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The Long Homology Sequence for Quasi-Banach Spaces, with Applications*

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Abstract. We establish the existence of long homology sequences in the category of quasi-Banach spaces, with values in a certain category of topological vector spaces. We derive from that new results about the structure of twisted sums of quasi-Banach and Banach spaces. Sample result: let A and B be subspaces of L_p, with $0<p<1$. If L_p/A and L_p/B are isomorphic then so are A^* and B^*. In particular, if A and B are finite dimensional, then L_p/A and L_p/B are isomorphic if and only if $\dim(A)=\dim(B)$.

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1. Introduction and statement of the main result

Our purpose in this paper is to establish the existence of long homology sequences in the category of quasi-Banach spaces, with values in a certain category of topological vector spaces. We derive from that new results about the structure of twisted sums of quasi-Banach and Banach spaces.

We will denote by **Q** the category in which the objects are quasi-Banach spaces (that is, complete locally bounded topological vector spaces) and the arrows are operators (linear continuous maps). The subcategory in which the objects are Banach spaces will be denoted **B**. We will also encounter non-Hausdorff spaces: a complete seminormed (not necessarily Hausdorff) space shall be called a semi-Banach space; analogously, a complete but not necessarily Hausdorff "quasi-normed" space shall be called a semi quasi-Banach space. The category of semi quasi-Banach and operators shall be denoted $\mathbf{Q_{1/2}}$.

An extension (or else a short exact sequence) in **Q** or **B** is a diagram $0\to Y\to X\to Z\to 0$ in the category such that the kernel of each arrow coincides with the image of the preceding. The open mapping theorem [12] guarantees that Y is a subspace of X such that the corresponding quotient X/Y is Z. The space X itself is called a twisted sum of Y and Z (in that order). Two extensions $0\to Y\to X_i\to Z\to 0$ $(i=1,2)$ are said to be equivalent if there exists an arrow T making

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commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_2 & \to & Z & \to & 0. \end{array}$$

By the three-lemma [6], and the open mapping theorem, T must be an isomorphism. An extension is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. Given two objects Y and Z in **Q** (resp. **B**), we denote by $\mathrm{Ext}_{\mathbf{Q}}(Z, Y)$ (resp. $\mathrm{Ext}_{\mathbf{B}}(Z, Y)$) the set of all possible extensions $0 \to Y \to X \to Z \to 0$ with X in Q (resp. in B) modulo equivalence. The basic result in the paper is

THEOREM 1. *Let* $0 \to Y \to X \to Z \to 0$ *be an exact sequence in* **Q** *and let* E *and* A *be quasi-Banach spaces. Then there exist exact sequences in* $\mathbf{Q}_{1/2}$ (*whose explicit construction is done in section* 7)

$$0 \to L(Z,E) \to L(X,E) \to L(Y,E) \to$$
$$\mathrm{Ext}_{\mathbf{Q}}(Z,E) \to \mathrm{Ext}_{\mathbf{Q}}(X,E) \to \mathrm{Ext}_{\mathbf{Q}}(Y,E).$$

and

$$0 \to L(A,Y) \to L(A,X) \to L(A,Z) \to$$
$$\mathrm{Ext}_{\mathbf{Q}}(A,Y) \to \mathrm{Ext}_{\mathbf{Q}}(A,X) \to \mathrm{Ext}_{\mathbf{Q}}(A,Z).$$

The existence of the preceding induced sequences in **B** with values in the category **V** of vector spaces and linear maps (and in particular the existence of a suitable linear structure in $\mathrm{Ext}_{\mathbf{B}}(Z, Y)$) is well-known and follows from the existence of projective (or injective) objects and the general theory of derived functors. For quasi-Banach spaces, injective or projective objects do not exist; so, even the fact that $\mathrm{Ext}_{\mathbf{Q}}(\cdot, \cdot)$ is a functor is not obvious at all. Nevertheless, once pull-backs and push-outs are shown to exist, the possibility of endowing an "intrinsic" vector space structure to the set $\mathrm{Ext}_{\mathbf{Q}}(Z, Y)$, and hence constructing the homology sequences with values in **V**, is well-known to algebraists. However, the highly abstract approach that texts in homological algebra adopt (see [15]) makes it advisable to display even the algebraic part of the proof in the quasi-Banach space setting; something that, to the best of our knowledge, has not been done so far. We have included an Appendix with a brief down-to earth description of the homology sequences in the pull-back and push-out terms; some acquiantaince with (say) the first section of [7] or the first chapter of [3] would help.

The homology sequences we shall construct are based on the description of extensions $0 \to Y \to X \to Z \to 0$ via quasi-linear maps $F: Z \to Y$. Roughly speaking, we represent $\mathrm{Ext}_{\mathbf{Q}}(Z, Y)$ as a certain space $\mathcal{Q}(Z, Y)$ of equivalence classes of maps $Z \to Y$. This representation makes obvious both the functorial and linear character of extensions and greatly simplifies the introduction of vector topologies in the space $\mathrm{Ext}_{\mathbf{Q}}(Z, Y)$. (An adaptation of the theory of quasi-linear maps to the locally convex setting can be found in [2].)

A proof that $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ (with its "own" linear structure) and $\mathcal{Q}(\cdot,\cdot)$ are naturally equivalent as functors $\mathbf{Q}\times\mathbf{Q}\rightsquigarrow\mathbf{V}$ is sketched in the Appendix.

2. Beginning of the Proof (with Precise Definitions)

The by now classical theory of Kalton and Peck [8; 10], see also the monograph [3], describes twisted sums of two quasi-Banach spaces Y and Z in terms of the so-called quasi-linear maps. These are homogeneous maps $F:Z\to Y$ satisfying that for some constant Q and all points $x,y\in Z$ one has

$$\|F(x+y)-F(x)-F(y)\|\leqslant Q(\|x\|+\|y\|).$$

The smallest constant Q satisfying the preceding inequality shall be denoted $Q(F)$. Given a quasi-linear map $F:Z\to Y$, it is possible to construct a twisted sum of Y and Z, which we shall denote by $Y\oplus_F Z$, endowing the product space $Y\times Z$ with the quasi-norm

$$\|(y,z)\|_F=\|y-Fz\|+\|z\|.$$

Clearly, the subspace $\{(y,0):y\in Y\}$ is isomorphic to Y and the corresponding quotient is isomorphic to Z, so that one has an exact sequence $0\to Y\to Y\oplus_F Z\to Z\to 0$. Conversely, every extension X of Y by Z comes (modulo equivalence) from a suitable quasi-linear map $F:Z\to Y$ that is obtained as follows: take $B:Z\to X$ a homogenous bounded selection for the quotient map and $L:Z\to X$ a linear selection for the quotient map. Then set $F=B-L$. Two quasi-linear maps $F,G:Z\to Y$ are said to be equivalent (or else, that F is a version of G) if they induce equivalent extensions; and this happens if and only if the difference $G-F$ can be written as the sum of a homogeneous bounded and a linear map both defined from Z into Y. In particular $0\to Y\to Y\oplus_F Z\to Z\to 0$ splits if and only if $F=B+L$, where $B:Z\to Y$ is bounded and $L:Z\to Y$ is linear. The space of all equivalence classes $[F]$ of quasi-linear maps $F:Z\to Y$ shall be denoted $\mathcal{Q}(Z,Y)$. The linear structure of $\mathcal{Q}(Z,Y)$ is obvious: $[F]+[G]=[F+G]$ and $\lambda[F]=[\lambda F]$.

Moreover, since the composition of quasi-linear maps with operators is again a quasi-linear map it is apparent that $\mathcal{Q}(Z,Y)$ depends functorially on Z and Y, so that $\mathcal{Q}(\cdot,\cdot):\mathbf{Q}\times\mathbf{Q}\rightsquigarrow\mathbf{V}$ is a (bi-) functor contravariant in the first variable and covariant in the second one.

At this point, we identify $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$ and $\mathcal{Q}(Z,Y)$ and, in particular, we transfer the functorial and linear structures from the spaces $\mathcal{Q}(Z,Y)$ to the sets $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$. (By the natural equivalence already mentioned, these structures coincide with the "intrinsic" ones of $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$ that are presented in the Appendix, but our proofs do not depend of this fact.) Thus, the following result proves the algebraic part of the first half of Theorem 1.

THEOREM 2. *Let* $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ *be a short exact sequence in* $\mathbf{Q}$ *and let* E *be a quasi-Banach space. Then the following sequence in* $\mathbf{V}$ *is exact*:

$$0 \to L(Z,E) \xrightarrow{q^*} L(X,E) \xrightarrow{j^*} L(Y,E) \xrightarrow{\omega} \mathcal{Q}(Z,E) \xrightarrow{q^*} \mathcal{Q}(X,E) \xrightarrow{j^*} \mathcal{Q}(Y,E).$$

Proof of Theorem 2. The maps q^* and j^* are, as usually, simple (right) composition with, respectively, q and j. These maps are just the (functorial) images of the maps of the starting sequence. To describe the "connecting" map ω, fix a quasi-linear map $F_0 : Z \to Y$ defining the starting sequence $0 \to Y \to X \to Z \to 0$. Then $\omega(T) = [T \circ F_0]$. The maps are well defined and linear. It remains to verify that it is an exact sequence. The exacteness at $L(Y,E)$ is proved in the following lemma.

LEMMA 1. *Let* $T : Y \to E$ *be an operator. Then* $T \circ F_0$ *is trivial if and only if* T *extends to an operator* $X \to E$.

Proof. Assume that $F_0 = B_0 - L_0$, where $B_0 : Z \to X$ is a bounded homogenous selection for the quotient map q, while $L_0 : Z \to X$ is a linear selection for q. If $T_X : X \to E$ is an extension of T, then the decomposition $T \circ F_0 = T_X \circ B_0 - T_X \circ L_0$ shows that $T \circ F_0$ is trivial. Conversely, suppose $T \circ F_0 = B + L$ with B a bounded map and L a linear map from Z to E. Then

$$T_X(x) = T(x - L_0(q(x)) - L(q(x))$$

is a linear extension of T. Moreover,

$$\begin{aligned} \|T_X(x)\| &= \|T(x - L_0 q(x)) - L(q(x))\| \\ &= \|T(x + F_0 q(x)) - B_0(q(x)) - L(q(x))\| \\ &= \|T(x - B_0 q(x)) + B(q(x))\| \\ &\leqslant \|T\|(1 + \|B_0\|\|q\| + \|B\|\|q\|)\|x\|, \end{aligned}$$

shows that T_X is bounded. □

We now prove the exactness at $\mathcal{Q}(Z,E)$.

LEMMA 2. *Let* $F : Z \to E$ *be a quasi-linear map. Then* $F \circ q$ *is trivial if and only if there is* $T \in L(Y,E)$ *such that* $T \circ F_0$ *is a version of* F.

Proof. The 'if' part is trivial. As for the converse, suppose $F \circ q$ trivial. Writting $F \circ q = B + L$ we see that $-B_{|Y} = L_{|Y}$ is a bounded operator $Y \to E$ which we shall call T. Clearly,

$$\begin{aligned} T \circ F_0 - F &= L_{|Y} \circ F_0 - F \circ q \circ B_0 \\ &= L \circ (B_0 - L_0) - (B + L) \circ B_0 \\ &= L \circ B_0 - L \circ L_0 - B \circ B_0 - L \circ B_0 \\ &= -L \circ L_0 - B \circ B_0, \end{aligned}$$

and the result follows.

Let us finally verify the exactness at $\mathcal{Q}(X,E)$

LEMMA 3. *Let $F:X\to E$ be a quasi-linear map. If the restriction of F to Y is trivial then there exists a quasi-linear map $G:Z\to E$ such that F is a version of $G\circ q$.*

Proof. That $j^*\circ q^*=(q\circ j)^*=0$ is obvious. Since $F\circ j$ is trivial, there is a decomposition $F\circ j=B_Y-L_Y$, where B_Y and L_Y are, respectively a bounded and a linear map $Y\to E$. Let $B:X\to E$ be a homogeneous bounded extension of B_Y and let $L:X\to E$ be a linear extension of L_Y. Now put

$$G=(B_Y-B_X)\circ B_0-(L_Y-L_X)\circ L_0.$$

We show that $G\circ q$ is a version of F:

$$\begin{aligned}&(B_Y-B_X)\circ B_0\circ q-(L_Y-L_X)\circ L_0\circ q-(B_Y-L_Y)\\&=(B_Y\circ B_0\circ q-L_Y\circ L_0\circ q)-(B_X\circ B_0\circ q-L_X\circ L_0\circ q)-(B_Y-L_Y)\\&=B_Y\circ(B_0\circ q-\mathrm{Id})-L_Y\circ(L_0\circ q-\mathrm{Id})-(B_X\circ B_0\circ q-L_X\circ L_0\circ q)\\&=B_X(B_0\circ q-\mathrm{Id})-L_X\circ(L_0\circ q-\mathrm{Id})-(B_X\circ B_0\circ q-L_X\circ L_0\circ q)\\&=L_X-B_X.\end{aligned}$$

□

The algebraic part of the second half of Theorem 1 can be stated as follows.

THEOREM 3. *Let $0\to Y\overset{j}{\to}X\overset{q}{\to}Z\to 0$ be a short exact sequence in $\mathbf{Q}$, and let A be a quasi-Banach space. Then there exists an exact sequence in $\mathbf{V}$*

$$0\to L(A,Y)\overset{j_*}{\to}L(A,X)\overset{q_*}{\to}L(A,Z)\overset{\alpha}{\to}\mathcal{Q}(A,Y)\overset{j_*}{\to}\mathcal{Q}(A,X)\overset{q_*}{\to}\mathcal{Q}(A,Z).$$

The proof is quite similar to the previous one and so we give only a brief sketch. The maps j_*,q_* and α are given by left composition with, respectively, j,q and F_0. The exactness of the sequence is given by the following simple dualization of the three previous lemmata.

LEMMA 4. *Let $T\in L(A,Z)$. Then T lifts to X if and only if $F_0\circ T$ is trivial.*

LEMMA 5. *Let $F:A\to Y$ be a quasi-linear map. Then $j\circ F$ is trivial if and only if there is $T\in L(A,Z)$ so that F is equivalent to $T\circ F_0$.*

LEMMA 6. *Let $F:A\to Z$ be a quasi-linear map. Then $q\circ F$ is trivial if and only if there exists a version of F taking values in Y.*

The details are left to the reader. □

3. End of the Proof

To complete the proof of Theorem 1, we introduce on the spaces $\mathcal{Q}(\cdot,\cdot)$ the semi quasi-norm

$$Q([F]) = \inf\{Q(G) : G \in [F]\}.$$

(Again, we transfer it to the spaces of extensions.) Let us remark that a certain ambiguity appears here in that we use $Q(\cdot)$ to denote the quasi-linearity constant of a map (in which case we write $Q(F)$) and the semi quasinorm in $\mathcal{Q}(Z,Y)$ (in which case we write $Q([F])$). One has:

LEMMA 7. *All maps appearing in the homology sequences are continuous when the spaces $\mathcal{Q}(\cdot,\cdot)$ are equipped with the semi quasi-norm $Q(\cdot)$.*

Proof. It follows from the obvious inequalities $Q(F\circ T) \leqslant Q(F)\|T\|$ and $Q(T\circ F) \leqslant \|T\|Q(F)$. □

We prove now that the spaces $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ are always complete.

THEOREM 4. *Let Y, Z be quasi-Banach spaces. Then $Q(\cdot)$ is a complete semi-quasi-norm on $\mathcal{Q}(Z,Y)$*

Proof. By the Aoki-Rolewicz theorem [12], we assume that Y is a p-normed space for some $0 < p \leqslant 1$. In that case, it is clear that $Q(\cdot)$ is a semi p-norm on $\mathcal{Q}(Z,Y)$. Although it might not be entirely obvious at first glance, to obtain the desired completeness it suffices to show that every absolutely p-convergent series converges in $\mathcal{Q}(Z,Y)$.

Suppose $\sum_{n=1}^{\infty}[F_n]$ is such that $\sum_{n=1}^{\infty} Q([F_n])^p < +\infty$. Choose bounded maps $B_n : Z \to Y$ so that $\sum_{n=1}^{\infty} Q(F_n - B_n)^p < +\infty$. Fix a Hamel basis $\{e_\alpha\}$ for Z and define linear maps $L_n : Z \to Y$ by

$$L_n\left(\sum \lambda_\alpha e_\alpha\right) = \sum_\alpha \lambda_\alpha (F_n(e_\alpha) - B_n(e_\alpha)).$$

It is not hard to see that there exists a function $\varrho : Z \to \mathbb{R}^+$ such that

$$\|F_n(x) - B_n(x) - L_n(x)\|_Y \leqslant Q(F_n - B_n)\varrho(x)$$

holds for all n and $x \in Z$. Thus one has

$$\sum_{n=1}^{\infty} \|F_n(x) - B_n(x) - L_n(x)\|_Y^p \leqslant \varrho(x)^p \sum_{n=1}^{\infty} Q(F_n - B_n)^p.$$

Since absolutely p-convergent series converge in Y, we can define a map $F:Z\to Y$ as

$$F(x)=\sum_{n=1}^{\infty}(F_n(x)-B_n(x)-L_n(x)).$$

Clearly, $Q(F)\leqslant(\sum_{n=1}^{\infty}Q(F_n-B_n)^p)^{1/p}$ is finite, so that F is quasi-linear. Finally, a straightforward verification shows that the series $\sum_n[F_n]=\sum_n[F_n-B_n]$ converges to $[F]$ in $(\mathcal{Q}(Z,Y),Q(\cdot))$. □

This completes the proof of Theorem 1.

4. Elementary Applications

We now discuss some applications. Recall that a property P is said to be a a three-space property if every twisted sum of two spaces with P has P. Our first application straightforwardly follows from Theorem 1.

COROLLARY 1. *The properties* $\operatorname{Ext}_{\mathbf{Q}}(A,\cdot)=0$ *and* $\operatorname{Ext}_{\mathbf{Q}}(\cdot,B)=0$ *are three-space properties.*

Strange as it seems, these properties have not been previously studied in connection with the "three-space" problem, in spite of being at the core of the problem. Recall from [8; 11; 12] that a quasi-Banach space X is said to be a K-space if $\operatorname{Ext}_{\mathbf{Q}}(X,\mathbb{K})=0$. Ribe's example [16] shows that ℓ_1 is not a K-space; hence, see [8], $\mathcal{L}_1$-spaces are not K-spaces. On the positive side, B-convex spaces and $\mathcal{L}_p$ spaces are K-spaces for all $0<p\leqslant\infty, p\neq 1$ (see [8; 13]). Taking $B=\mathbb{K}$ in the preceding result, one has

COROLLARY 2. *To be a K-space is a three-space property.*

Amongst the new examples of K-spaces this result provides one finds the Z_p spaces [10] for $0<p<1$. Another interesting consequence is that all twisted sums of Hilbert spaces and $\mathcal{L}_\infty$ spaces, in particular those constructed in [14], are K-spaces. Observe that such spaces are not necessarily included in the previously known cases (i.e., they need not be either B-convex or quotients of $\mathcal{L}_\infty$-spaces).

In [12, theorems 5.1 and 5.2] the following result is proved, for which a straightforward proof follows from the homology sequence.

PROPOSITION 1. *Let X be a quasi-Banach K-space and let Y be a closed subspace of X. Then X/Y is a K-space if and only if every linear continuous functional on Y can be extended to a linear continuous functional on X* (*that is, if Y has the Hahn-Banach extension property in X, or HBEP in short*). *In particular, quotients of Banach K-spaces are K-spaces.*

Proof. Simply observe that the sequence

$$0 \to Z^* \to X^* \to Y^* \to \mathrm{Ext}_{\mathbf{Q}}(Z, \mathbb{K}) \to \mathrm{Ext}_{\mathbf{Q}}(X, \mathbb{K}) \to \mathrm{Ext}_{\mathbf{Q}}(Y, \mathbb{K})$$

is exact. Since X is a K-space, $\mathrm{Ext}_{\mathbf{Q}}(X, \mathbb{K}) = 0$. It is therefore clear that $\mathrm{Ext}_{\mathbf{Q}}(Z, \mathbb{K}) = 0$ if and only if $X^* \to Y^*$ is surjective. □

5. Dierolf's Result Revisited

THEOREM 5. (DIEROLF, SEE [4]). *Let Z be a Banach space. If some nonlocally convex twisted sum of a Banach space Y and Z exists then some nontrivial* (*hence nonlocally convex*) *twisted sum of the ground field and Z exists.*

Before entering into the proof, perhaps it is worth to state as a lemma the following remark.

LEMMA 8. *Let* $0 \to Y \to X \to Z \to 0$ *be an exact sequence in which Y and Z are Banach spaces. Then X is a Banach space if and only if Y has the HBEP in X.*

Proof. Let $co(X)$ denote the containing Banach space of X (that is the Banach space that X generates in X^{**}). It is clear that operators on X with values in locally convex spaces factorize through $co(X)$, and so does the quotient map. Observe thus the diagram

$$\begin{array}{ccccccc} 0 \to Y \to & X & \to Z \to & 0 \\ & \downarrow & \| & \\ & co(X) & \to Z \to & 0. \end{array}$$

Now, if Y has the HBEP in X then the sequence $0 \to Z^* \to X^* \to Y^* \to 0$ is exact, and so is the bidual sequence $0 \to Y^{**} \to X^{**} \to Z^{**} \to 0$. Observe that the topology that Y receives from Y^{**} is the same it receives from X^{**}, namely, from $co(X)$. This completes the diagram to

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & Y & \to & co(X) & \to & Z & \to & 0. \end{array}$$

and the three-lemma provides the isomorphism betwen X and $co(X)$. □

We pass to a proof of Dierolf's result.

Proof. Let $0 \to Y \to X \to Z \to 0$ be an exact sequence. It induces the sequence

$$0 \to Z^* \to X^* \to Y^* \xrightarrow{\omega} \mathrm{Ext}_{\mathbf{Q}}(Z, \mathbb{K}) \to \cdots$$

If X is not locally convex then some functional $y^* \in Y^*$ cannot be extended to X by 8, hence $\omega(y^*)$ represents a non-trivial extension of Z by $\mathbb{K}$. □

6. Further Applications

It should be noted that only the algebraic nature of the homology sequences was exploited so far. We give now some applications in which the topology of the spaces of extensions plays a major role. We need the following "uniform boundedness" type result of Kalton.

LEMMA 9. [8, proposition 3.3] *Suppose Y and Z are quasi-Banach spaces such that* $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)=0$. *Then there exists a constant C such that to each quasi-linear map $F:Z\to Y$ there corresponds a linear map $L:Z\to Y$ with $\|F(x)-G(x)\|\leqslant CQ(F)\|x\|$ for all $x\in Z$.*

We do not know if there exist Banach spaces Y and Z for which the space $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$ is Hausdorff, apart from the trivial case in which $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)=0$. Sometimes this difficulty can be surmounted when one passes to the quasi-Banach setting. Recall that a quasi-Banach space is said to be an ultrasummand [9] if it is complemented in its ultrapowers.

THEOREM 6. *Let Y be a subspace of a quasi-Banach space X and let E be an ultrasummand. Assume that $L(X,E)=0$ and that* $\mathrm{Ext}_{\mathbf{Q}}(X,E)=0$. *Then* $\mathrm{Ext}_{\mathbf{Q}}(X/Y,E)$ *is Hausdorff.*

Proof. Let us write $Z=X/Y$ and let $q:X\to Z$ be the quotient map. Suppose $F:Z\to E$ is a quasi-linear map such that $Q([F])=0$. Then there is a sequence $B_n:Z\to E$ of bounded maps such that

$$\lim_{n\to\infty} Q(F-B_n)=0.$$

We claim that there exists a constant M such that $\|B_n(z)\|\leqslant M\|z\|$ for all $n\in\mathbb{N}$ and $z\in Z$.

Observe that $Q(B_n)\leqslant 1+Q(F)$ for large n. Hence, if we consider the maps $B_n\circ q:X\to E$ then it is clear that $\sup_{n\in\mathbb{N}} Q(B_n\circ q)<+\infty$. By the preceding lemma, since $\mathrm{Ext}_{\mathbf{Q}}(X,E)=0$ there exist linear maps $L_n:X\to E$ and some finite constant M such that for all $x\in X$

$$\|B_n\circ q(x)-L_n(x)\|\leqslant M\|x\|.$$

Since each $B_n\circ q$ is bounded, these estimates imply that $L_n\in L(X,E)$, hence $L_n=0$ for all $n\in\mathbb{N}$. Thus, $\|B_n(q(x))\|\leqslant M\|x\|$, and since q is a quotient map we arrive to $\|B_n(z)\|\leqslant M\|z\|_Z$, as we claimed.

To conclude the proof, take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define the map B: $Z\to E_{\mathcal{U}}$ by $B(z)=[B_n(z)]$. Now take a linear continuous projection $\pi:E_{\mathcal{U}}\to$

E. Clearly $Q(F-\pi\circ B)=0$, which means that $F-\pi\circ B$ is linear. Therefore F is the sum of a bounded and a linear map and $[F]=0$. This completes the proof. □

COROLLARY 3. *Let X be a quasi-Banach space and E an ultrasummand. Assume that $L(X,E)=0$ and $\mathrm{Ext}_{\mathbf{Q}}(X,E)=0$. If Y is a quasi-Banach subspace of X then $L(Y,E)=\mathrm{Ext}_{\mathbf{Q}}(X/Y,E)$ as quasi-Banach spaces.*

Proof. We set $Z=X/Y$. In view of the exactness of $0\to L(Z,E)\to L(X,E)\to L(Y,E)\to \mathrm{Ext}_{\mathbf{Q}}(Z,E)\to \mathrm{Ext}_{\mathbf{Q}}(X,E)\to \mathrm{Ext}_{\mathbf{Q}}(Y,E)$ and since $L(X,E)=0$ and $\mathcal{Q}(X,E)=0$, it follows that $\omega: L(Y,E)\to \mathrm{Ext}_{\mathbf{Q}}(Z,E)$ is a continuous bijection. Since $\mathrm{Ext}_{\mathbf{Q}}(Z,E)$ is Hausdorff, the open mapping theorem yields that ω is a topological isomorphism. □

These results introduce new invariants in (quasi-) Banach space theory. For instance, given a K-space with trivial dual, $\mathrm{Ext}_{\mathbf{Q}}(X/\cdot,\mathbb{R})$ acts as an invariant since it distinguishes subspaces of X in the following sense:

COROLLARY 4. *Let X be quasi-Banach K-space with trivial dual. If Y is a subspace of X then $Y^*=\mathrm{Ext}_{\mathbf{Q}}(X/Y,\mathbb{K})$, as Banach spaces.*

Examples of quasi-Banach K-spaces with trivial dual are provided by the L_p-spaces for $0<p<1$. Thus we obtain a new proof (for A and B finite dimensional) and extend (for the general case) some results of Kalton and Peck [11].

COROLLARY 5. *If A and B are subspaces of $L_p, 0<p<1$, such that L_p/A and L_p/B are isomorphic, then A^* and B^* are isomorphic. In particular, if A and B are finite-dimensional subspaces of L_p of distinct dimension, then L_p/A and L_p/B are not isomorphic.*

REMARK 1. For $0<p<1$ the spaces L_p/J_p and L_p/H^p are isomorphic [1], while J_p and H^p are not [9]. Thus one cannot expect that the isomorphism between L_p/A and L_p/B forces A and B to be isomorphic, even when both spaces have separating duals.

These results can be dualized applying the homology sequence in the second variable.

THEOREM 7. *Let Y be a subspace of a quasi-Banach space X. Let A be a quasi-Banach space. If $\mathcal{Q}(A,X)=0$ and $L(A,X)=0$ then $L(A,X/Y)$ and $\mathcal{Q}(A,Y)$ are isomorphic quasi-Banach spaces.*

Proof. Set $Z=X/Y$ and let $\alpha: L(A,Z)\to \mathcal{Q}(A,Y)$ be the map given by the homology sequence, that is, $\alpha(T)=[F_0\circ T]$. Clearly, α is a continuous bijection, but since we cannot guarantee a priori that $\mathcal{Q}(Z,Y)$ is Hausdorff, some work must be done.

We explicitly describe α^{-1} and prove that it is continuous. Let $G: A\to Y$ be a quasi-linear map and consider the composition $j\circ G$, where $j: Y\to X$

is the inclusion map. Since $\mathrm{Ext}_{\mathbf{Q}}(A,X)=0$ one has $j\circ G=B+L$, where $B:A\to X$ is bounded and $L:A\to X$ is linear. The hypothesis $L(A,X)=0$ makes the decomposition unique. (Indeed, let $j\circ G=B_1+L_1=B_2+L_2$ be two decompositions. Then $B_2-B_1=L_1-L_2$ is simultaneously linear and bounded and, therefore, $B_2=B_1$ and $L_2=L_1$.)

Observe that, by Lemma 9, one has

$$\|j\circ G(x)-L(x)\|=\|B(x)\|\leqslant CQ(G)\|x\|$$

for all x, where C depends only on A and X. Even if L may be unbounded, the linear map $T:A\to Z$ given by $T=q\circ L$ is continuous: let $a\in A$. One has

$$\begin{aligned}\|Ta\|_Z&=\inf\{\|x\|_X:x\in X, qx=Ta\}\leqslant\|j\circ G(a)-L(a)\|_X\\&=\|Ba\|_X\leqslant CQ(G)\|a\|_A.\end{aligned}$$

Next, we show that $-\alpha(T)=[G]$. This clearly implies that T depends only on the class of G and also that $\|T\|\leqslant CQ([G])$ (not merely $\|T\|\leqslant CQ(G)$) which means that α^{-1} is continuous. Obviously, $-\alpha(T)-[G]=[-F_0\circ T-G]$. Since

$$\begin{aligned}-F_0\circ T-G&=(L_0-B_0)\circ q\circ L-(\mathrm{Id}-B_0\circ q)\circ j\circ G\\&=L_0\circ q\circ L-B_0\circ q\circ L-(\mathrm{Id}-B_0\circ q)\circ(B+L)\\&=L_0\circ q\circ L-B_0\circ q\circ L-B-L+B_0\circ q\circ B+B_0\circ q\circ L\\&=(L_0\circ q\circ L-L)+(B_0\circ q\circ B-B)\\&=(L_0\circ q-\mathrm{Id})\circ L+(B_0\circ q-\mathrm{Id})\circ B,\end{aligned}$$

the proof is complete. □

Let us remark that it is also possible to obtain in this way a proof of the isomorphism in Corollary 3. Curiously enough, this proof allows one to drop the assumption "E ultrasummand" on Theorem 6. Nevertheless, we feel that the two proofs are complementary and that each of them contains some information that, in some sense, is not explicit in the other. For this reason, we present a sketch of this new proof.

THEOREM 8. *Let Y be a subspace of a quasi-Banach space X. Let E be a quasi-Banach space. If $Q(X,E)=0$ and $L(X,E)=0$ then $L(X/Y,E)$ and $Q(X/Y,E)$ are isomorphic quasi-Banach spaces.*

Proof. Write $X/Y=Z$ and let $\omega:L(Y,E)\to Q(X/Y,E)$ be the continuous bijection $\omega(T)=[T\circ F_0]$ given by the homology sequence. Just as before, we identify ω^{-1} and prove that it is continuous. Let $G:Z\to E$ be a quasi-linear map. Since $G\circ q:X\to E$ is trivial there exists a unique decomposition $G\circ q=B+L$, with B bounded and L linear. Moreover, $\|B(x)\|\leqslant CQ(G)\|x\|$, for some constant depending only on X and E. Now $G\circ q$ vanishes on Y hence $B_{|Y}=-L_{|Y}$ is a bounded operator $Y\to E$ which we shall call T. Obviously, $\|T\|\leqslant CQ(G)$. To conclude the proof we show that $-\omega(T)=[G]$. Clearly, $-\omega(T)-[G]=[-T\circ F_0-G]$. Since $-T\circ F_0-G=L\circ L_0-B\circ B_0$ the result follows. □

COROLLARY 6. *Suppose A and B are subspaces of $L_p, 0<p<1$, such that L_p/A and L_p/B are isomorphic. If E is a q-Banach space for some $q>p$ then the spaces of operators $L(A,E)$ and $L(B,E)$ are isomorphic.*

Proof. That $L(L_p,E)=0$ is clear. That $\mathrm{Ext}_{\mathbf{Q}}(L_p,E)=0$ for any q-Banach space E with $q>p$ was proved by Kalton in [8, theorem 3.6]. □

7. Appendix: the Classical Algebraic Approach

The algebraic constructions of pull-back and push-out can be found in [6]; their realizations in **Q** and **B** can be seen in [3] or [7]. We sketch both.

Pull-back construction. Given an exact sequence $0\to Y\overset{j}{\to}X\overset{q}{\to}Z\to 0$ and an operator $T:M\to Z$ the pull-back exact sequence induced by the couple $\{q,T\}$ is the lower row in the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \uparrow & & \uparrow T & & \\ 0 & \to & Y & \to & PB & \to & M & \to & 0 \end{array}$$

The pull-back space PB is the closed subspace of $X\oplus M$ defined as $PB=\{(x,m)\in X\oplus M: qx=Tm\}$. The arrows $PB\to M$ and $PB\to X$ are the restrictions to PB of the canonical projections. The inclusion $Y\to PB$ is given by $y\to(y,0)$. It is clear that the pull-back sequence splits if and only if T lifts to X. Observe that the pull-back construction shows that every operator $T:M\to Z$ induces a map $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)\to\mathrm{Ext}_{\mathbf{Q}}(M,Y)$ for each fixed quasi-Banach space Y. Thus, the set $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ depends functorially on the first variable.

Push-out construction. Given an exact sequence $0\to Y\overset{j}{\to}X\overset{q}{\to}Z\to 0$ and an operator $T:Y\to M$ the push-out exact sequence induced by the couple $\{T,j\}$ is the lower row in the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & T\downarrow & & \downarrow & & \| & & \\ 0 & \to & M & \to & PO & \to & Z & \to & 0 \end{array}$$

The push-out space PO is the quotient space $(M\oplus X)/\Delta$ where $\Delta=\{(Ty,y)\in M\oplus X\}$. The arrows $M\to PO$ and $X\to PO$ are the composition of the natural injections into $M\oplus X$ with the quotient map. The quotient map $PO\to Z$ is given by $(m,x)+\Delta y\to qx$. The push-out sequence splits if and only if T extends to X. The push-out construction shows that set $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ depends functorially on the second variable.

In this way, $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ becomes a (bi-) functor on **Q** with values is the category of sets and maps. It is contravariant in the first variable and covariant in the second one.

Linear structure. The addition in $\mathrm{Ext}_{\mathbf{Q}}(B,A)$ can be defined as follows (Baer's sum): if $0\to A\to X\to B\to 0$ and $0\to A\to X_1\to B\to 0$ are two exact sequences then the sum is defined as the exact sequence obtained after considering the sequence $0\to A\oplus A\to X\oplus X_1\to B\oplus B\to 0$ and making pull-back with the map $\Delta: B\to B\oplus B$ given by $\Delta(x)=(x,x)$ and then push-out with $S: A\oplus A\to A$ given by $S(x,y)=x+y$. The product of a sequence $0\to Y\to X\to Z\to 0$ by the number λ is obtained making pull-back with the dilation $\lambda\mathrm{Id}: Z\to Z$ (or else making push out with $\lambda\mathrm{Id}: Y\to Y$). Under these operations the sets $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$ become linear spaces and the functor $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)$ takes values in $\mathbf{V}$.

The natural transformation $\mathrm{Ext}_{\mathbf{Q}}(\cdot,\cdot)\leftrightsquigarrow\mathcal{Q}(\cdot,\cdot)$. The process for obtaining a quasi-linear map from an exact sequence and vice versa establishes a natural transformation between the two functors. This means that under that transformation the pull-back sequence obtained from an exact sequence defined by a quasi-linear map F and an operator T is the sequence corresponding to the quasi-linear map $F\circ T$, that the push-out sequence obtained from an exact sequence defined by a quasi-linear map F and an operator T is the sequence corresponding to the quasi-linear map $T\circ F$ and, finally, that the correspondence is linear in the sense that the sum of quasi-linear maps corresponds to the sum of extensions.

With the notation of Theorem 1, let us establish the existence and exactness of a sequence

$$\begin{aligned}&0\to L(Z,E)\overset{q^*}{\to}L(X,E)\overset{j^*}{\to}L(Y,E)\overset{\omega}{\to}\\&\mathrm{Ext}_{\mathbf{Q}}(Z,E)\overset{\alpha}{\to}\mathrm{Ext}_{\mathbf{Q}}(X,E)\overset{\beta}{\to}\mathrm{Ext}_{\mathbf{Q}}(Y,E).\end{aligned}$$

We define the connecting morphism ω. If $T: Y\to E$ is an operator, then $\omega(T)$ is the push-out sequence generated by the couple $\{j,T\}$ as in the diagram:

$$\begin{array}{ccccccccc}0&\to&Y&\overset{j}{\to}&X&\to&Z&\to&0\\&&T\downarrow&&\downarrow&&\|&&\\0&\to&E&\to&PO&\to&Z&\to&0.\end{array}$$

The maps α and β are the (functorial) images of the maps q and j of the starting sequence. Thus α is making pull-back with q, that is, the image of the (equivalence class of the) extension $0\to E\to W\to Z\to 0$ under α is the lower exact sequence in the pull-back square

$$\begin{array}{ccccccccc}0&\to&E&\to&W&\to&Z&\to&0\\&&\|&&\uparrow&&\uparrow q&&\\0&\to&E&\to&PB&\to&X&\to&0.\end{array}$$

Analogously, β transforms the extension $0\to E\to V\to X\to 0$ into the lower exact sequence in the pull-back square

$$\begin{array}{ccccccccc}0&\to&E&\to&V&\to&X&\to&0\\&&\|&&\uparrow&&\uparrow j&&\\0&\to&E&\to&PB&\to&Y&\to&0.\end{array}$$

It is not hard to verify that these maps are well defined and linear. Let us see that the sequence so obtained is exact.

Proof that Im j^* = Ker ω. This is just a different statement of the already mentioned splitting criterion for push-out sequences.

Proof that Im ω = Ker α. The containment Im $\omega \subset$ Ker α is consequence of the fact that making push-out and then pull-back produces an equivalent sequence to that obtained making first pull-back and then push-out; now, the pull-back sequence

$$\begin{array}{ccccccc} 0 \to & Y & \overset{j}{\to} & X & \overset{q}{\to} & Z & \to 0 \\ & \| & & \uparrow & & \uparrow q & \\ 0 \to & Y & \to & PB & \to & X & \to 0 \end{array}$$

splits: since $PB = \{(x, x') \in X \oplus X : qx = qx'\}$ the map $\pi(x, x') = (x - x', 0)$ is a linear continuous projection onto $Y = \{(y, 0) : y \in Y\}$. We prove then that Ker $\alpha \subset$ Im ω. Let $0 \to E \to W \to Z \to 0$ be an element of $\mathrm{Ext}_{\mathbf{Q}}(Z, E)$ belonging to Ker α. One has the diagram

$$\begin{array}{ccccccc} 0 \to & E & \to & W & \to & Z & \to 0 \\ & \| & & \uparrow & & \uparrow q & \\ 0 \to & E & \overset{\curvearrowleft}{\to} & PB & \to & X & \to 0 \\ & & & \uparrow & & \uparrow & \\ & & & Y & = & Y & , \end{array}$$

in which the second row splits. Hence the upper row is (equivalent to) the push-out sequence corresponding to the operator $T : Y \to E$ defined by composing the inclusion $Y \to PB$ with any bounded linear projection of PB onto E.

Proof that Im α = Ker β. It is clear that $\beta \circ \alpha = 0$ since one is making pull-back with 0. Hence Im $\alpha \subset$ Ker β. On the other hand, consider an element of $\mathrm{Ext}_{\mathbf{Q}}(X, E)$ belonging to Ker β. We then have an exact sequence $0 \to E \to V \to X \to 0$ that becomes trivial when restricted to Y. Therefore Y is a subspace of V. Since the diagram

$$\begin{array}{ccccccccc} 0 \to & E & \to & V & \to & X & \to & 0 \\ & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & 0 & \to & Y & = & Y & \to & 0 \end{array}$$

is commutative with exact rows, it can be completed with the exact row of the quotients yielding

$$\begin{array}{ccccccc} 0 \to & E & \to & V/Y & \to & Z & \to 0 \\ & \| & & \uparrow & & \uparrow & \\ 0 \to & E & \to & V & \to & X & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & 0 & \to & Y & = & Y & \to 0 \end{array}$$

which shows that our extension $0 \to E \to V \to X \to 0$ is a pull-back of $0 \to E \to V/Y \to Z \to 0$.

Proceeding in a completely analogous (dual) fashion one obtains the homology sequence corresponding to the covariant case.

8. Concluding Remarks and Questions

It seems interesting to know if the results here presented can be extended to the category **F** of F-spaces, that is, complete metrizable topological vector spaces (or any other reasonable category including the space L_0 among its members). The category **F** is as reasonable as it can be and contains most of the natural examples. However, the representation we have considered for $\mathrm{Ext}(\cdot,\cdot)$ simply does not work; nevertheless, the abstract viewpoint developed in the Appendix obviously does (pull-backs and push-outs clearly exist and the open mapping theorem still holds in **F**). Hence the results of the paper that depend only on the linear structure of extensions (namely Theorem 1, Corollaries 1 and 2, Proposition 1, the whole of Section 5 defining the containing Fréchet space of an F-space via the Mackey topology as in [12], and, finally, Corollary 5 for finite-dimensional A and B) can be translated to F-spaces. We leave the details to the reader.

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