PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 3, Pages 697–698
S 0002-9939(03)07180-6
Article electronically published on September 29, 2003

A SIMPLE PROOF THAT SUPER-REFLEXIVE SPACES ARE K-SPACES

FÉLIX CABELLO SÁNCHEZ

(Communicated by Jonathan M. Borwein)

ABSTRACT. We demonstrate the title.

A quasi-Banach space Z is called a K-space [3] if every extension of Z by the ground field splits; that is, whenever X is a quasi-Banach space having a line L such that X/L is isomorphic to Z, L is complemented in X (and so, $X = L \oplus Z$). These spaces play an important rôle in the theory of extensions of (quasi) Banach spaces [1], [2].

The property of being a K-space is closely related to the behaviour of quasi-linear functionals. Recall that a homogeneous functional $f : Z \to \mathbb{K}$ is said to be quasi-linear if there is a constant Q such that

$$|f(x+y) - f(x) - f(y)| \leq Q(\|x\| + \|y\|) \qquad (x, y \in Z).$$

The least possible constant in the preceding inequality shall be denoted by $Q(f)$.

It is well known [1] that Z is a K-space if and only if each quasi-linear functional on Z can be approximated by a true linear (but not necessarily continuous!) functional $\ell : Z \to \mathbb{K}$ in the sense that the distance

$$\operatorname{dist}(f, \ell) \stackrel{\text{def}}{=} \inf\{K \geq 0 : |f(x) - \ell(x)| \leq K\|x\| \text{ for all } x \in Z\}$$

is finite.

The main examples of K-spaces are supplied by Kalton and co-workers: for instance, $\mathcal{L}_p$ spaces $(0 < p \leq \infty)$ are K-spaces if and only if $p \neq 1$ ([1], [4], [5], [6]). Also, B-convex spaces (Banach spaces having nontrivial type $p > 1$) are K-spaces and so are quotients of Banach K-spaces.

In this short note, we present a very simple proof that super-reflexive Banach spaces are K-spaces. Of course this is contained in Kalton's result for B-convexity. Nevertheless, I believe that a simpler proof for this particular case is interesting because, in the presence of some unconditional structure (e.g., for Banach lattices), B-convexity is equivalent to super-reflexivity.

Mini-Theorem. *Every super-reflexive space is a K-space.*

Proof. Suppose on the contrary that Z is super-reflexive and there exists a quasi-linear function $f : Z \to \mathbb{K}$ such that $\operatorname{dist}(f, \ell) = \infty$ for all linear maps $\ell : Z \to \mathbb{K}$.

Received by the editors June 20, 2001.
2000 *Mathematics Subject Classification.* Primary 46B03, 46B08, 39B82.
Key words and phrases. K-space, super-reflexivity, ultraproduct.
Supported in part by DGICYT project BMF 2001—083.

Let $\mathcal{F}$ denote the family of all finite-dimensional subspaces of Z. For each $E \in \mathcal{F}$, let f_E denote the restriction of f to E. It is clear that $Q(f_E) \leq Q(f)$. Put

$$\delta_E = \text{dist}(f_E, E^*) = \inf\{\text{dist}(f_E, \ell) : \ell \in E^*\}.$$

Obviously, δ_E is finite for all $E \in \mathcal{F}$. The hypothesis means that $\delta_E \to \infty$ with respect to the natural (inclusion) order in $\mathcal{F}$. In particular, $\delta_E > 0$ for E large enough. Now, for each $E \in \mathcal{F}$, take $\ell_E \in E^*$ such that $\text{dist}(f_E, \ell_E) = \delta_E$ and let $g_E = \delta_E^{-1}(f_E - l_E)$ (if $\delta_E = 0$, take $g_E = 0$). Clearly, $|g_E(x)| \leq \|x\|$ provided $x \in E$. Also, it is clear that $Q(g_E) \to 0$ as E increases in $\mathcal{F}$.

Let $\mathfrak{V}$ be any ultrafilter refining the Fréchet filter on $\mathcal{F}$, and let $\mathcal{F}_{\mathfrak{V}}$ denote the ultraproduct of $\mathcal{F}$ with respect to $\mathfrak{V}$. Define $g : \mathcal{F}_{\mathfrak{V}} \to \mathbb{K}$ by

$$g[(x_E)]_{\mathfrak{V}} = \lim_{\mathfrak{V}(E)} g_E(x_E),$$

where $[(x_E)]_{\mathfrak{V}}$ denotes the class of (x_E) in $\mathcal{F}_{\mathfrak{V}}$.

Obviously, g is a (well-defined) bounded linear functional on $\mathcal{F}_{\mathfrak{V}}$ and, in fact, $\|g\| \leq 1$. Since Z is super-reflexive, $\mathcal{F}_{\mathfrak{V}}$ is reflexive and we have $(\mathcal{F}_{\mathfrak{V}})^* = (\mathcal{F}^*)_{\mathfrak{V}}$, where $\mathcal{F}^* = \{E^* : E \in \mathcal{F}\}$ (see [7]). It follows that $g = [(\ell'_E)]_{\mathfrak{V}}$, where $\ell'_E \in E^*$ and $\|\ell'_E\| \leq 1$ for all E. Hence,

$$\lim_{\mathfrak{V}(E)} g_E(x_E) = \lim_{\mathfrak{V}(E)} \ell'_E(x_E)$$

and so

$$\lim_{\mathfrak{V}(E)} \text{dist}(g_E, \ell'_E) = 0.$$

In particular, for every $\varepsilon > 0$, the set $\mathcal{S} = \{E \in \mathcal{F} : 0 < \text{dist}(g_E, \ell'_E) < \varepsilon\}$ belongs to $\mathfrak{V}$. But, for $E \in \mathcal{S}$, one has

$$\text{dist}(f_E, \ell_E + \delta_E \ell'_E) \leq \varepsilon \delta_E < \delta_E,$$

a contradiction. □

References

[1] N. Kalton, The three space problem for locally bounded F-spaces, Compositio Mathematica 37 (1978) 243–276. MR **80j:**46005

[2] N. Kalton, *Nonlinear commutators in interpolation theory*, Memoirs of the American Mathematical Society, vol. 73, no. 385, 1988. MR **89h:**47104

[3] N. Kalton, N. T. Peck, and J. W. Roberts, *An F-space sampler,* London Mathematical Society Lecture Note Series 89, Cambridge University Press, Cambridge, 1984. MR **87c:**46002

[4] N. Kalton and J. W. Roberts, Uniformly exhaustive submeasures and nearly additive set functions, Trans. Amer. Math. Soc. 278 (1983) 803–816. MR **85f:**28006

[5] M. Ribe, Examples for the nonlocally convex three space problem. Proc. Amer. Math. Soc. 73 (1979) 351–355. MR **81a:**46010

[6] J. W. Roberts, A nonlocally convex F-space with the Hahn-Banach approximation property, in: Banach spaces of analytic functions, Springer Lecture Notes in Mathematics 604, Berlin-Heidelberg-New York (1977) 76–81. MR **58:**30008

[7] B. Sims, *“Ultra”-techniques in Banach space theory.* Queen’s Papers in Pure and Applied Mathematics 60, Queen’s University, Kingston, Ontario, Canada, 1982. MR **86h:**46032

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain

E-mail address: `fcabello@unex.es`