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UNIFORM BOUNDEDNESS AND TWISTED SUMS OF BANACH SPACES

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Abstract. We construct a Banach space X admitting an uncomplemented copy of l_1 so that $X/l_1 = c_0$. To do that we study the uniform boundedness principles that arise when one considers exact sequences of Banach spaces; as well as several elements of homological algebra applied to the construction of nontrivial twisted sum of Banach spaces. The combination of both elements allows one to determine the existence of nontrivial twisted sums for almost all combinations of classical Banach spaces.

1. Introduction

The objective of this paper is to be able to construct a nontrivial twisted sum of l_1 and c_0; precisely, to show the existence of a Banach space X admitting an uncomplemented copy of l_1 so that $X/l_1 = c_0$. To do that we need to develop certain uniform boundedness principles that arise when one considers exact sequences of Banach spaces and some homological constructions applied to the construction of twisted sum of Banach spaces. The combination of both approaches will show the existence of nontrivial twisted sums for almost all combinations of classical Banach spaces.

This is not the first time that the existence "uniform boundedness principles" for twisted sums of (quasi) Banach spaces has been considered. Domański [12] developed similar principles; while some of these topics have been recently considered in [20]. There, some kind of uniform boundedness principle is used to attack the problem of which spaces admit a nontrivial twisted sum with l_2. However,

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those papers are not easy to read, the former because of the lack of an algebraic approach forces the author to develop time after time some ingenious devices of his own to resolve situations; and the latter because of the rather laconic form in which the difficult point of exactly how one can paste together the pieces (or proceed to obtain local information) is treated. Moreover, the approaches of those two papers are almost disjoint: while Kalton and Pełczyński use hard local theory of Banach spaces (behaviour of cotype constants, Maurey's theorem ...), Domański has a more operator-ideal oriented point of view. In conclusion, that while there is no doubt that some experts are aware of the existence of some principles (see, e.g. [14], where the author mentions that "some localization technique" allows to conclude his argument) we think that a comprehensive theory for such principles has not been yet developed. Moreover, none of the known principles is able to produce a nontrivial twisted sum of l_1 and c_0. Such a construction can be seen (extremely) implicit in [2]; and also in [3].

In this paper we present a unified, and far simpler, approach to the topic based on the use of the elements of homological algebra and the theory of quasi-linear maps.

Let us briefly outline the contents of the paper. Section 2 of the paper introduces the background required (some elements of homological algebra and an adaptation to the locally convex setting of the theory of quasi-linear maps). Section 3 contains the uniform boundedness principles. The section 4 contains nontrivial (in both senses of the word) examples of twisted sums of classical Banach spaces, culminating in the promised nontrivial exact sequence $0 \to l_1 \to X \to c_0 \to 0$. The last section of the paper characterizes $\mathcal{L}_1$-spaces showing that Z is a $\mathcal{L}_1$-space if and only if $\mathrm{Ext}(R, Z) = 0$ for every reflexive space R.

2. Background, with some applications

In the sequel, the word operator means linear continuous map. For general information about exact sequences the reader can consult [13]. Information about categorical constructions in the Banach space setting can be found in the monograph [5] or in Johnson paper [14]. A diagram $0 \to Y \to X \to Z \to 0$ of Banach spaces and operators is said to be an exact sequence if the kernel of each arrow coincides with the image of the preceding. This means, by the open mapping theorem, that Y is (isomorphic to) a closed subspace of X and the corresponding quotient is (isomorphic to) Z. We shall also say that X is a twisted sum of Y and Z or an extension of Y by Z. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there is an operator

$T : X \to X_1$ making the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

commutative. The "three-lemma" (see [13, lemma 1.1]) and the open mapping theorem imply that T must be an isomorphism. The exact sequence $0 \to Y \to X \to Z \to 0$ is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This already implies that the twisted sum X is isomorphic to the direct sum $Y \oplus Z$ (the converse is not true). Given two Banach spaces Y and Z we denote by $\operatorname{Ext}(Z, Y)$ the set of exact sequences $0 \to Y \to X \to Z \to 0$ modulo equivalence. Thus, $\operatorname{Ext}(Z, Y) = 0$ means that every exact sequence $0 \to Y \to X \to Z \to 0$ splits.

The pull-back square. Let $A : U \to Z$ and $B : V \to Z$ be two operators. The pull-back of $\{A, B\}$ is the space $PB = \{(u, v) : Au = Bv\} \subset U \times V$ endowed with the relative product topology, together with the restrictions of the canonical projections of $U \times V$ onto, respectively, U and V. If $0 \to Y \to X \to Z \to 0$ is an exact sequence with quotient map q and $T : V \to Z$ is an operator and PB denotes the pull-back of the couple $\{q, T\}$ then the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \uparrow & & \uparrow T & & \\ 0 & \to & Y & \to & PB & \to & V & \to & 0 \end{array}$$

is commutative with exact rows. The three-lemma implies that the operator $PB \to X$ is onto if and only T is. It is well-known that the pull-back sequence splits if and only if the operator T can be lifted to X; i.e., there exists an operator $\tau : V \to X$ such that $q \circ \tau = T$.

2.1. **Application: nontrivial twisted sums of c_0 and l_∞.** Consider a nontrivial sequence $0 \to c_0 \to JL_2 \to l_2(I) \to 0$, where I has the power of continuum (see [15]). The pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & c_0 & \to & JL_2 & \to & l_2(I) & \to & 0 \\ & & \| & & \uparrow & & \uparrow q & & \\ 0 & \to & c_0 & \to & PB & \to & l_\infty & \to & 0 \end{array}$$

where $q : l_\infty \to l_2(I)$ is a quotient map gives a nontrivial exact sequence as we now show. The space PB cannot be isomorphic to $c_0 \oplus l_\infty$ since JL_2 cannot be a quotient of $c_0 \oplus l_\infty$: if Q is such a quotient map, since no nonseparable subspace of JL_2 is isomorphic to a subspace of l_∞ ([15]), then by Rosenthal's theorem [24]

$Q|_{l_\infty}$ is weakly compact and thus its range is separable, and the same occurs to $Q|_{c_0}$.

2.2. **Application: nontrivial twisted sums of c_0 and $L_1(\mu)$.** Start with a nontrivial exact sequence $0 \to c_0 \to JL_\infty \to c_0(I) \to 0$ (see again [15]), and observe that for some finite measure μ there exists a quotient map from $L_1(\mu)$ onto $c_0(I)$ (see [24]). The space $L_1(\mu)$ is WCG since μ is finite. The pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & c_0 & \to & JL_\infty & \to & c_0(I) & \to & 0 \\ & & \| & & \uparrow & & \uparrow q & & \\ 0 & \to & c_0 & \to & PB & \to & L_1(\mu) & \to & 0 \end{array}$$

produces a non WCG space PB that cannot therefore be isomorphic to the product $c_0 \oplus L_1(\mu)$.

The push-out square. Let $A : K \to X$ and $B : K \to M$ be two operators. The push-out of $\{A, B\}$ is the space $PO = (M \oplus X)/\overline{\Delta}$, where $\Delta = \{(Ak, Bk) : k \in K\}$ endowed with the quotient topology, together with the restrictions of the quotient map to, respectively, M and X. If $0 \to Y \to X \to Z \to 0$ is an exact sequence with injection j and $T : Y \to M$ is an operator and PO denotes the push-out of the couple $\{j, T\}$ then the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & T\downarrow & & \downarrow & & \| & & \\ 0 & \to & M & \to & PO & \to & Z & \to & 0 \end{array}$$

is commutative with exact rows. Note that the operator $X \to PO$ is an isomorphic embedding if and only if T is. The push-out sequence splits if and only if the operator T can be extended to X; i.e., there exists an operator $\tau : X \to M$ such that $\tau \circ j = T$.

Zero-linear maps and extensions. The by now classical theory of Kalton and Peck [16, 19] describes short exact sequences of quasi-Banach spaces in terms of the so-called quasi-linear maps. The theory of quasi-linear maps can be easily adapted to the locally convex (Banach) setting as follows (see [4]). A map $F : Z \to Y$ acting between Banach spaces is said to be zero-linear if it is homogeneous and satisfies that, for some constant K and all points $x_i \in Z$, one has

$$\left\| F\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} F(x_i) \right\| \leq K \left(\sum_{i=1}^{n} \|x_i\|\right).$$

The infimum of the constants K as above is denoted $Z(F)$ and referred to as the zero-linearity constant of F. As in the quasi-linear case, zero-linear maps give rise to twisted sums: given a zero-linear map $F : Z \to Y$, it is possible to construct a twisted sum, which we shall denote by $Y \oplus_F Z$, endowing the product space $Y \times Z$ with the quasi-norm $\|(y,z)\|_F = \|y - Fz\| + \|z\|$ which is always equivalent to a norm that makes $Y \oplus_F Z$ into a Banach space. Clearly, the subspace $\{(y,0) : y \in Y\}$ is isometric to Y and the corresponding quotient is isomorphic to Z. The fundamental result [19, theorem 2.5] establishes that if F and G are zero-linear maps from Z to Y then the induced sequences $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y \to Y \oplus_G Z \to Z \to 0$ are equivalent if and only if the difference $G - F$ is at finite distance from some linear (not necessarily continuous!) map $L : Z \to Y$. Here, the (possibly infinite) distance between two homogeneous maps A and B (acting between the same Banach spaces) is given by

$$dist(A,B) = \inf\{C : \|A(x) - B(x)\| \leq C\|x\| \text{ for all } x\}.$$

Consequently $0 \to Y \to Y \oplus_F Z \to Z \to 0$ splits if and only if F is at finite distance from some linear map $Z \to Y$ (in which case we shall say that F is trivial). Conversely, zero-linear maps arise from exact sequences: given a short exact sequence $0 \to Y \to X \to Z \to 0$ of Banach spaces a zero-linear map $F : Z \to Y$ can be obtained taking a linear (possibly non-continuous) selection $L : Z \to X$ for the quotient map $q : X \to Z$ and a bounded homogeneous (possibly non-linear nor continuous) selection $B : Z \to X$ for q. The difference $F = B - L$ is then zero-linear and takes values in Y since $q \circ (B - L) = 0$. Moreover the original sequence $0 \to Y \to X \to Z \to 0$ is equivalent to the sequence $0 \to Y \to Y \oplus_F Z \to Z \to 0$ induced by F.

3. Uniform boundedness principles. First applications

The introduction of zero-linear maps permits us to quantify the fact that every extension $0 \to Y \to X \to Z \to 0$ is trivial without making any reference to the unknown middle spaces X. For instance, if Y and Z are finite-dimensional Banach spaces, then obviously $\mathrm{Ext}(Z,Y) = 0$. However, in a sense, this information is irrelevant. From the "local" point of view, it is more interesting to have some information about the relation between the zero-linearity constant of the maps $F : Z \to Y$ and the corresponding distance to the space of linear maps $Z \to Y$. The most important result in this line is the following version of [16, proposition 3.3] for zero-linear maps. It can be proved as [16, proposition 3.3]. It asserts that if Z and Y are Banach spaces such that every zero-linear map $F : Z \to Y$ is at finite distance from some linear maps $Z \to Y$, then that distance depends only on $Z(F)$. Precisely:

Theorem 1 (Mainly Kalton [16])**.** *Let Y and Z be Banach spaces such that* $\mathrm{Ext}(Z,Y)=0$. *Then there exists a constant C such that, for every zero-linear map $F: Z \to Y$, one has $dist(F,L) \le CZ(F)$ for some linear map $L: Z \to Y$.*

The result can be used to construct nontrivial twisted sums with "smaller" spaces once nontrivial twisted sums with "larger" spaces having the same local structure has been obtained. Some "compactness" condition is necessary to be able to past the pieces together: for instance that the target space is complemented in the bidual.

Let $\mathcal{E}$ be a family of finite dimensional Banach spaces. A Banach space X is said to be locally $\mathcal{E}$ if there is some constant C so that every finite dimensional subspace of X is contained in another finite dimensional subspace F of X such that for some $E \in \mathcal{E}$ one has $d(F,E) \le C$, where $d(\cdot,\cdot)$ denotes the (multiplicative) Banach-Mazur distance. The space X is said to contain $\mathcal{E}$ uniformly complemented if there is a constant C such that every element of $\mathcal{E}$ is C-isomorphic to some C-complemented subspace of X. The class of all Banach spaces that are locally $\mathcal{E}$ shall be denoted $\Lambda(\mathcal{E})$. The class of all Banach spaces containing $\mathcal{E}$ uniformly complemented shall be denoted $\Pi(\mathcal{E})$.

Theorem 2. *Let Y be a Banach space complemented in its bidual and let $\mathcal{E}$ be a family of finite dimensional Banach spaces. If* $\mathrm{Ext}(W,Y)=0$ *for some $W \in \Pi(\mathcal{E})$ then* $\mathrm{Ext}(Z,Y)=0$ *for every $Z \in \Lambda(\mathcal{E})$.*

PROOF. The hypothesis $\mathrm{Ext}(W,Y)=0$, together with Theorem 1 implies the existence of a constant C such that, for every $E \in \mathcal{E}$ and every zero-linear map $F: E \to Y$, there exists a linear map $L: E \to Y$ with $dist(F,L) \le C \cdot Z(F)$.

Now, let $F: Z \to Y$ be a zero-linear map. Since Z is locally in $\mathcal{E}$, there is a cofinal subnet $\mathcal{G}$ of the net of all finite-dimensional subspaces of Z such that, for every $G \in \mathcal{G}$, there is $E \in \mathcal{E}$ with $d(G,E) \le M$. For each $G \in \mathcal{G}$, let F_G denote the restriction of F to G. Clearly, F_G is zero-linear, with $Z(F_G) \le Z(F)$. So, we can choose linear maps $L_G: G \to Y$ so that $dist(F_G, L_G) \le C \cdot M \cdot Z(F)$ for all $G \in \mathcal{G}$.

Let U be an ultrafilter refining the Fréchet (= order) filter on $\mathcal{G}$. We can define a linear map $L: Z \to Y^{**}$ taking

$$L(z) = {}^{*}\text{weak-}\lim_{U(G)} L_G(z).$$

The definition makes sense because, for every $z \in Z$, the net $L_G(z)$ is well-defined for G large enough and bounded: $\|L_G(z)\| \le \|F(z)\| + MCZ(F)\|z\|$.

Finally, let π be a bounded linear projection of Y^{**} onto Y. Then the composition $\pi \circ L$ is a linear map from Z to Y, with

$$dist(F, \pi \circ L) \leq MCZ(F)\|\pi\|.$$

Hence $\mathrm{Ext}(Z, Y) = 0$. This completes the proof. □

The preceding proof contains the finite dimensional versions that we state separately for the sake of clarity. Given families $\mathcal{E}$ and $\mathcal{F}$ of Banach spaces we shall say that $\mathrm{Ext}(\mathcal{F}, \mathcal{E}) = 0$ *uniformly* if there is a constant C such that, for every couple of spaces $A \in \mathcal{E}$ and $B \in \mathcal{F}$, one has that for every zero-linear map $F : B \to A$, there is a linear map $L : B \to A$ so that $dist(F, L) \leq CZ(F)$.

Theorem 3. *Let $\mathcal{E}$ and $\mathcal{F}$ be families of finite-dimensional Banach spaces such that* $\mathrm{Ext}(\mathcal{F}, \mathcal{E}) = 0$ *uniformly. Suppose $Y \in \Lambda(\mathcal{E})$ and $Z \in \Lambda(\mathcal{F})$. If Y is complemented in its bidual then* $\mathrm{Ext}(Z, Y) = 0$.

Theorem 4. *Let $\mathcal{E}$ and $\mathcal{F}$ be families of finite-dimensional Banach spaces. Let Y and Z be Banach spaces such that $Y \in \Pi(\mathcal{E})$ and $Z \in \Pi(\mathcal{F})$. Then* $\mathrm{Ext}(Z, Y) = 0$ *implies that* $\mathrm{Ext}(\mathcal{F}, \mathcal{E}) = 0$ *uniformly.*

Remark 1. There is, of course, a version of these results for quasi-Banach spaces. One has only to amend two things: one, "zero-linear" by "quasi-linear" (with the corresponding change of constants: $Q(\cdot)$ instead of $Z(\cdot)$, and the use of the original result of Kalton instead of its version given in Theorem 1); two, "complemented in its bidual" has to be replaced by "quasi-Banach ultrasummand" (see [12, 18]). We leave the details to the reader.

Remark 2. The roles of the classes $\Lambda(\mathcal{E})$ and $\Pi(\mathcal{E})$ cannot be reversed: $\mathrm{Ext}(L_1, l_2) = 0$ ([21]), while $\mathrm{Ext}(l_2 \oplus l_1, l_2) \neq 0$. Also, the hypothesis that Y must be complemented in its bidual is essential: $\mathrm{Ext}(l_1, c_0) = 0$ and $\mathrm{Ext}(c_0, c_0) = 0$ while $\mathrm{Ext}(L_1(\mu), c_0) \neq 0$ for some μ and $\mathrm{Ext}(l_\infty, c_0) \neq 0$ (see Section 2).

There is a simpler "uniform boundedness principle" which follows from the projective description of the functor Ext which we briefly sketch now. (This shall be used in Section 5.) The interested reader can consult [13] or [5, section 1.4]. Given a Banach space Z, a projective presentation of Z, i.e., an exact sequence $0 \to K \to P \to Z \to 0$ in which P is a certain $l_1(I)$ space induces (via push-out) an exact sequence

$$0 \to L(Z, Y) \to L(P, Y) \to L(K, Y) \to \mathrm{Ext}(Z, Y) \to 0.$$

The map $L(P,Y) \to L(K,Y)$ is plain restriction to K and its image is not, usually, a closed subspace of $L(K,Y)$. When $\mathrm{Ext}(Z,Y)=0$ then the restriction map is surjective, hence open, and one gets the following.

Proposition 1. *Let Y and Z be Banach spaces. Then $\mathrm{Ext}(Z,Y)=0$ if and only if for every projective presentacion $0 \to K \to P \to Z \to 0$ there exists a constant C such that every operator $T : K \to Y$ admits an extension $\tilde{T} : P \to Y$ with $\|\tilde{T}\| \le C\|T\|$.* □

4. Examples

4.1. **Nontrivial twisted sums of l_1 and l_2.** Let $0 \to l_2 \to Z_2 \to l_2 \to 0$ be Kalton and Peck's solution to Palais problem [19]. Let $j : l_2 \to L_1$ be an isomorphic embedding and consider the push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & l_2 & \to & Z_2 & \to & l_2 & \to & 0 \\ & & j\downarrow & & \downarrow & & \| & & \\ 0 & \to & L_1 & \to & PO & \to & l_2 & \to & 0 \end{array}$$

The space Z_2 has not cotype 2 (see [5, theorem 5.1.b] or [19]); hence, it cannot be a subspace of $L_1 \oplus l_2$. Therefore the space PO is not isomorphic to the direct product $L_1 \oplus l_2$, which means, in particular, that the sequence is nontrivial. Applying now Theorem 2, the existence of a nontrivial twisted sum of l_1 and l_2 is clear.

Let us point out now a special feature of this example: the quotient map in the sequence $0 \to L_1 \to PO \to l_2 \to 0$ is strictly singular. And it is so because given any infinite dimensional subspace H of l_2 the pull-back sequence

$$\begin{array}{ccccccccc} 0 & \to & L_1 & \to & PO & \to & l_2 & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & L_1 & \to & PB(PO) & \to & H & \to & 0 \end{array}$$

is equivalent to the exact sequence obtained starting with the Kalton-Peck sequence and making first pull-back with $H \to l_2$ and then push-out with $l_2 \to L_1$. Now, it is essentially contained in [19, theorem 3.6] that the first of these extensions contains a copy of Z_2, and thus it cannot be a subspace of $L_1 \oplus H$.

4.2. **Nontrivial twisted sums of l_2 and c_0.** This construction is dual of the preceding. The space Z_2 cannot be a quotient of a $C(K)$-space since it is isomorphic to its own dual (see [19]). Hence, considering the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & l_2 & \to & Z_2 & \to & l_2 & \to & 0 \\ & & \| & & \uparrow & & \uparrow j^* & & \\ 0 & \to & l_2 & \to & PB & \to & L_\infty & \to & 0. \end{array}$$

we see that $\mathrm{Ext}(L_\infty, l_2) \neq 0$. Now, that $\mathrm{Ext}(c_0, l_2) \neq 0$ is consequence of Theorem 2.

Let us remark a different way to obtain a nontrivial twisted sum of l_2 and c_0 mentioned to us by D. Yost: we have a nontrivial extension $0 \to l_1 \to X \to l_2 \to 0$. Starting with a dual and ending with a reflexive space, this sequence is the transpose of a (necessarily nontrivial) sequence $0 \to l_2 \to X_* \to c_0 \to 0$ (see [11, 4]).

4.3. **Nontrivial twisted sums of l_1 and c_0.** We focus our attention in the construction of a nontrivial sequence $0 \to L_1 \to X \to L_\infty \to 0$. (By Theorem 2 this implies that $\mathrm{Ext}(c_0, l_1) \neq 0$.) To this end, we start again with the sequence $0 \to l_2 \to Z_2 \to l_2 \to 0$ and construct the push-out and pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & l_2 & \to & Z_2 & \to & l_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & L_1 & \to & PO & \to & l_2 & \to & 0 \\ & & \| & & \uparrow & & \uparrow q & & \\ 0 & \to & L_1 & \to & PB & \to & L_\infty & \to & 0 \end{array}$$

where q is a quotient map (take for instance the transpose of the isomorphic embedding $l_2 \to L_1$). We prove that the lowest sequence does not split. Or, what is the same, that q cannot be lifted to an operator $Q : L_\infty \to PO$.

Assume on the contrary that $Q : L_\infty \to PO$ is a lifting for $q : L_\infty \to l_2$. Let (x_n) be a bounded sequence in L_∞ such that $q(x_n) = e_n$, the usual basis of l_2. Taking into account that PO has finite cotype [17] and that Q is weakly compact, we infer from [9, theorem 2.3] that the set $\{Q(x_n)\}$ is relatively weakly-2-compact. So, some subsequence $(Q(x_{n(k)}))_k$ is weakly-2-convergent (necessarily to a point $f \in L_1$ since (e_n) is weakly null in l_2). Thus, $(Q(x_{n(k)}) - f)_k$ would be weakly-2-summable in PO; or, which is the same, the continuous image of $(e_{n(k)})_k$. This would imply that the quotient map $PO \to l_2$ is invertible on the subspace spanned by the sequence $(e_{n(k)})_k$, which is impossible: $PO \to l_2$ is strictly singular.

In spite of our efforts we have been unable to prove that a nontrivial twisted sum of l_1 and c_0 cannot be (or has to be) isomorphic to the product $l_1 \oplus c_0$. In [6] it can be seen a nontrivial twisted sum of two totally incomparable spaces which is actually isomorphic to its product: just consider $0 \to K \to l_1 \to c_0 \to 0$ a projective presentation of c_0 and obtain a nontrivial sequence $0 \to K \oplus c_0 \to l_1 \oplus c_0 \oplus K \to c_0 \oplus K \to 0$. It is easy to see that $K = K \oplus l_1 = K \oplus K$ what makes the middle space $K \oplus c_0$ isomorphic to the product of the other two. In [10] it has been asked whether such a twisted sums has to have the Dunford-Pettis property, or even the hereditary Dunford-Pettis property.

Reduced to its basic elements, one has proved the following.

Corollary 1. *Let Y be a cotype 2 space containing l_2. Then* $\mathrm{Ext}(l_2, Y) \neq 0$ *and* $\mathrm{Ext}(l_\infty, Y) \neq 0$. *If, moreover, Y is complemented in its bidual then also* $\mathrm{Ext}(c_0, Y) \neq 0$.

In fact, the hypothesis about the containment of l_2 can be removed. To do it, we need to show that if $\mathrm{Ext}(Z, Y) = 0$, then every Banach twisted sum of Y and Z is near to the direct sum $Y \oplus_1 Z$ in the Banach-Mazur distance. Precisely:

Theorem 5. *Let Y and Z be Banach spaces such that* $\mathrm{Ext}(Z, Y) = 0$. *Then there is a function $f(\cdot,\cdot,\cdot,\cdot)$ such that, for every short exact sequence*

$$0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$$

in which X is a Banach space (with convex norm), one has

$$d(X, Y \oplus_1 Z) \leq f\left(\|j\|, \|j^{-1}\|, \|q\|, \|(q^*)^{-1}\|\right).$$

Sketch of the Proof. First of all, we remark that the number $\|(q^*)^{-1}\|$ measures how many open q is. In fact, it is easily seen that

$$\|(q^*)^{-1}\| = \sup_{\|z\|_Z = 1} \inf_{qx = z} \|x\|_X,$$

so that for each fixed $K > \|(q^*)^{-1}\|$ there is a bounded selection $B : Z \to X$ for q with $\|B\| \leq K$ (that is, such that $\|B(z)\| \leq K\|z\|$ for all $z \in Z$).

Now, if $0 \to Y \to X \to Z \to 0$ is an extension, let us construct another equivalent extension $0 \to Y \to Y \oplus_F Z \to Z \to 0$, where $F = j^{-1} \circ (B - L)$ is taken as in Section 2. Clearly, one has $Z(F) \leq 2\|j^{-1}\|\|B\|$. Consider the isomorphism $T : X \to Y \oplus_F Z$ given by $Tx = (j^{-1}(x - Lqx), qx)$. It is easily seen that $\|T\| \leq \|j^{-1}\|\,(1 + \|q\|(1 + \|B\|))$ and $\|T^{-1}\| \leq \max\{\|B\|, \|j\|\}$. Thus, we have

$$d(X, Y \oplus_F Z) \leq \|j^{-1}\| \cdot (1 + \|q\|(1 + \|B\|)) \cdot \max\{\|B\|, \|j\|\}.$$

Now, suppose $dist(F, A) < \infty$ for some linear map $A : Z \to Y$. Then one can define an isomorphism $S : Y \oplus_F Z \to Y \oplus_1 Z$ by $S(y, z) = (y - A(z), z)$. It is easily seen that $\|S\| \leq 1 + dist(F, A)$. For the inverse isomorphism (which is given by $(y, z) \mapsto (y + A(z), z)$) one also has $\|S^{-1}\| \leq 1 + dist(F, A)$, so

$$d(Y \oplus_F Z, Y \oplus_1 Z) \leq (1 + dist(F, A))^2 .$$

It is now clear that, if $\mathrm{Ext}(Z, Y) = 0$ and C is the constant given by Theorem 1, then the function

$$f(s, t, u, v) = (1 + 2Ctv)^2 t(1 + u(1 + v)) \max\{s, v\}$$

does what announced. □

Remark 3. Ribe example [22] shows that the preceding Theorem may fail if one allows quasi-norms on X, even if they are equivalent to convex norms.

Corollary 2. *Let Y be an infinite dimensional space with cotype 2. Then*

$$\mathrm{Ext}(l_2, Y) \neq 0.$$

Proof. Suppose, on the contrary, that $\mathrm{Ext}(Y, l_2) = 0$. By Dvoreztky's theorem, for each $n \geq 1$, there is an operator $T : l_2^n \to Y$ with $\|T\|\|T^{-1}\| \leq 1/n$. Let $P : l_2 \to l_2^n$ be the obvious projection and set $R = P \circ T$. The push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & l_2 & \to & Z_2 & \to & l_2 & \to & 0 \\ & & R\downarrow & & \downarrow & & \| & & \\ 0 & \to & Y & \to & PO_n & \to & l_2 & \to & 0 \end{array}$$

and the fact that $d(PO_n, Y \oplus_1 l_2)$ is bounded by a constant independent on n would imply that Z_2 is (crudely) finitely representable in $Y \oplus_1 l_2$, which has cotype 2: a contradiction. □

Corollary 3. [20, theorem 4.1] *Let Z be an infinite dimensional space such that Z^* has cotype 2. Then* $\mathrm{Ext}(Z, l_2) \neq 0$.

Proof. We know that there is a nontrivial exact sequence $0 \to Z^* \to X \to l_2 \to 0$. By [11] (see also [4]) this sequence is the adjoint on a (necessarily nontrivial) sequence $0 \to l_2 \to X_* \to Z \to 0$. □

5. Characterization of $\mathcal{L}_1$-spaces

As a further application of the uniform boundedness technique let us provide some improvement for a classical characterization of $\mathcal{L}_1$-spaces. Let us recall a result of Lindenstrauss [21]: Every twisted sum of a Banach space complemented in its bidual and a $\mathcal{L}_1$-space splits. The result can be completed by showing that the converse is also true: If $\operatorname{Ext}(Z, Y) = 0$ for all Banach spaces Y complemented in its bidual then Z is a $\mathcal{L}_1$-space. We improve this result.

Proposition 2. *A Banach space Z is a $\mathcal{L}_1$-space if and only if for each reflexive space R one has* $\operatorname{Ext}(Z, R) = 0$.

Proof. Consider a sequence M_n of finite dimensional Banach spaces that is dense in the space of all finite dimensional Banach spaces with respect to the Banach-Mazur distance. Clearly, $\operatorname{Ext}(Z, l_2(M_n)) = 0$. Hence, if $0 \to K \to P \to Z \to 0$ is a projective presentation of Z there exists a constant C such that every operator $\phi : K \to F$, where F is a finite dimensional Banach space admits an extension $\psi : P \to F$ with norm at most C_1. In particular, for some C_1 and every finite dimensional subspace F of P the induced sequence $0 \to K \to K + F \to (K + F)/K \to 0$ splits and there exists a projection $K + F \to K$ with norm at most C. This means that the sequence $0 \to K \to P \to Z \to 0$ *locally splits* using the terminology of [18], what means that the dual sequence splits. Hence Z^* is a $\mathcal{L}_\infty$-space and Z is a $\mathcal{L}_1$ space. □

It is clear that the smaller class of W_2-spaces introduced in [8] (reflexive spaces in which every weakly null sequence admits a weakly 2-summable subsequence) can replace the class of reflexive spaces since $l_2(M_n)$ is an W_2-space (see [7]). On the other hand, "Hilbert" spaces are not enough: let M be an uncomplemented copy of l_1 inside l_1 (see [1]). The sequence $0 \to M \to l_1 \to Z \to 0$ does not split and thus Z is not a $\mathcal{L}_1$-space. Nevertheless, $\operatorname{Ext}(Z, l_2) = 0$ since every operator $\phi : M \to l_2$ is 2-summing and thus it extends to l_1 (this example also appears in [20]). Our guess is that "superreflexive" spaces are enough.

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