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Answer to a Question of S. Rolewicz

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Abstract. We exhibit examples of F-spaces with trivial dual which are isomorphic to its quotient by a line, thus solving a problem in Rolewicz's *Metric Linear Spaces*.

In this short note we make some comments about the following problem raised by Rolewicz in [6, p. 197, Problem 4.2.9].

Suppose that X is an F-space with trivial dual. Can X be isomorphic to its quotient by a line?

As we shall show, the answer is affirmative. The problem was motivated by the obvious fact that if X has no nonzero functionals, it cannot be isomorphic to its product by a line. It is well-known that the answer is negative for $X = L_p$, with $0 \leq p < 1$: if E and F are finite dimensional subspaces of L_p, then L_p/E and L_p/F are isomorphic (if and) only if E and F have the same dimension (see [3]).

To clarify this point, let us recall that, given F-spaces Z and Y, an extension of Z by Y is a short exact sequence $0 \to Y \to X \to Z \to 0$ in which X is an F-space. The open mapping Theorem [3] guarantees that Y is a subspace of X such that the corresponding quotient X/Y is Z. Two extensions $0 \to Y \to X_i \to Z \to 0$ $(i = 1, 2)$ are said to be *equivalent* if there exists an operator T making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

By the three-lemma [2], and the open mapping theorem, T must be an isomorphism. An extension $0 \to Y \to X \to Z \to 0$ is said to *split* if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This just means that Y is complemented in X, and implies that X is isomorphic to the direct sum $Y \oplus Z$ (the converse is not true).

Given two F-spaces Y and Z, we denote by $\operatorname{Ext}(Z, Y)$ the set of all possible extensions $0 \to Y \to X \to Z \to 0$ modulo equivalence. It is a standard fact that $\operatorname{Ext}(Z, Y)$ carries a "natural" linear structure and it was proved in [1] that if Y and Z are quasi-Banach spaces (locally bounded F-spaces in [6]), then $\operatorname{Ext}(Z, Y)$ becomes a linear topological space in a functorial way. Moreover if $0 \to Y \to X \to Z \to 0$ is an exact sequence and W is an F-space then there exists an exact sequence of linear maps

$$0 \to L(Z, W) \to L(X, W) \to L(Y, W) \to \operatorname{Ext}(Z, W) \to \operatorname{Ext}(X, W) \to \operatorname{Ext}(Y, W).$$

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In the locally bounded case the maps are even continuous operators.

Suppose now that X is a K-space (this means that $\mathrm{Ext}(X, \mathbb{K}) = 0$, that is, every extension of X by a line splits) with trivial dual (that is, $X^* = L(X, \mathbb{K}) = 0$) and let Y be a closed subspace of X. Taking $Z = X/Y$ and $W = \mathbb{K}$ in the homology sequence, we see that $\mathrm{Ext}(X/Y, \mathbb{K})$ is isomorphic to Y^*. Hence, if $Y^* \neq 0$, the quotient X/Y cannot be isomorphic to X. This applies to the typical examples of K-spaces with trivial dual $X = L_p$ for $0 \leq p < 1$ (see again [3]).

Thus, a possible counterexample for Rolewicz's problem must be a non-K-space. The simplest way of obtaining non-K-spaces (with no nonzero functionals) is taking the quotient of any space with trivial dual X by a closed subspace Y such that $Y^* \neq 0$. Then Y^* embeds in $\mathrm{Ext}(X/Y, \mathbb{K})$ (in the pure linear sense) and X/Y is not a K-space.

Observe that if X is a K-space with trivial dual and Y is finite dimensional, then X/Y cannot be isomorphic to its quotient by a line L since the homology sequence yields $\mathrm{Ext}(X/Y, \mathbb{K}) = Y^*$, while, for the same reason, $\mathrm{Ext}\big((X/Y)/L, \mathbb{K}\big) = (Y \oplus L)^*$.

Thus, Y has to be an infinite dimensional subspace of X. Let $X = L_p$, for some $0 \leq p < 1$, and $(I_n)_{n \geq 0}$ a partition of $[0, 1]$, into sets of positive measure. Let f_n denote the characteristic function of I_n and write Y for the closed subspace spanned by $(f_n)_{n=2}^{\infty}$. It is easily seen that Y is then isomorphic to ℓ_p (where $\ell_0 = \omega$ is the space of all sequences). Obviously X/Y has trivial dual yet it is isomorphic to its quotient by the line spanned by f_1 since

$$\begin{aligned} X/Y &= \ell_p\big(L_p(I_0 \oplus I_1), L_p(I_2)/[f_2], L_p(I_3)/[f_3], \dots\big) \\ &\cong \ell_p\big(L_p(I_0), L_p(I_1)/[f_1], L_p(I_2)/[f_2], L_p(I_3)/[f_3], \dots\big) \\ &= X/(Y \oplus [f_1]) \\ &= (X/Y)/[f_1]. \end{aligned}$$

From an abstract viewpoint, the preceding isomorphism is obtained after the following observation. Let T be an automorphism of the F-space X and let Y be a closed subspace of X where T acts as a shift (that is, there is a line $L \in Y$ such that $Y = TY \oplus L$). Then X/Y is isomorphic to its quotient by the line $T^{-1}L$ since

$$X/Y = X/(TY \oplus L) \cong X/T^{-1}(TY \oplus L) = X/(Y \oplus T^{-1}L) = (X/Y)/T^{-1}L.$$

Another interesting example can be obtained taking $X = L_p(\mathbb{T})$ for $0 < p < 1$ and $Y = H^p$, the corresponding Hardy class (that is, the closed subspace spanned by the polynomials). Let T be given by $Tf(z) = zf(z)$. Then T is an isometry on $L_p(\mathbb{T})$ and since $TH^p = H^p_0$ we have $H^p = TH^p \oplus [1]$ and $L_p(\mathbb{T})/H^p$ is isometric to its quotient by the line spanned by the function $z \mapsto z^{-1}$.

Finally, we show that, for every $0 \leq p < 1$, there is an isomorphic embedding $\ell_2 \to L_p$ such that also L_p/ℓ_2 is isomorphic to its quotient by a line. To this end, let $\Delta = \{1, -1\}^{\mathbb{Z}}$ be the Cantor group endowed with its Haar measure. We regard the points of Δ as functions x on $\mathbb{Z}$, with $x(k) = \pm 1$ for all $k \in \mathbb{Z}$. For $n = 1, 2, \dots$ the n-th Rademacher function $r_n\colon \Delta \to \mathbb{K}$ is given by $r_n(x) = x(n)$. In this way (r_n) becomes a sequence of independent Bernoulli variables with mean zero on the probability space Δ which forms an ℓ_2-basis in $L_p(\Delta)$: this is straightforward for $p > 0$ while for $p = 0$ it follows from Chebyshev's inequality.

Consider the shift $\sigma\colon \Delta \to \Delta$ given by $\sigma(x)(k) = x(k+1)$ for all $x \in \Delta$ and $k \in \mathbb{Z}$. Clearly, σ is a measure preserving automorphism, so that the operator T given by $T(f) = f \circ \sigma$ is an isometry of $L_p(\Delta)$ for all p. Obviously $Tr_n = r_{n+1}$ for all $n \geq 1$, hence, if H denotes the closed subspace spanned by the sequence r_n, we have $H = TH \oplus [r_1]$ and so $L_p(\Delta)/H$ is isometric to its quotient by the line $[T^{-1}r_1]$.

We close the paper with the following remark. We have seen that there exist F-spaces with trivial dual which are isomorphic to its quotient by a specified line. We do not know whether they are isomorphic to their quotients by all lines, apart from the case $X = L_0/\omega$: this easily follows from the fact, proved by Peck and Starbird [4], that L_0 is ω-transitive (this means that if i and j are isomorphic embeddings of ω into L_0 then there is an automorphism T of L_0 such that $j = T \circ i$).

We remark, however, that there are F-spaces (Ribe's space [5] is such an example) which are isomorphic to its quotient by some lines, but not by all lines.

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