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Quasi-additive mappings ☆

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0. Introduction and preliminaries

This note deals with the stability of homomorphisms on the topological groups underlying certain topological vector spaces.

Suppose $\mathcal{G}$ and $\mathcal{H}$ are topological Abelian groups written additively. Then a mapping $\omega : \mathcal{G} \to \mathcal{H}$ is said to be a quasi-homomorphism if $\omega(0) = 0$ and the Cauchy difference

$$\Delta(\omega)(x, y) \stackrel{\text{def}}{=} \omega(x + y) - \omega(x) - \omega(y)$$

is continuous at the origin, as a map $\mathcal{G} \times \mathcal{G} \to \mathcal{H}$. If $\mathcal{G}$ and $\mathcal{H}$ are typical additive groups, we speak of a quasi-additive map. Here, the basic question is if a given quasi-homomorphism is *approximable* by a true homomorphism $a : \mathcal{G} \to \mathcal{H}$ is the sense that $\omega - a$ is continuous at the origin (no more can be expected).

The main result of the paper is the following

Theorem 1. *Every quasi-additive map from the real line into a quasi-Banach space is approximable.*

I believe this theorem is much more natural than other known results whose hypotheses usually involve quite special estimates. Actually, the above result is a common generalization of two recent results obtained by completely different methods. In [1, Theorem 2] it was proved that each function from the line into a quasi-Banach space $\mathfrak{Y}$ satisfying an estimate

$$\big\| \Delta(\omega)(s, t) \big\|_{\mathfrak{Y}} \leqslant \varepsilon\big(|s| + |t|\big) \quad (s, t \in \mathbb{R}) \tag{1}$$

is approximable ($\| \cdot \|_{\mathfrak{Y}}$ denotes the quasi norm of $\mathfrak{Y}$). In [2, Theorem 1(b)] it is proved that quasi-additive functions from the line into Banach spaces are approximable. The method

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of proof used in [1] is essentially direct, but strongly depends on the estimate (1) and it is unclear how it could be generalized to deal with quasi-additive functions. On the other hand, the proof in [2] is very intricate and the last part of the argument depends on the existence of enough linear continuous functionals on the range space, a condition which often fails in quasi-Banach spaces.

The paper is organized as follows. Section 1 contains the proof of Theorem 1. As the reader will see, the argument given below is much simpler than those of [1,2]. The crucial steps are, in fact, simple generalizations of two venerables oldies by Hyers and Skof.

In Section 2 we prove that every quasi-additive map from L_0 (explanations follow) into a quasi-Banach space is approximable. The same is true if L_0 is replaced by a topological vector space (TVS) with a weak topology. Finally, as an application of our main result, we obtain that being a quasi-Banach space is a three space property for Abelian groups: if $\mathcal{E}$ is a topological Abelian group having a subgroup $\mathcal{H}$ such that both $\mathcal{H}$ and $\mathcal{E}/\mathcal{H}$ are topologically isomorphic to (the groups underlying) a quasi-Banach space, then so is $\mathcal{E}$. This solves a problem raised in [2].

For the reader's convenience, we recall here some facts from the linear theory. First of all, an F-space is a complete metrizable TVS. A locally bounded space is a TVS having a bounded neighborhood of zero. Recall that a bounded set is one which is absorbed by each neighborhood of zero (and not a set of finite diameter). It is well known that if $\mathfrak{X}$ is locally bounded if and only if there is a function $\|\cdot\|$ on $\mathfrak{X}$ satisfying

- $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$;
- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, $x \in \mathfrak{X}$;
- $\|x + y\| \leqslant K(\|x\| + \|y\|)$ for some constant $K \geqslant 1$ and all $x, y \in \mathfrak{X}$

(that is, a quasi-norm), and giving the topology of $\mathfrak{X}$, in the sense that the balls

$$B(\varepsilon) = \{x \in \mathfrak{X}\colon\ \|x\| \leqslant \varepsilon\} \quad (\varepsilon > 0)$$

form a neighborhood base at zero. A locally bounded (= quasi-normed) space is locally convex if and only if it is (linearly isomorphic to) a normed space. A quasi-Banach space is a complete, locally bounded space. The so-called p-norms ($0 < p \leqslant 1$) are quasi-norms satisfying the inequality $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$. By the Aoki–Rolewicz theorem [9,12], every locally bounded space admits an equivalent p-norm, for some $0 < p \leqslant 1$. Clearly, if $\|\cdot\|$ is a p-norm for $\mathfrak{X}$, then the formula $d(x, y) = \|x - y\|^p$ defines an invariant metric for $\mathfrak{X}$ and $\|\cdot\|^p$ is a p-homogeneous F-norm for $\mathfrak{X}$. The well-known fact that a set in a quasi-normed space is bounded if and only if it is "metrically bounded" for the corresponding quasi-norm will be used without further mention.

Important examples are the Lebesgue spaces L_p for $0 \leqslant p < \infty$. Let μ be a measure on Δ. Then, for $0 < p < \infty$, L_p is the space of all measurable functions $f : \Delta \to \mathbb{K}$ for which the quasi-norm

$$\|f\|_p = \left(\int_\Delta |f(t)|^p \, d\mu(t)\right)^{1/p}$$

is finite, with the usual convention about identifying functions equal almost everywhere. Finally, L_0 is the space of all measurable functions on Δ with the topology of convergence in measure on sets of finite measure (the topology of convergence in measure fails to be linear on L_0 if μ is infinite; see [10, Exercise 2.18(c)]). If μ is finite this topology is given by the F-norm

$$\|f\|_0 = \int_\Delta \frac{|f(t)|}{1+|f(t)|}\, d\mu(t).$$

Let us remark that, for $1 \leqslant p < \infty$, the spaces L_p are Banach spaces (i.e., locally bounded and locally convex F-spaces); for $0 < p < 1$ they are not locally convex, yet locally bounded (i.e., quasi-Banach spaces) and that L_0 is not locally convex (unless μ is purely atomic) nor locally bounded (unless Δ consists of a finite number of atoms).

1. Proof of the main result

The proof of Theorem 1 consists of three steps. The first one is a straightforward generalization of a classical result by Hyers (see, e.g., [6]).

Lemma 1 (Mainly Hyers). *Let $\mathcal{G}$ be an Abelian group (no topology in assumed) and $\mathfrak{Y}$ a locally bounded F-space. Let $\omega : \mathcal{G} \to \mathfrak{Y}$ be a mapping such that the set of Cauchy differences $\{\Delta(\omega)(x, y)\colon x, y \in \mathcal{G}\}$ is bounded in $\mathfrak{Y}$. Then there exists an additive mapping $a : \mathcal{G} \to \mathfrak{Y}$ such that the range of $\omega - a$ is bounded in $\mathfrak{Y}$. If $\mathcal{G}$ is torsion free, then a is unique.*

Proof. The proof closely follows [5]. We may and do assume that the F-norm of $\mathfrak{Y}$ is p-homogeneous for some $0 < p \leqslant 1$. The hypothesis means that $\|\Delta(\omega)(x, y)\| \leqslant \varepsilon$ for some $\varepsilon \geqslant 0$ and all $x, y \in \mathcal{G}$.

Fix $x \in \mathcal{G}$ and consider the sequence $(\omega(2^n x)/2^n)$. A straightforward induction argument yields $\|\omega(2^n x) - 2^n \omega(x)\| \leqslant (2^{pn} - 1)\varepsilon$ for all $n \in \mathbb{N}$. Thus, for $n, m \in \mathbb{N}$, we have $\|\omega(2^{n+m} x) - 2^m \omega(2^n x)\| \leqslant 2^{pm}\varepsilon$. Dividing by 2^{n+m}, we obtain the estimate

$$\left\| \frac{\omega(2^{n+m} x)}{2^{n+m}} - \frac{\omega(2^n x)}{2^n} \right\| \leqslant \frac{\varepsilon}{2^{pn}},$$

so that $(\omega(2^n x)/2^n)$ is a Cauchy sequence for each fixed $x \in \mathcal{G}$. Put

$$a(x) = \lim_{n\to\infty} \frac{\omega(2^n x)}{2^n}.$$

Obviously, $\|\omega(2^n x)/2^n - \omega(x)\| \leqslant \varepsilon$ for all n and x, and, therefore, $\|\omega(x) - a(x)\| \leqslant \varepsilon$ for all x. It remains to see that a is additive. But

$$\begin{aligned} \big\| a(x+y) - a(x) - a(y) \big\| &= \lim_{n\to\infty} \left\| \frac{\omega(2^n(x+y))}{2^n} - \frac{\omega(2^n x)}{2^n} - \frac{\omega(2^n y)}{2^n} \right\| \\ &\leqslant \lim_{n\to\infty} \frac{\varepsilon}{2^{pn}} = 0. \end{aligned}$$

Since the ‘uniqueness part’ is clear, the proof is complete. □

Lemma 2 (Compare to [1]). *Let $\omega:\mathcal{G}\to\mathfrak{Y}$ be a quasi-additive mapping, where $\mathcal{G}$ is an Abelian topological group and $\mathfrak{Y}$ a locally bounded F-space. If ω maps a neighborhood of the origin in $\mathcal{G}$ into a bounded set in $\mathfrak{Y}$, then it is continuous at the origin of $\mathcal{G}$.*

Proof. Let $\mathcal{O}_{\mathcal{G}}$ denote the filter of neighborhoods of the origin in $\mathcal{G}$. Again, we assume that $\mathfrak{Y}$ has p-homogeneous norm for some $0 < p \leqslant 1$. The hypothesis implies that there is $V \in \mathcal{O}_{\mathcal{G}}$ such that $\|\omega(x)\| \leqslant K$ for some K and all $x \in V$. Fix $\varepsilon > 0$. Choose n such that $K/2^{np} < \varepsilon/2$. Now, take $W \in \mathcal{O}_{\mathcal{G}}$, with $W \subset V$, such that

$$\bigl\|\omega(x+y)-\omega(x)-\omega(y)\bigr\| \leqslant \frac{\varepsilon}{2} \quad (x, y \in W)$$

and let $U \in \mathcal{O}_{\mathcal{G}}$ be such that $U + \overset{2^n\text{times}}{\cdots} + U \subset W$. A straightforward induction on $k = 1, 2, \ldots, n$ yields

$$\bigl\|\omega(2^k x) - 2^k\omega(x)\bigr\| \leqslant (2^{kp}-1)\frac{\varepsilon}{2} \quad (x \in U).$$

Taking $k = n$ and dividing by 2^n, we obtain

$$\left\|\frac{\omega(2^n x)}{2^n} - \omega(x)\right\| \leqslant \frac{\varepsilon}{2} \quad (x \in U),$$

and so,

$$\bigl\|\omega(x)\bigr\| \leqslant \frac{\varepsilon}{2} + \frac{K}{2^{np}} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (x \in U).$$

This completes the proof. □

Lemma 3 (Mainly Skof; see [13] or [6]). *Let $\mathfrak{Y}$ be a locally bounded space with p-homogeneous norm $\|\cdot\|$ and $\delta > 0$. Suppose $\omega:[0, 2\delta]\to\mathfrak{Y}$ is any function such that $\|\Delta(\omega)(s,t)\| \leqslant \varepsilon$ for some ε and all $0 \leqslant s, t \leqslant \delta$. Then there exists $\tilde\omega:\mathbb{R}^+\to\mathfrak{Y}$ such that $\tilde\omega(s) = \omega(s)$ for $0 \leqslant s \leqslant \delta$ with $\|\Delta(\tilde\omega)(s,t)\| \leqslant 2\varepsilon$ for all $s, t \in \mathbb{R}^+$.*

Proof. There is no loss of generality in assuming $\delta = 1$. Let us write $s = \lfloor s\rfloor + s^\star$, where $\lfloor\cdot\rfloor$ is the integer part function. Now, extend ω from $[0, 1]$ to the whole half-line taking $\tilde\omega(s) = \lfloor s\rfloor\omega(1) + \omega(s^\star)$. Let us estimate $\Delta(\tilde\omega)$. Take $s, t \geqslant 0$. If $\lfloor s+t\rfloor = \lfloor s\rfloor + \lfloor t\rfloor$, then $(s+t)^\star = s^\star + t^\star$ and so $\Delta(\tilde\omega)(s,t) = \Delta(\omega)(s^\star, t^\star)$. Otherwise, $\lfloor s+t\rfloor = \lfloor s\rfloor + \lfloor t\rfloor + 1$ and so $(s+t)^\star = s^\star + t^\star - 1$. Hence,

$$\begin{aligned}\Delta(\tilde\omega)(s,t) &= \omega(1) + \omega(s^\star + t^\star - 1) - \omega(s^\star) - \omega(t^\star)\\ &= -\Delta(\omega)(1, s^\star + t^\star - 1) + \Delta(\omega)(s^\star, t^\star),\end{aligned}$$

and therefore $\|\Delta(\tilde\omega)(s,t)\| \leqslant 2\varepsilon$. □

Proof of Theorem 1. Let $\omega:\mathbb{R}\to\mathfrak{Y}$ be a quasi-additive function, where $\mathfrak{Y}$ is a locally bounded space with p-homogeneous norm $\|\cdot\|$. There is no loss of generality in assuming ω odd: the even part of a quasi-additive mapping is continuous at the origin. Take $\delta > 0$ so that $\|\Delta(\omega)(s,t)\| \leqslant 1$ for $|s|, |t| \leqslant \delta$. By Lemma 3 there is $\tilde\omega:\mathbb{R}\to\mathfrak{Y}$ with $\tilde\omega(s) = \omega(s)$ for $|s| \leqslant \delta$ and such that $\|\Delta(\tilde\omega)(s,t)\| \leqslant 2$ for all real s and t. Now use Lemma 1 to get an additive $a:\mathbb{R}\to\mathfrak{Y}$ such that $\|\tilde{\omega}(s)-a(s)\|\leqslant 2$ for all s. Obviously $\omega-a$ is quasi-additive and bounded on $[-\delta,\delta]$. Finally, according to Lemma 2 the difference $\omega-a$ is continuous at zero, which completes the proof. □

Suppose $\omega:\mathbb{R}\to\mathfrak{Y}$ is a quasi-additive odd function, where $\mathfrak{Y}$ a p-Banach space with p-homogeneous norm. Let $\delta(\omega,\varepsilon)$ be the greatest constant δ such that $\|\Delta(\omega)(s,t)\|\leqslant\varepsilon$ whenever $|s|,|t|<\delta$. Let $\delta_1<\delta(\omega,1)$. Then $\|\Delta(\omega)(s,t)\|\leqslant 1$ for $|s|,|t|\leqslant\delta_1$. Then, in Lemma 3, we have $\|\Delta(\tilde{\omega})(s,t)\|\leqslant 1$ for all $s,t\in\mathbb{R}$ and $\tilde{\omega}=\omega$ on $[-\delta_1,\delta_1]$. Therefore the additive function of Lemma 1 satisfies $\|\tilde{\omega}(s)-a(s)\|\leqslant 2$ for all s and so $\|\omega(s)-a(s)\|\leqslant 2$ for $s\in[-\delta_1,\delta_1]$. Hence according to the proof of Lemma 2, given $\varepsilon>0$, one has

$$\big\|\omega(s)-a(s)\big\|\leqslant\varepsilon\quad\text{provided } |s|<\frac{\delta(\omega,\varepsilon/2)}{2^n},$$
$$\text{where } n\geqslant\log_2\left(\frac{4}{\varepsilon}\right)^{1/p}=\frac{2-\log_2\varepsilon}{p}.$$

We have proved the following

Corollary 1. *Let Ω be a family of quasi-additive mappings $\omega:\mathbb{R}\to\mathfrak{Y}_\omega$, where $\mathfrak{Y}_\omega$ are possibly different p-Banach spaces, with p fixed. Suppose Ω is uniformly quasi-additive in the sense that for every $\varepsilon>0$ there is $\delta>0$ such that*

$$\big\|\omega(s+t)-\omega(s)-\omega(t)\big\|_\omega\leqslant\varepsilon\quad\big(|s|,|t|\leqslant\delta\big)$$

for all $\omega\in\Omega$—here $\|\cdot\|_\omega$ denotes the F-norm of $\mathfrak{Y}_\omega$. Then Ω is uniformly approximable in the sense that to each $\omega\in\Omega$ there corresponds an additive map $a_\omega:\mathbb{R}\to\mathfrak{Y}_\omega$ in such a way that the family $\{\omega-a_\omega\colon\ \omega\in\Omega\}$ is equicontinuous at zero, that is, for every $\varepsilon>0$ there is $\delta>0$ such that

$$\big\|\omega(s)-a_\omega(s)\big\|_\omega\leqslant\varepsilon\quad\big(|s|\leqslant\delta\big)$$

for all $\omega\in\Omega$.

2. Applications

Quasi-additive functions on some infinite dimensional spaces

It will be clear to those acquainted with the theory of extensions of TVSs that Theorem 1 fails if $\mathbb{R}$ is replaced by any infinite dimensional quasi-Banach space (see [3]). One has, however, the following

Theorem 2. *Let μ be a nonatomic σ-finite measure and $\mathfrak{Y}$ a locally bounded F-space. Then every quasi-additive map $\omega:L_0\to\mathfrak{Y}$ is approximable.*

Proof. We may assume that μ is a probability. For if not, taking a probability ν with the same null sets, we have that $L_0(\mu)$ and $L_0(\nu)$ are linearly isomorphic (hence topologically isomorphic with respect to the underlying group structures). Let $\omega : L_0 \to \mathfrak{Y}$ be quasi-additive. As before, we assume that $\mathfrak{Y}$ has p-homogeneous norm for some $0 < p \leqslant 1$. Choose δ_0 such that $\|\omega(f+g) - \omega(f) - \omega(g)\| \leqslant 1$ whenever $\|f\|_0, \|g\|_0 \leqslant \delta_0$.

Let $\Delta = \bigoplus_{i=1}^r \Delta_i$ be a partition into measurable sets, with $\mu(\Delta_i) \leqslant \delta_0$ for all $1 \leqslant i \leqslant r$. Then $L_0 = \prod_{i=1}^r L_0(\Delta_i)$, as a topological direct product, and $\|f\| \leqslant \delta_0$ for all $f \in L_0(\Delta_i)$ and all $1 \leqslant i \leqslant r$, since $\|f\|_0 \leqslant \mu\{t \in \Delta\colon f(t) \neq 0\}$.

Write ω_i for the restriction of ω to $L_0(\Delta_i)$. Since $\|\omega_i(f+g) - \omega_i(f) - \omega_i(g)\| \leqslant 1$ for all $f, g \in L_0(\Delta_i)$ and all $1 \leqslant i \leqslant r$, we can apply Lemma 1 to obtain unique additive maps $a_i : Z_i \to Y$ such that $\|\omega_i(f) - a_i(f)\| \leqslant 1$ for $f \in L_0(\Delta_i)$.

By Lemma 2, we have that each $\omega_i - a_i$ is continuous at the origin of $L_0(\Delta_i)$ and so, the additive map $a : L_0 \to \mathfrak{Y}$ given by

$$a(f) = \sum_{i=1}^r a_i(f_i) \quad \left(f = \sum_{i=1}^r f_i,\ f_i \in L_0(\Delta_i) \right)$$

approximates ω near the origin. □

Actually, the approximating map a turns out to be unique, since if μ is nonatomic there are no continuous additive map from L_0 into any quasi-Banach space, apart from the null map. Let us mention that Theorem 2 implies that every extension of L_0 by a locally bounded space splits (in view of [4, Lemmas 2.2.a and 3.2]; see also [2]). This is one of the main results in [8].

We now introduce a class of TVSs sharing many properties with discrete L_0-spaces. Let $\mathfrak{X}$ be a linear space and $\mathfrak{F}$ a linear subspace of its algebraic dual. The sets

$$\{x \in \mathfrak{X}\colon |f(x)| \leqslant 1\} \quad (f \in \mathfrak{F})$$

are a subbase for a linear topology on $\mathfrak{X}$, which is often denoted $\sigma(\mathfrak{X}, \mathfrak{F})$ and referred to as the *weak* topology induced by $\mathfrak{F}$. It is clear that $\sigma(\mathfrak{X}, \mathfrak{F})$ is Hausdorff if and only if $\mathfrak{F}$ is total (in the sense that $f(x) = 0$ for all $f \in \mathfrak{F}$ only if $x = 0$). Note that these topologies need not be (and usually are not) metrizable nor complete; however, they are always locally convex. Typical examples are

- The usual weak topology of a Banach space $\mathfrak{X}$, corresponding to $\sigma(\mathfrak{X}, \mathfrak{X}^*)$, where $\mathfrak{X}^*$ is the topological dual of $\mathfrak{X}$;
- The weak* topology in the topological dual $\mathfrak{X}^*$ of a Banach space $\mathfrak{X}$, that is, the topology $\sigma(\mathfrak{X}^*, \mathfrak{X})$;
- The product topology on $\mathbb{K}^{\mathbb{N}}$ is the weak topology induced by the linear span of the coordinate functionals;
- The direct sum topology in $\bigoplus_{n=1}^\infty \mathbb{K}$ is the topology induced by the space of all linear functionals on it.

Theorem 3. *Let $\mathfrak{Z}$ be a linear space with a (Hausdorff) weak topology. Then every quasi-additive map from $\mathfrak{Z}$ into a quasi-Banach space is approximable.*

Proof. Let $\omega : \mathfrak{Z} \to \mathfrak{Y}$ be quasi-additive, where $\mathfrak{Y}$ is a quasi-Banach space. Choose $U \in \mathcal{O}_{\mathfrak{Z}}$ such that

$$\left\|\omega(x+y)-\omega(x)-\omega(y)\right\|_{\mathfrak{Y}} \leqslant 1 \quad (x, y \in U).$$

There is no loss of generality in assuming that

$$U = \left\{x \in \mathfrak{Z}\colon \left|f_i(x)\right| \leqslant 1 \text{ for all } i = 1, \ldots, k\right\},$$

where $f_i \in \mathfrak{F}$. Let $\mathfrak{Z}_0 = \{x \in \mathfrak{Z}\colon f_i(x) = 0 \text{ for all } i = 1, \ldots, k\}$. It is clear that $\mathfrak{Z}_0$ is a closed finite codimensional subspace of $\mathfrak{Z}$, with $\dim(\mathfrak{Z}/\mathfrak{Z}_0) \leqslant k$, so that $\mathfrak{Z} = \mathfrak{Z}_0 \times \mathfrak{Z}_1$, where $\dim(\mathfrak{Z}_1) \leqslant k$. For $i = 0, 1$, let ω_i denote the restriction of ω to $\mathfrak{Z}_i$. Clearly, ω_1 is approximable ($\mathfrak{Z}_1$ is a product of no more than k lines). On the other hand, ω_0 has bounded Cauchy difference: it follows from Lemmas 1 and 2 that it is approximable too. Hence ω is approximable. □

Theorem 3 implies that the hypothesis of μ being nonatomic can be removed from Theorem 2: indeed, if μ is a σ-finite measure, then $L_0(\mu) = L_0(\nu) \times \mathbb{K}^{\mathfrak{a}}$, where ν is the nonatomic part of μ and $\mathfrak{a}$ is the set of its atoms.

Extensions of topological groups

As explained in [2] there are close connections between quasi-homomorphism and extensions of topological groups, as they are between extensions of topological vector spaces and quasi-linear maps (see [4]).

Let us recall from [2] that, given (not necessarily Abelian) topological groups $\mathcal{G}$ and $\mathcal{H}$, an extension of $\mathcal{G}$ by $\mathcal{H}$ is a short exact sequence

$$0 \to \mathcal{H} \xrightarrow{\imath} \mathcal{E} \xrightarrow{\pi} \mathcal{G} \to 0$$

in which $\mathcal{E}$ is another topological group and the arrows are relatively open continuous homomorphisms. Less technically, we can regard $\mathcal{E}$ as a topological group containing $\mathcal{H}$ as a normal subgroup in such a way that $\mathcal{E}/\mathcal{H}$ is (topologically isomorphic to) $\mathcal{G}$.

The following result follows from Corollary 1 exactly as Theorem 2 of [2] follows from Corollary 2 there.

Theorem 4. *Let $0 \to \mathfrak{Y} \to \mathcal{E} \to \mathfrak{Z} \to 0$ be an extension of topological groups in which both $\mathfrak{Z}$ and $\mathfrak{Y}$ are locally bounded F-spaces. If the middle group $\mathcal{E}$ is Abelian then it admits an outer multiplication $\star$ by real numbers which makes it into a TVS is such a way that $0 \to \mathfrak{Y} \to (\mathcal{E}, +, \star) \to \mathfrak{Z} \to 0$ becomes an extension of TVSs* (*that is, a sequence of TVSs and relatively open linear continuous operators*).

By [11] the space $(\mathcal{E}, +, \star)$ is actually locally bounded and complete (i.e., a quasi-Banach space). Since extensions of quasi-Banach spaces come from homogeneous quasi-additive maps [7], Theorem 4 implies that every quasi-additive map $\omega : \mathfrak{Z} \to \mathfrak{Y}$ between quasi-Banach spaces admits a decomposition $\omega = \eta + a + \epsilon$, where a is additive, ϵ is continuous at the origin and η homogeneous (and clearly quasi-additive too). Hence η satisfies an estimate

$$\left\|\eta(x+y)-\eta(x)-\eta(y)\right\|_{\mathfrak{Y}} \leqslant \varepsilon\left(\|x\|_{\mathfrak{Z}}+\|y\|_{\mathfrak{Z}}\right)$$

for some $\varepsilon \geqslant 0$ and all $x, y \in \mathfrak{Z}$ (here $\|\cdot\|_{\mathfrak{X}}$ stands for the corresponding quasi-norms).

Also, Theorem 4 implies that two quasi-Banach spaces $\mathfrak{Z}$ and $\mathfrak{Y}$ have only approximable quasi-additive maps $\omega : \mathfrak{Z} \to \mathfrak{Y}$ if and only if every extension of quasi-Banach spaces $0 \to \mathfrak{Y} \to \mathfrak{X} \to \mathfrak{Z} \to 0$ splits. In this way many results in the theory of extension of quasi-Banach spaces can be regarded as stability results. Sample: every quasi-additive map from L_p (or ℓ_p) into a q-Banach space with $0 < p < q \leqslant 1$ is approximable [7]. We will refrain from entering into further details here.

Problem. Is 'being topologically isomorphic to an F-space' a three-groups property for Abelian groups? Is at least every quasi-additive map from the real line into an F-space approximable?

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