Z,

$$\begin{aligned} &\lim_{z^*\to 0}(\omega(z^*)-a^*(z^*)) \\ &\quad= \lim_{z^*\to 0}\left(\lim_{z\to z^*}(\omega(z^*)-\omega(z^*-z)-a(z))\right) \\ &\quad= \lim_{z^*,z\to 0}(\omega(z^*)-\omega(z^*-z)-\omega(z)+\omega(z)-a(z)) \qquad (z^*\in Z, z\in G) \\ &\quad= \lim_{z^*,z\to 0}(\omega(z^*)-\omega(z^*-z)-\omega(z))+\lim_{z\to 0}(\omega(z)-a(z)) \quad (z^*\in Z, z\in G) \\ &\quad= 0 \qquad \text{(since } \omega \text{ is quasi-additive on } Z \text{ and approximable by } a \text{ on } G). \end{aligned}$$

This completes the proof. ∎

2. QUASI-ADDITIVE MAPS ON THE DYADIC GROUP

We now prove the main result of the paper.

THEOREM 2. *Let $\omega: Z \to Y$ be a quasi-additive map, where Z is a finite-dimensional (quasi-normed) space and Y is a quasi-Banach space. Then there is an additive map $a: Z \to Y$ such that $\omega - a$ is continuous at the origin of Z.*

The following result reduces the proof of Theorem 2 to the case in which Z is one-dimensional. Its proof is straightforward and is left to the reader.

LEMMA 2. *Let Z_1, Z_2 and Y be quasi-normed spaces and consider the product space $Z = Z_1 \times Z_2$ equipped with any product quasi-norm. Let $\omega: Z \to Y$ be a quasi-additive map and write ω_i for the restriction of ω to Z_i, for $i = 1, 2$. Then each ω_i is quasi-additive. Moreover, ω is approximable if and only if each ω_i is.*

Let Δ denote the dyadic group, that is, the group of all real numbers having an expansion of the form

$$t = \sum_{k \in \mathbb{Z}} \epsilon_k 2^k,$$

where $\epsilon_k \in \{-1, 0, 1\}$ and $\epsilon_k = 0$ for all but finitely many k's. It is clear that Δ is a dense subgroup of $\mathbb{R}$. So, in view of Lemma 1, Theorem 2 is an obvious consequence of the following.

THEOREM 3. *Every quasi-additive map from Δ into any quasi-Banach space is continuous at zero.*

We break the proof in three lemmas. Note that by the Aoki–Rolewicz theorem, we may assume that Y is a p-Banach space for some $0 < p \leq 1$.

LEMMA 3. *Let ω: $G \to Y$ be a quasi-additive map, where Y is a p-Banach space and G is a subgroup of a quasi-Banach space Z. Then the odd part of ω*

$$\omega_o(z) = \frac{\omega(z) - \omega(-z)}{2}$$

is quasi-additive, with $Q(\omega_o) \leq 2^{1/p-1} Q(\omega)$. Moreover, $\|\omega(z) - \omega_o(z)\| \leq Q(\omega)\|z\|$ for all $z \in G$. ∎

LEMMA 4 (Kalton [9]). *Let ω: $G \to Y$ as above. Then for all $z_1, \dots, z_n$ in G, one has*

$$\left\| \omega\left(\sum_{k=1}^{n} z_k\right) - \sum_{k=1}^{n} \omega(z_k) \right\|^p \leq Q(\omega)^p \sum_{k=1}^{n} k \|z_k\|^p.$$

Proof. Induction on n. ∎

LEMMA 5. *Let ω: $\Delta \to Y$ be a quasi-additive map, where Y is a p-Banach space. If $\omega(1) = 0$, then*

$$\|\omega(2^k)\|^p \leq Q(\omega)^p |k| 2^{pk} \qquad (k \in \mathbb{Z}).$$

Proof. There is no loss of generality in assuming $Q(\omega) = 1$. For $k = 0$, the result is obvious. We prove it for $k > 0$. Suppose we know that $\|\omega(2^{k-1})\|^p \leq (k-1)2^{p(k-1)}$. Then

$$\begin{aligned} \|\omega(2^k)\|^p &\leq \|\omega(2^k) - 2\omega(2^{k-1})\|^p + 2^p \|\omega(2^{k-1})\|^p \\ &\leq (2^{k-1} + 2^{k-1})^p + 2^p (k-1) 2^{p(k-1)} \\ &= k 2^{pk}. \end{aligned}$$

As for negative k, suppose $\|\omega(2^{k+1})\|^p \leq |k+1| 2^{p(k+1)}$. Then, from $\|\omega(2^{k+1}) - 2\omega(2^k)\|^p \leq 2^{p+1}$, we obtain

$$\|\omega(2^k)\|^p \leq 2^{pk} + |k+1| 2^{pk} = |k| 2^{pk},$$

which completes the proof. ∎

End of the Proof of Theorem 3. We may and do assume that ω is odd. Moreover, there is no loss of generality in assuming that $\omega(1) = 0$ (for, if not, $\tilde{\omega}(t) = \omega(t) - t\omega(1)$ is a quasi-additive map, with $Q(\tilde{\omega}) = Q(\omega)$, vanishing at 1 and $\tilde{\omega} - \omega$ is continuous) and $Q(\omega) = 1$.

Let $t \in \Delta$, with $0 < t < 1$. Then t can be written as

$$t = \sum_{k=l(t)}^{u(t)} \delta_k 2^k,$$

where $\delta_k \in \{0, 1\}$, $\delta_{u(t)} = \delta_{l(t)} = 1$, and $u(t) < 0$. One has

$$\|\omega(t)\|^p \leq \left\| \omega(t) - \sum_{k=l(t)}^{u(t)} \delta_k \omega(2^k) \right\|^p + \sum_{k=l(t)}^{u(t)} \delta_k \|\omega(2^k)\|^p. \tag{3}$$

Let us estimate the second summand in the right-hand side of (3). By Lemma 5, one has

$$\begin{aligned}
\sum_{k=l(t)}^{u(t)} \delta_k \|\omega(2^k)\|^p &\leq \sum_{k=l(t)}^{u(t)} \delta_k |k| 2^{pk} \\
&\leq \sum_{n=0}^{\infty} (n - u(t)) 2^{p(u(t)-n)} \\
&= 2^{pu(t)} \sum_{n=0}^{\infty} (n - u(t)) 2^{-pn} \\
&= 2^{pu(t)} \left(\sum_{n=0}^{\infty} n 2^{-pn} - u(t) \sum_{n=0}^{\infty} 2^{-pn} \right) \\
&= 2^{pu(t)} \left(\frac{2^{-p}}{(1 - 2^{-p})^2} - \frac{u(t)}{1 - 2^{-p}} \right).
\end{aligned}$$

That is,

$$\sum_{k=l(t)}^{u(t)} \delta_k \|\omega(2^k)\|^p \leq \frac{t_0^p}{1 - 2^{-p}} \left(|\log_2 t_0| + \frac{2^{-p}}{1 - 2^{-p}} \right),$$

where $t_0 = 2^{u(t)} \leq t$.

As for the first one, we can apply Lemma 4 to get

$$\begin{aligned}
&\left\| \omega(t) - \sum_{k=l(t)}^{u(t)} \delta_k \omega(2^k) \right\|^p \\
&\quad \leq 2^{pu(t)} + 2 \cdot 2^{p(u(t)-1)} + 3 \cdot 2^{p(u(t)-2)} + 4 \cdot 2^{p(u(t)-3)} + \cdots \\
&\quad = 2^{pu(t)} (1 + 2 \cdot 2^{-p} + 3 \cdot (2^{-p})^2 + 4 \cdot (2^{-p})^3 + \cdots) \\
&\quad = \frac{t_0^p}{(1 - 2^{-p})^2}.
\end{aligned}$$

Putting all this together, we have

$$\|\omega(t)\|^p \leq \frac{t_0^p}{1-2^{-p}}\left(|\log_2 t_0| + \frac{1+2^{-p}}{1-2^{-p}}\right) \qquad (t \in \Delta \cap (0,1)).$$

Taking into account that the function $s \mapsto s^p(|\log_2 s| + (1+2^{-p})/(1-2^{-p}))$ is increasing on $(0,1]$ for all $0 < p \leq 1$, we get

$$\|\omega(t)\|^p \leq \frac{t^p}{1-2^{-p}}\left(|\log_2 t| + \frac{1+2^{-p}}{1-2^{-p}}\right) \qquad (t \in \Delta \cap (0,1)), \tag{4}$$

and, in particular, we have $\omega(t) \to 0$ as $t \to 0^+$ in Δ. Since ω is odd, the proof is complete. ∎

We now prove that the estimate (4) holds for all $t \in \Delta$.

PROPOSITION 1. *Let* $\omega\colon \Delta \to Y$ *be a quasi-additive map, where* Y *is a* p*-Banach space. If* $\omega(1) = 0$, *then*

$$\|\omega(t)\|^p \leq Q(\omega)^p \frac{t^p}{1-2^{-p}}\left(|\log_2 t| + \frac{1+2^{-p}}{1-2^{-p}}\right) \tag{5}$$

for each positive $t \in \Delta$.

Proof. The result has been already proved for $0 < t < 1$. We prove it for $t > 1$. Without loss of generality, we may assume $Q(\omega) = 1$. Write

$$t = 2^n + \delta_{n-1}2^{n-1} + \cdots + \delta_1 2^1 + \delta_0 2^0 + \delta_{-1}2^{-1} + \cdots + \delta_{-l}2^{-l} \qquad (n, l \geq 0,\ \delta_k \in \{0,1\}).$$

As before, one has

$$\begin{aligned}
\|\omega(t)\|^p &\leq \left\|\omega(t) - \sum_{k=-l}^{n} \delta_k \omega(2^k)\right\|^p + \sum_{k=-l}^{n} |k| 2^{pk} \\
&\leq \big(2^{pn} + 2\cdot 2^{p(n-1)} + 3\cdot 2^{p(n-2)} + \cdots + n2^p + (n+1)2^0 \\
&\qquad + (n+2)2^{-p} + (n+3)2^{-2p} + (n+4)2^{-3p} + \cdots\big) \\
&\qquad + \big(n2^{pn} + (n-1)2^{p(n-1)} + (n-2)2^{p(n-2)} + \cdots + 2^p + 0 \\
&\qquad + 2^{-p} + 2\cdot 2^{-2p} + 3\cdot 2^{-3p} + \cdots\big) \\
&= (n+1)2^{pn} + (n+1)2^{p(n-1)} + (n+1)2^{p(n-2)} + \cdots + (n+1)2^p \\
&\qquad + (n+3)2^{-p} + (n+5)2^{-2p} + 3(n+7)2^{-3p} + \cdots \\
&= (n+1)\frac{2^{p(n+1)}}{2^p - 1} + (n+1)\left(\frac{1}{1-2^{-p}} - 1\right) + 2\frac{2^{-p}}{(1-2^{-p})^2}
\end{aligned}$$

$$\begin{aligned}
&\le (\log_2 t + 1)\left(\frac{2^p t^p - 1}{2^p - 1} + \frac{1}{1 - 2^{-p}} + \frac{2^{-p} - 1}{1 - 2^{-p}}\right) + \frac{2^{1-p}}{(1 - 2^{-p})^2} \\
&= \frac{t^p}{1 - 2^{-p}}\left(\log_2 t + 1 + \frac{2^{1-p}}{(1 - 2^{-p})^2}\right) \\
&= \frac{t^p}{1 - 2^{-p}}\left(|\log_2 t| + \frac{1 + 2^{-p}}{1 - 2^{-p}}\right),
\end{aligned}$$

as desired. ∎

In fact, these estimates show that quasi-additive maps on the line are "uniformly approximable". In the following result, we write

$$f_p(t) = \frac{t^p}{1 - 2^{-p}}\left(|\log_2 t| + \frac{1 + 2^{-p}}{1 - 2^{-p}}\right) \tag{6}$$

for all real positive s.

COROLLARY 1. *Let $\omega\colon \mathbb{R} \to Y$ be a quasi-additive odd map, where Y is a p-Banach space. Then there is an additive map $a\colon \mathbb{R} \to Y$ such that*

$$\|w(s) - a(s)\| \le Q(\omega)^p(|s|^p + f_p(|s|)) \qquad (s \in \mathbb{R}),$$

where $Q(\omega)$ is the quasi-additivity constant of ω.

Proof. There is no loss of generality if we suppose $Q(\omega) = 1$. By Theorem 2, there is an additive map a such that $\omega - a$ is continuous at zero. By adding a suitable linear map to a if necessary, we may assume that $\omega - a$ vanishes at 1. Let $\tilde{\omega} = \omega - a$. Clearly, $Q(\tilde{\omega}) = 1$. We have

$$\|\tilde{\omega}(s) - \tilde{\omega}(s - t) - \tilde{\omega}(t)\|^p \le |s - t|^p + |t|^p \qquad (s, t \in \mathbb{R}).$$

Hence,

$$\|\tilde{\omega}(s)\|^p \le |s - t|^p + |t|^p + \|\tilde{\omega}(s - t)\|^p + \|\tilde{\omega}(t)\|^p \qquad (s, t \in \mathbb{R}).$$

And so,

$$\begin{aligned}
\|\tilde{\omega}(s)\|^p &\le \limsup_{t \to s^+}(|s - t|^p + |t|^p + \|\tilde{\omega}(s - t)\|^p + \|\tilde{\omega}(t)\|^p) \\
&\le s^p + f_p(s) \qquad (t \in \Delta),
\end{aligned}$$

which completes the proof. ∎

We now classify all quasi-additive maps from the line into a Banach space up to an asymptotic term. First, note that if $\theta\colon \mathbb{R} \to Y$ is a Lipschitz map (that is, $\|\theta(s) - \theta(t)\| \le \mathrm{Lip}(\theta) \cdot |s - t|$ for all $s, t \in \mathbb{R}$), then the (vector-valued) Kalton–Peck map $\omega_\theta\colon \mathbb{R} \to Y$ given by $\omega_\theta(s) = s\theta(\log_2 |s|)$ is quasi-additive, with $Q(\omega) \le \mathrm{Lip}(\theta)$. To see this, suppose s and t have the same sign. Then

$$\begin{aligned}
&\|\omega_\theta(s+t) - \omega_\theta(s) - \omega_\theta(t)\| \\
&\quad = \|(s+t)\theta(\log_2(s+t)) - s\theta(\log_2 s) - t\theta(\log_2 t)\| \\
&\quad = \|s(\theta(\log_2(s+t)) - \theta(\log_2 s)) + t(\theta(\log_2(s+t)) - \theta(\log_2 t))\| \\
&\quad = (s+t)\left\| \frac{s}{s+t}(\theta(\log_2(s+t)) - \theta(\log_2 s)) \right. \\
&\qquad\qquad \left. + \frac{t}{s+t}(\theta(\log_2(s+t)) - \theta(\log_2 t)) \right\| \\
&\quad \le \mathrm{Lip}(\theta)(s+t)\left(\left| \frac{s}{s+t} \log_2 \frac{s}{s+t} \right| + \left| \frac{t}{s+t} \log_2 \frac{t}{s+t} \right| \right) \\
&\quad \le \mathrm{Lip}(\theta)(|s| + |t|),
\end{aligned}$$

since the maximum value of $|t \log_2 t|$ on $(0, 1)$ is $1/2$ (which is attained at $t = 1/2$).

Finally, if s and t have different signs, we may and do assume that $s > 0$, $t < 0$, and $s + t > 0$. Taking into account that ω_θ is odd, we have

$$\begin{aligned}
\|\omega_\theta(s+t) - \omega_\theta(s) - \omega_\theta(t)\| &= \|\omega_\theta(s) - \omega_\theta(-t) - \omega_\theta(s+t)\| \\
&\le \mathrm{Lip}(\theta)(|-t| + |s+t|) \\
&= \mathrm{Lip}(\theta)(|s| + |t|).
\end{aligned}$$

THEOREM 4 (Compare to Kalton–Peck [11, theorem 3.7]). *Let $\omega\colon \mathbb{R} \to Y$ be a quasi-additive map, where Y is a Banach space. Then there exist an additive map $a\colon \mathbb{R} \to Y$ and a Lipschitz map $\theta\colon \mathbb{R} \to Y$, with $\mathrm{Lip}(\theta) \le Q(\omega)$, such that*

$$\|\omega(s) - a(s) - s\theta(\log_2 |s|)\| \le 19Q(\omega)|s| \qquad (s \in \mathbb{R}).$$

Proof. There is no loss of generality in assuming that $Q(\omega) = 1$. First, suppose ω odd. By Theorem 2, there is an additive map $a\colon \mathbb{R} \to Y$ such that $\omega - a$ is continuous at zero, with $\omega(1) = a(1)$. Put $\tilde{\omega} = \omega - a$. Clearly, $\|\tilde{\omega}(2^{k+1}) - 2\tilde{\omega}(2^k)\| \le 2^{k+1}$ for all $k \in \mathbb{Z}$. Hence,

$$\left\| \frac{\tilde{\omega}(2^{k+1})}{2^{k+1}} - \frac{\tilde{\omega}(2^k)}{2^k} \right\| \le 1 \qquad (k \in \mathbb{Z}).$$

Define $\theta\colon \mathbb{Z} \to Y$, taking $\theta(k) = 2^{-k}\tilde{\omega}(2^k)$. Clearly, θ is 1-Lipschitz and, therefore, it can be extended to a 1-Lipschitz map $\theta\colon \mathbb{R} \to Y$. Let ω_θ be the Kalton–Peck map induced by θ and let us estimate $\omega^* = \tilde{\omega} - \omega_\theta$. It is clear that ω^* is continuous at zero and odd (both $\tilde{\omega}$ and ω_θ are) and also that $Q(\omega^*) \leq 2$. Take a positive $t \in \Delta$ and write $t = \sum_{k\leq n} \delta_k 2^k$, with $\delta_k \in \{0, 1\}$ and $\delta_n = 1$.

Since $\omega^*(2^k) = 0$ for all $k \in \mathbb{Z}$, we have

$$\|\omega^*(t)\| = \left\|\omega^*(t) - \sum_{k\leq n} \delta_k \omega^*(2^k)\right\| \leq Q(\omega^*) \sum_{j=1}^{\infty} j 2^{n-j+1} = \frac{2^{n+1}}{(1-2^{-1})^2} \leq 8t.$$

Arguing as in the proof of Corollary 1, one obtains $\|\omega^*(t)\| \leq Q(\omega^*)(|t| + 8|t|) \leq 18|t|$ for all real t, which ends the proof for odd ω. An appeal to Lemma 3 ends the proof. ∎

In view of the result just proved, we obtain that if ω is a quasi-additive map from the line into a Banach space Y, then there is an additive map $a\colon \mathbb{R} \to Y$ such that

$$\|\omega(s)-a(s)\| \leq 19Q(\omega)|s| + Q(\omega)|s||\log_2 s| = Q(\omega)|s|(|\log_2 s| + 19), \tag{7}$$

while the case $p = 1$ of Corollary 1 yields

$$\begin{aligned} \|\omega(s) - a(s)\| &\leq 2Q(\omega)|s| + 2Q(\omega)|s|(|\log_2 s| + 3) \\ &= 2Q(\omega)|s|(|\log_2 s| + 4). \end{aligned} \tag{8}$$

Curiously enough, the estimate (7) is better than (8) for s either small or very large, while (8) is better than (7) for s near 1.

3. QUASI-ADDITIVE MAPS ON INFINITE-DIMENSIONAL SPACES

It will be apparent for those acquainted with the theory of extensions of (quasi) Banach spaces that the results of the preceding section cannot be extended to the case in which Z is infinite-dimensional.

Let us recall the minimal background that one needs to understand what follows. Fix two quasi-Banach spaces, Z and Y. An extension of Z by Y (in that order!) is a topological linear space X containing Y as a closed subspace in such a way that the corresponding quotient is (linearly isomorphic with) Z. By a result of Roelcke [18] (see also [9, theorem 1.1]), such an X must be also a quasi-Banach space. We can regard an extension as a short exact sequence

$$0 \longrightarrow Y \stackrel{i}{\longrightarrow} X \stackrel{q}{\longrightarrow} Z \longrightarrow 0 \tag{9}$$

in which the arrows represent (linear, continuous) operators. An extension is said to be trivial (or to split) if Y is a complemented subspace of X (that is, $X = Y \oplus Z$). It was known from the very beginning of the theory that extensions can be represented by homogeneous quasi-additive maps (quasi-linear maps, for short). To see this, suppose $\omega\colon Z \to Y$ is a quasi-linear map. Then we can construct an extension of Z by Y as follows. Consider the functional defined on the product $Y \times Z$ by

$$\|(y, z)\|_\omega = \|y - \omega(z)\| + \|z\|. \tag{10}$$

It is easily verified that $\|(\cdot,\cdot)\|_\omega$ is a quasi-norm: quasi-additivity of ω implies the "weak" triangle inequality for $\|(\cdot,\cdot)\|_\omega$ and homogeneity of ω gives $\|(\lambda y, \lambda z)\|_\omega = |\lambda| \cdot \|(y, z)\|_\omega$ for all $(y, z) \in Y \times Z$ and $\lambda \in \mathbb{K}$. From now on, we shall write $Y \oplus_\omega Z$ for the product space equipped with the quasi-norm given by (10). Observe that the map $Y \to Y \oplus_\omega Z$ given by $y \mapsto (y, 0)$ is an isometric embedding and that $(Y \oplus_\omega Z)/Y$ is isometric to Z. Therefore, $Y \oplus_\omega Z$ is an extension of Z by Y and, in particular, it is complete.

Actually, all extensions come from quasi-linear maps: given an extension (9), we can obtain a quasi-linear map $\omega\colon Z \to Y$ as follows. First, by the open mapping theorem, there is a homogeneous map $\varrho\colon Z \to X$ such that $q(\varrho(z)) = z$ (that is, a "selection" for q) with $\|\varrho(z)\| \le K\|z\|$ for some constant K and all $z \in Z$. On the other hand, q admits a (generally discontinuous) linear selection $L\colon Z \to X$. Put $\omega\colon \varrho - L$. Then ω takes values in Y (instead of X) since $q(\omega(z)) = q(\varrho(z)) - q(L(z)) = z - z = 0$. Moreover, $\omega\colon Z \to Y$ is quasi-linear since it is obviously homogeneous and satisfies

$$\begin{aligned}\|\omega(x+y) - \omega(x) - \omega(y)\| &= \|\varrho(x+y) - \varrho(x) - \varrho(y)\| \\ &\le \Delta_X(\|\varrho(x+y)\| + \Delta_X(\|\varrho(x)\| + \|\varrho(y)\|)) \\ &\le K\Delta_X(\|x+y\| + \Delta_X(\|x\| + \|y\|)) \\ &\le K\Delta_X(\Delta_z(\|x\| + \|y\|) + \Delta_X(\|x\| + \|y\|)) \\ &\le K\Delta_X(\Delta_Z + \Delta_X)(\|x\| + \|y\|).\end{aligned}$$

It can be proved (see [9] or [5]) that the extension induced by ω is equivalent to (9) in the sense that there exists a topological isomorphism $X \to Y \oplus_\omega Z$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\omega Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}.$$

Thus, extensions of Z by Y are in correspondence with quasi-linear maps $Z \to Y$. Moreover, the splitting of extensions is closely related to the asymptotic stability of the associated quasi-linear maps:

LEMMA 6 (Kalton [9]). *Let* $\omega: Z \to Y$ *be a quasi-linear map acting between quasi-Banach spaces. Then the associated extension* $0 \to Y \to Y \oplus_\omega Z \to Z \to 0$ *is trivial if and only if there is a true linear (but not necessarily continuous!) map* $L: Z \to Y$ *such that* $\|\omega(z) - L(z)\| \leq C\|z\|$ *for some* C *independent on* $z \in Z$.

In particular, all extensions of Z by Y are trivial if and only if all quasi-linear maps $\omega: Z \to Y$ are asymptotically linear in the sense of Kalton's Lemma. The attentive reader will notice that if ω is a quasi-linear map admitting an additive approximation a near the origin (in the sense that $\omega - a$ is continuous at zero), then a is already linear over $\mathbb{R}$ (by homogeneity of ω) and one actually has an estimate $\|\omega(z) - a(z)\| \leq C\|z\|$ for some constant $C \geq 0$. Hence each nontrivial extension of Z by Y is associated to a quasi-linear (hence quasi-additive) map $Z \to Y$ which cannot be approximated by any additive map. In particular, every uncomplemented subspace Y of a (quasi) Banach space X gives rise to a non-approximable quasi-additive map $X/Y \to Y$.

The main result of this section relates quasi-additive maps between quasi-Banach spaces to the linear theory and will provide us with many infinite-dimensional (quasi) Banach spaces Z and Y having only approximable quasi-additive maps.

THEOREM 5. *Let* Z *and* Y *be quasi-Banach spaces. The following statements are equivalent:*

(a) *Every quasi-additive map* $\omega: Z \to Y$ *is approximable.*

(b) *Every quasi-linear map* $\omega: Z \to Y$ *is asymptotically linear.*

(c) *Every extension* $0 \to Y \to X \to Z \to 0$ *splits.*

The implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) have been already proved. To show that (c) implies (a), we need to revisit some ideas from the theory of "twisted sums."

Suppose $\omega: Z \to Y$ is a quasi-additive odd map between quasi-Banach spaces. As in the linear case, we consider the function on $Y \times Z$ given by

$$\|(y, z)\|_\omega = \|y - \omega(z)\| + \|z\|.$$

It is clear that $\|\cdot\|_\omega$ is symmetric (that is, an even function). Moreover, one has

$$\begin{aligned}&\|(y + y', z + z')\|_\omega \\ &\qquad = \|y - \omega(z) + y' - \omega(z') + \omega(z) + \omega(z') - \omega(z + z')\| + \|z + z'\|\end{aligned}$$

$$
\begin{aligned}
&\leq \Delta_Y(\|y-\omega(z)+y'-\omega(z')\| + \|\omega(z+z')-\omega(z)-\omega(z')\|) \\
&\quad + \Delta_Z(\|z\|+\|z'\|) \\
&\leq \Delta_Y^2(\|y-\omega(z)\| + \|y'-\omega(z')\|) + (\Delta_Y Q(\omega)+\Delta_Z)(\|z\|+\|z'\|) \\
&\leq \max\{\Delta_Y^2, \Delta_Y Q(\omega)+\Delta_Z\}\cdot(\|(y,z)\|_\omega + \|(y',z')\|_\omega),
\end{aligned}
$$

so that $\|\cdot\|_\omega$ is a group quasi-norm (in the terminology of [14] and [3]), with constant $\max\{\Delta_Y^2, \Delta_Y Q(\omega)+\Delta_Z\}$. This clearly implies that there is a unique group topology on the additive group $Y\times Z$ for which the sets

$$B(\varepsilon) = \{(y,z)\colon \|(y,z)\|_\omega \leq \varepsilon\} \qquad (\varepsilon > 0)$$

form a neighborhood base at the origin. Let $Y\oplus_\omega Z$ denote the corresponding topological group. This is consistent with the notation used in the linear case. Again, $Y\oplus_\omega Z$ contains a closed subgroup topologically isomorphic to Y (via $i\colon y\mapsto (y,0)$) and the quotient group $(Y\oplus_\omega Z)/Y$ is topologically isomorphic to Z (via $q\colon (y,z)\mapsto z$). Hence

$$0 \longrightarrow Y \stackrel{i}{\longrightarrow} Y\oplus_\omega Z \stackrel{q}{\longrightarrow} Z \longrightarrow 0$$

is a short exact sequence of topological groups with relatively open continuous homomorphisms. We now give a nonhomogeneous version of Kalton's Lemma 6.

LEMMA 7. *Let $\omega\colon Z\to Y$ be a quasi-additive odd map between quasi-Banach spaces. The following statements are equivalent*:

(a) *ω is approximable by some additive map $a\colon Z\to Y$ (in the sense that $\omega - a$ is continuous at zero).*

(b) *There is a continuous group homomorphism which is a left inverse for the natural embedding $i\colon Y\to Y\oplus_\omega Z$.*

Proof. We first prove that (b) implies (a). Suppose $P\colon Y\oplus_\omega Z\to Y$ is a continuous projection homomorphism. Clearly, we can write $P(y,z) = y - a(z)$, where a is a (not necessarily continuous) additive map from Z to Y. Nevertheless, P is continuous and so, $\|P(y,z)\|\to 0$ as $\|(y,z)\|_\omega\to 0$. Taking $y=\omega(z)$, we have $P(y,z)=\omega(z)-a(z)$ and $\|(y,z)\|_\omega = \|z\|$. Thus,

$$\|\omega(z)-a(z)\|_\omega \to 0 \qquad \text{as} \quad \|z\|\to 0,$$

which shows that $\omega - a$ is continuous at zero.

As for the converse, suppose some additive map $a\colon Z\to Y$ such that $\omega - a$ is continuous at zero exists. Define $P(y,z) = y - a(z)$. It is clear that P is a left inverse homomorphism for the inclusion $Y \to Y \oplus_\omega Z$. We prove it is continuous. It clearly suffices to show continuity at zero. One has

$$\begin{aligned}\lim_{\|(y,z)\|_\omega \to 0} \|P(y,z)\| &= \lim_{\|(y,z)\|_\omega \to 0} \|y - a(z)\| \\ &\leq \Delta_Y \left(\lim_{\|(y,z)\|_\omega \to 0} \|y - \omega(z)\| + \lim_{\|(y,z)\|_\omega \to 0} \|\omega(z) - a(z)\| \right) \\ &= \Delta_Y \lim_{\|z\| \to 0} \|\omega(z) - a(z)\| = 0,\end{aligned}$$

which completes the proof. ∎

Lemma 8. *With the above notations, if Z and Y are complete, then $Y \oplus_\omega Z$ is complete.*

Proof. Clearly, the topology of $Y \oplus_\omega Z$ is metrizable. Thus, it suffices to prove that $Y \oplus_\omega Z$ is sequentially complete. Let (y_n, z_n) be a Cauchy sequence in $Y \oplus_\omega Z$. Then (z_n) is Cauchy in Z and converges, say to z. Obviously, $(\omega(z_n - z), z_n - z)$ converges to zero in $Y \oplus_\omega Z$. Therefore,

$$(y_n, z_n) - (0, z) - (\omega(z_n - z), z_n - z) = (y_n - \omega(z_n - z), 0)$$

is a Cauchy sequence in $Y \oplus_\omega Z$. Hence $y_n - \omega(z_n - z)$ converges to some point y in Y. Finally, the starting sequence converges to (y, z) in $Y \oplus_\omega Z$ since

$$\lim_n \|(y_n, z_n) - (y, z)\| = \lim_n \|y_n - \omega(z_n - z) - y\| + \lim_n \|z_n - z\| = 0.$$

∎

The main step in the proof of Theorem 5 appears now.

Proposition 2. *The topological group $Y \oplus_\omega Z$ admits an outer multiplication by real numbers which makes it into a topological vector space.*

Proof. Once again, we assume that Y is a p-Banach space. Clearly, $Y \oplus_\omega Z$ is a vector space on $\mathbb{Q}$. We claim it is a *topological* vector space over $\mathbb{Q}$, that is, that the multiplication

$$\mathbb{Q} \times (Y \oplus_\omega Z) \to Y \oplus_\omega Z \qquad (q, (y, z)) \mapsto (qy, qz) \tag{11}$$

is (jointly) continuous. This amounts to verifying its continuity at the origin of $\mathbb{Q} \times (Y \oplus_\omega Z)$ together with the following two conditions (see [1, Châpitre III, remarque suivant la définition 3]):

- For each fixed $x \in Y \oplus_\omega Z$, the map $q \mapsto qx$ is continuous at zero.
- For each fixed $q \in \mathbb{Q}$, the map $x \mapsto qx$ is continuous at the origin of $Y \oplus_\omega Z$.

Since $\|(qy, qz)\|_\omega \leq |q|\|(y, z)\|_\omega + \|q\omega(z) - \omega(qz)\|$ and $\|(y, z)\|_\omega \geq \|z\|_Z$, one only has to show that

$$\|q\omega(z) - \omega(qz)\| \to 0$$

in the following three cases:

- as $(q, z) \to (0, 0) \in \mathbb{Q} \times Z$;
- as $q \to 0$, for each fixed $z \in Z$; and
- as $z \to 0$, for each fixed $q \in \mathbb{Q}$.

For $z \in Z$, define $\omega_z: \mathbb{R} \to Y$ by $\omega_z(t) = \omega(tz)$. It is clear that each ω_z is quasi-additive, with $Q(\omega_z) \leq Q(\omega)\|z\|$. By Corollary 1, for each $z \in Z$, we can select an additive map $a_z: Z \to Y$ such that

$$\|\omega_z(t) - a_z(t)\|^p \leq Q(\omega)^p \|z\|^p (|t|^p + f_p(|t|)) \qquad (t \in \mathbb{R}),$$

where f_p is the function defined in (6). Since each a_z is $\mathbb{Q}$-linear, we have

$$\begin{aligned} \|q\omega(z) - \omega(qz)\|^p &= \|q\omega_z(1) - \omega_z(q)\|^p \\ &\leq \|q\omega_z(1) - qa_z(1) + a_z(q) - \omega_z(q)\|^p \\ &\leq \|q\omega_z(1) - qa_z(1)\|^p + \|a_z(q) - \omega_z(q)\|^p \\ &\leq Q(\omega)^p \|z\|^p (|q|^p (1 + f_p(1)) + |q|^p + f_p(|q|)). \end{aligned}$$

This proves our claim.

Since $Y \oplus_\omega Z$ is complete (see Lemma 8), it is clear (or see [1, Châpitre III, Section 6.6 et le théorème 1 du Section 6.5]) that the outer multiplication in (11) extends to a (jointly continuous) multiplication

$$\star: \mathbb{R} \times (Y \oplus_\omega Z) \to Y \oplus_\omega Z,$$

which makes $Y \oplus_\omega Z$ into a topological vector space. ∎

We remark that this product by real numbers $\star$ in $Y \oplus_\omega Z$ need not be the "obvious one" $t(y, z) = (ty, tz)$, which is generally discontinuous. We suggest identifying $\star$ on $\mathbb{R} \oplus_\omega \mathbb{R}$ when ω is a discontinuous additive function on the line.

End of the Proof of Theorem 5. It remains to prove the implication (c) $\Rightarrow$ (a). Let $\omega: Z \to Y$ be a quasi-additive map. We want to see that ω is approximable. We may assume that ω is odd. In this case, one only has to show that there is a continuous group homomorphism $P: Y \oplus_\omega Z \to Y$ which is a left inverse for the natural embedding $i: Y \to Y \oplus_\omega Z$. Let $\star$ be the outer multiplication in $Y \oplus_\omega Z$ given by Proposition 2. It is clear that $\star$ extends the product of Y. Thus, the topological linear space $(Y \oplus_\omega Z, \star)/Y$ is (isomorphic to) Z, as a topological group: it follows that it is linearly homeomorphic with Z, although it is not Z since $t \star (0, z)$ may be different from $(0, tz)$.

Hence $(Y \oplus_\omega Z, \star)$ is itself locally bounded and the sequence

$$0 \to Y \to (Y \oplus_\omega Z, \star) \to (Y \oplus_\omega Z, \star)/Y \to Z \to 0$$

represents a quasi-Banach extension of Y by Z. If (a) holds, this extension must be trivial; that is, there is a linear operator P: $(Y \oplus_\omega Z, \star) \to Y$ such that $P \circ i$ is the identity on Y. The trivial observation that P is a continuous homomorphism for the underlying additive structures completes the proof. ∎

Following [9, 10, 12], let us say that a quasi-Banach space Z is a K-space if the condition (c) of Theorem 5 holds when $Y = \mathbb{K}$. The main examples of K-spaces are supplied by Kalton and co-workers. For instance, (infinite-dimensional) $\mathscr{L}_p$ spaces $(0 < p \leq \infty)$ are K-spaces if and only if $p \neq 1$ (see [9, 10, 13, 16, 17]). In particular, c_0 and ℓ_∞ are. Also, B-convex spaces (that is, Banach spaces having nontrivial type $p > 1$) are K-spaces [9]. This includes all super-reflexive Banach spaces. Also, quasi-Banach extensions of K spaces are K-spaces and so are quotients of Banach K-spaces [4].

Corollary 2. *Let Z be a (real) quasi-Banach K-space. Then every quasi-additive function ω: $Z \to \mathbb{R}$ is approximable.*

Of course, Theorem 5 can be used to obtain new results on stability from old ones on extensions. Sample results: every quasi-additive map from L_p $(0 < p < 1)$ into a q-Banach space, with $q > p$, is approximable (see [9]); every quasi-additive map from a separable quasi-Banach K-space into c_0 is approximable (this is Sobczyk's theorem [21]). In particular, every quasi-additive map from c_0 into itself is approximable. And so on. We refrain from entering into further details here.

We remark, however, that the proofs given in [9, 16] that ℓ_1 (hence L_1) fails to be a K-space explicitly construct quasi-linear maps ω which are not asymptotically linear (hence nonapproximable). The fact that the induced extensions

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus_\omega \ell_1 \longrightarrow Y \longrightarrow 0$$

are not trivial shows that $\mathbb{R} \oplus_\omega \ell_1$ cannot be locally convex (otherwise the Hahn–Banach extension theorem would give a continuous projection of $\mathbb{R} \oplus_\omega \ell_1$ onto $\mathbb{R}$). One may wonder which quasi-linear maps ω: $Z \to Y$ (between Banach spaces) induce locally convex (that is, Banach) extensions $Y \oplus_\omega Z$. It turns out [5, 2] that $Y \oplus_\omega Z$ is locally convex if and only if ω satisfies the stronger estimate

$$\left\| \omega\left(\sum_{k=1}^{n} x_k \right) - \sum_{k=1}^{n} \omega(x_k) \right\| \leq K \sum_{k=1}^{n} \|x_k\|, \tag{12}$$

for some K and all points $x_1, \ldots, x_n$ in Z.

Curiously enough, maps satisfying (12) have been previously studied by a number of authors in the setting of stability theory [8, 6, 20]; see also [7, pp. 25–30].

4. CONCLUDING REMARKS

We close with some questions. First, it would be interesting to obtain the "largest" quasi-additive odd function ω on the dyadic group with $Q(\omega) = 1$ and $\omega(1) = 0$. A reasonable candidate could be the map given by

$$\Omega(t) = \sigma(t) \sum_k |k| \delta_k 2^k,$$

where $t = \sum_k \delta_k 2^k$ is the unique decomposition of t with $\delta_k \in \{0, 1\}$ for all $k \in \mathbb{Z}$ and σ is the signum function. Of course, the difficult point is the computation of the exact value of $Q(\Omega)$. The same question can be asked for quasi-additive functions on the line.

Also, a classification of quasi-additive maps from the line into quasi-Banach spaces would be desirable. We do not know under which conditions on a map $\theta\colon \mathbb{R} \to Y$ is the "Kalton–Peck" map $s \mapsto s\theta(\log_2 |s|)$ quasi-additive. Here Y can be assumed to be a p-Banach space $(0 < p \leq 1)$.

Regarding Proposition 2, it would be interesting to know whether or not "being topologically isomorphic to a topological vector space" is a three-groups property for abelian groups. In other words, must each abelian topological group G having a closed subgroup S, such that both S and G/S are topologically isomorphic to (topological) vector spaces, be topologically isomorphic to a (topological) vector space? Notice that there is a noncommutative topological group G containing $\mathbb{R}$ as a closed normal subgroup and such that $G/\mathbb{R}$ is also topologically isomorphic to $\mathbb{R}$. If the estimates given in Corollary 1 were sharp, one could obtain a counterexample by constructing a nonapproximable "quasi-additive" map from the line into a locally pseudo-convex space.

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