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CONVEX TRANSITIVE NORMS ON SPACES OF CONTINUOUS FUNCTIONS

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Abstract

A norm on a Banach space X is called maximal when no equivalent norm has a larger group of isometries. If, besides this, there is no equivalent norm with the same isometries (apart from its scalar multiples), the norm is said to be uniquely maximal, which is equivalent to the convex-transitivity of X: the convex hull of the orbits under the action of the isometry group on the unit sphere is dense in the unit ball of X. The main result of the paper is that the complex $C_0(\Omega)$ is convex-transitive in its natural supremum norm if Ω is a connected manifold (without boundary). As a complement, it is shown that if Ω is a connected manifold of dimension at least two, then the diameter norm is convex transitive on the corresponding space of real functions.

Introduction

We deal in this paper with maximal symmetric norms on Banach spaces of continuous functions. Recall that a norm on a Banach space X is called *maximal* when no equivalent norm has a larger group of (linear, surjective) isometries. Since each bounded group of automorphisms of X can be regarded as a subgroup of the isometry group of some renorming of X, it is clear that a norm is maximal if and only if its isometry group is maximal among the bounded groups of automorphisms of X.

We are primarily concerned with uniquely maximal norms. These are maximal norms with the additional property that there is no equivalent norm with the same isometry group, apart from its scalar multiples.

While proofs of maximality are often rather indirect and delicate, things become simpler for uniquely maximal norms. This is so because the latter property is known [**10**] to be equivalent to the convex-transitivity of X: the convex hull of the orbits under the action of the isometry group on the unit sphere is dense in the unit ball of X. If the orbits themselves are dense (in the unit sphere) then X is termed *almost isotropic*, but this is a different tale [**4**, **6**].

Although the maximality of (the usual supremum norm on) the spaces $C_0(\Omega)$ spurred a considerable interest in the past [**15–17, 21, 22**], not much is known about the convex-transitivity of these spaces [**2**].

The plan of the paper is as follows. Section 1 contains preliminaries. Section 2 deals with complex spaces: our main result in this line is that the complex $C_0(\Omega)$ is convex-transitive if Ω is a connected manifold (without boundary). Note that by previous results of Kalton and Wood, one has maximality when Ω is a manifold with boundary [**15**, Theorem 8.2].

In Section 3 we consider spaces of real-valued functions. First, we give a simple proof of the convex-transitivity of $\ell_\infty/c_0, L_\infty(0,1)$ and $C(\Delta)$, where Δ is the Cantor

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set. The remainder of the section studies the 'diameter' norm. This is an unusual norm introduced by Győry and Molnár in [**13**] (see also [**7**, **11**, **19**]). Among other things, we show that the diameter norm is convex-transitive on $C_0(\mathbb{R}^n)$ for all $n \geqslant 2$. This is probably the most surprising result of the paper, since it suggests that the usual supremum norm is not the 'right' norm of $C_0(\mathbb{R}^n)$ with regard to the isometries.

1. *Preliminaries*

Throughout the paper, Ω denotes a locally compact (Hausdorff) space and $C_0(\Omega)$ or $C_0^{\mathbb{K}}(\Omega)$ the space of all continuous functions $f:\Omega \longrightarrow \mathbb{K}$ vanishing at infinity, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The usual supremum norm on $C_0(\Omega)$ is given by

$$\|f\|_\infty = \sup_{t\in\Omega} |f(t)|.$$

By the Banach–Stone theorem, every isometry of $C_0(\Omega)$ has the form

$$T(f) = u \cdot (f \circ \varphi),$$

where $u : \Omega \longrightarrow \mathbb{K}$ is a continuous unimodular function and φ is a homeomorphism of Ω. The group of homeomorphisms of Ω is denoted in this paper by $\Gamma(\Omega)$.

The Riesz representation theorem identifies the conjugate space $C_0(\Omega)^*$ with the space $M(\Omega)$ of all regular Borel measures on Ω with values in the ground field. The duality is given by

$$\langle \mu, f \rangle = \int_\Omega f(t)d\mu(t).$$

Moreover, the norm of a measure acting as a linear functional equals its total variation:

$$\sup_{\|f\|_\infty \leqslant 1} |\langle \mu, f\rangle| = \|\mu\|_1 \stackrel{\text{def}}{=} |\mu|(\Omega),$$

where $|\mu|$ is the semi-variation of μ. Also, it is well known that the extreme points of the unit ball of $M(\Omega)$ are the functionals $\tau \cdot \delta_t$, where τ is a scalar of modulus one and δ_t is the unit mass at the point $t \in \Omega$.

As we mentioned in the introduction, the main advantage of unique maximality is that it can be characterized by the 'size' of the orbits of the isometry group. The following lemma puts together results by Cowie [**10**] and Becerra Guerrero and Rodríguez Palacios [**4**, Proposition 2.28]. A proof is sketched for the sake of simplicity. We write $\mathcal{G}$ or $\mathcal{G}_X$ for the isometry group of X.

Lemma 1. *For a Banach space X, the following statements are equivalent.*

(a) *The norm of X is uniquely maximal: each equivalent norm whose isometry group contains $\mathcal{G}$ is a multiple of the original norm.*

(b) *Each continuous semi-norm invariant under $\mathcal{G}$ is a multiple of the original norm of X.*

(c) *X is convex-transitive: for every unit norm $x \in X$ the convex hull of the orbit $\mathcal{G}(x) = \{T(x) : T \in \mathcal{G}\}$ is strongly dense in the unit ball of X.*

(d) *For every unit norm $x^* \in X^*$ the convex hull of $\mathcal{G}^*(x^*) = \{T^*(x^*) : T \in \mathcal{G}_X\}$ is weakly* dense in the unit ball of X^*.*

(e) *For every unit norm $x^* \in X^*$ the weak* closure of $\mathcal{G}^*(x^*)$ contains all extreme points of the unit ball of X^*.*

Sketch of the proof. Each statement is clearly equivalent to the adjacent ones: (a) implies (b) because if ϱ is a continuous semi-norm on X, then $\|\cdot\|_X + \varrho(\cdot)$ is an equivalent norm on X; (b) and (c) are equivalent because continuous $\mathcal{G}$-invariant semi-norms are in exact correspondence (by polarity) with $\mathcal{G}$-invariant closed convex balanced neighbourhoods of the origin; the equivalence between (c) and (d) is a simple application of the Hahn–Banach theorem; finally, that (e) implies (d) follows from the Kreĭn–Milman theorem. $\square$

2. *Complex functions*

After these preliminaries we are ready for the proof of our main result on complex spaces. By a *manifold*, we mean a topological manifold without boundary; that is, a Hausdorff topological space Ω where every point has a neighbourhood homeomorphic to $\mathbb{R}^n$ for some fixed $n \geqslant 1$, which is called the *dimension* of Ω.

THEOREM 1. *Let Ω be a connected manifold. The usual supremum norm is convex transitive on $C_0^{\mathbb{C}}(\Omega)$.*

The following result asserts that complex measures (which clearly allow 'polar decompositions' [**12**] with measurable argument) always allow 'nearly polar decompositions' [**8**] with continuous arguments. Note that if g is a Borel bounded function on Ω and μ is a Borel measure, the product $g \cdot \mu$ should be interpreted as the measure $A \mapsto \int_A g d\mu$.

LEMMA 2. *Let Ω be a locally compact space. For each $\mu \in M(\Omega)$ and $\varepsilon > 0$, there exists a continuous function $u : \Omega \longrightarrow \mathbb{T}$ such that*

$$\| \mu - u \cdot |\mu| \|_1 < \varepsilon.$$

Proof. Since each Borel measure is absolutely continuous with respect to its semi-variation, the Radon–Nikodým theorem yields a measurable $\sigma : \Omega \longrightarrow \mathbb{C}$ such that $\mu(A) = \int_A \sigma d|\mu|$; that is, $\mu = \sigma \cdot |\mu|$. Since

$$|\mu|(A) = \int_A |\sigma| \, d|\mu|$$

for every Borel $A \subset \Omega$ we see that $|\sigma(t)| = 1$ almost $|\mu|$-everywhere. Therefore we may assume that σ is a Borel unimodular function. Write

$$\sigma(t) = e^{i\theta(t)},$$

where $\theta : \Omega \longrightarrow (-\pi, \pi]$ is another Borel function. By Luzin's theorem [**14**, Theorem 11.36] there is a continuous $\vartheta : \Omega \longrightarrow (-\pi, \pi]$ such that

$$|\mu| \left(\{ t \in \Omega : \vartheta(t) \neq \theta(t) \} \right) < \varepsilon.$$

The function $u = e^{i\vartheta}$ is clearly continuous and one has

$$\| \mu - u \cdot |\mu| \|_1 = \| \sigma \cdot |\mu| - u \cdot |\mu| \|_1 \leqslant \int_\Omega |\sigma - u| \, d|\mu| \leqslant 2\varepsilon,$$

as desired. $\square$

Now, to prove the convex-transitivity of $C_0^{\mathbb{C}}(\Omega)$ one has only to verify that Ω satisfies a certain shrinking property: the precise statement is given by condition (g) in Lemma 3 below. Note that if T is an endomorphism of the form

$$T(f) = g \cdot (f \circ \varphi) \qquad (f \in C_0(\Omega)),$$

the adjoint map T^* acts on $M(\Omega)$ by the rule

$$T^*(\mu) = (g \cdot \mu) \circ \varphi^{-1}.$$

Lemma 3. *For a locally compact space Ω, the following are equivalent.*

(f) *The supremum norm is convex-transitive on $C_0^{\mathbb{C}}(\Omega)$.*

(g) *Given a regular Borel probability $\mu \in M(\Omega)$, a non-empty open $U \subset \Omega$ and $\varepsilon > 0$, there exist a (compact) $K \subset \Omega$ with $\mu(\Omega \backslash K) \leqslant \varepsilon$ and $\varphi \in \Gamma(\Omega)$ such that $\varphi(K) \subset U$.*

(h) *For all probabilities $\mu \in M(\Omega)$, the weak* closure of the set $\{\mu\circ\varphi : \varphi \in \Gamma(\Omega)\}$ contains all evaluation functionals δ_t with $t \in \Omega$.*

Proof. That (h) implies the convex transitivity of $C_0(\Omega)$ follows from Lemmas 1 and 2, taking into account the fact that the set of those x^* for which condition (e) – or (c) – of Lemma 1 holds is norm closed in the unit sphere of X^*.

Let us show the implication (f) $\Rightarrow$ (g). Fix μ, U and ε as in (g). Pick $p \in U$ and let $V \subset U$ be a compact neighbourhood of p. Choose a continuous $f : \Omega \longrightarrow [0, 1]$ so that $f(p) = 1$ and $f \equiv 0$ outside V. By convex transitivity, there is an isometry T of $C_0(\Omega)$ such that

$$|\langle T^*(\mu), f\rangle - \langle \delta_p, f\rangle| < \varepsilon;$$

that is,

$$\left|1 - \int_\Omega u(t) f(\varphi(t)) d\mu(t)\right| < \varepsilon,$$

where $u \in C_0(\Omega)$ is unimodular and $\varphi \in \Gamma(\Omega)$. Therefore,

$$1-\varepsilon < \left|\int_\Omega u(t) f(\varphi(t)) d\mu(t)\right| \leqslant \int_\Omega |u(t) f(\varphi(t))| d\mu(t) = \int_{\varphi^{-1}(V)} f \circ \varphi \, d\mu \leqslant \mu(\varphi^{-1}(V)).$$

Hence (g) holds, taking $K = \varphi^{-1}(V)$.

We end the proof by showing that (g) implies (h).

Fix a probability μ on Ω and $p \in \Omega$. Take $f \in C_0(\Omega)$ and $\varepsilon > 0$. We see that there is a $\varphi \in \Gamma(\Omega)$ such that

$$\left|\langle \delta_p - \mu \circ \varphi^{-1}, f\rangle\right| < \varepsilon.$$

There is no loss of generality in assuming that f is non-negative, with $\|f\|_\infty \leqslant 1$. Let U be a neighbourhood of p such that

$$|f(p) - f(t)| < \varepsilon \qquad (t \in U),$$

and let K be a compact set such that $\mu(K) > 1 - \varepsilon$ and $\varphi(K) \subset U$ for some $\varphi \in \Gamma(\Omega)$.

One has

$$\begin{aligned}
\left|\langle \delta_p - \mu\circ\varphi^{-1}, f\rangle\right| &= \left| f(p) - \int_\Omega f\, d\left(\mu\circ\varphi^{-1}\right)\right| \\
&= \left| f(p) - \int_\Omega f(\varphi(t)) d\mu(t)\right| \\
&\leqslant \varepsilon + \left|\mu(\Omega\backslash K)f(p) + \mu(K)f(p) - \int_K f(\varphi(t))d\mu(t)\right| \\
&= 2\varepsilon + \left|\int_K (f(p) - f(\varphi(t))d\mu(t)\right| \\
&\leqslant 3\varepsilon.
\end{aligned}$$

This completes the proof. □

It is now clear that, for instance, $C_0^{\mathbb{C}}(\mathbb{R}^n)$ has convex-transitive norm. Actually, $\mathbb{R}^n$ has a shrinking property stronger than (g); namely, that if $U \neq \varnothing$ is open and K is compact, then there is a $\varphi\in\Gamma(\mathbb{R}^n)$ mapping K into U.

As for other 'non-flat' manifolds such as $\mathbb{T}^n$ or $\mathbb{S}^n$, a moment's reflection shows that they satisfy condition (g) of Lemma 3 – although not the stronger form just mentioned.

Our immediate aim is to show that every connected manifold has property (g). The crucial steps are the following two lemmas. We write $\mathbb{B}^n$ for the open unit ball of $\mathbb{R}^n$, centred at the origin; that is, the set $\{x\in\mathbb{R}^n : \|x\|<1\}$; $\overline{\mathbb{B}}^n$ and $\mathbb{S}^{n-1}$ will denote its closure and its boundary (in $\mathbb{R}^n$), respectively.

Lemma 4. (i) *For every $p\in\mathbb{B}^n$ there is a $\varphi\in\Gamma(\overline{\mathbb{B}}^n)$ such that $\varphi(0)=p$ and $\varphi(x)=x$ for all $x\in\mathbb{S}^{n-1}$.*

(ii) *For every $0<\varepsilon, r<1$ there is a $\varphi\in\Gamma(\overline{\mathbb{B}}^n)$ such that $\varphi(r\overline{\mathbb{B}}^n)=\varepsilon\overline{\mathbb{B}}^n$ and $\varphi(x)=x$ for all $x\in\mathbb{S}^{n-1}$.*

Proof. (i) For the first part, it suffices to consider the map $\varphi(x)=x+(1-\|x\|)p$.

(ii) As for the second one, let $g:[0,1]\longrightarrow[0,1]$ be the only function that is affine on $[0,r]$ and $[r,1]$ and takes the values 0, ε and 1 at 0, r and 1, respectively. Then the desired selfmap can be defined as

$$\varphi(x) = g(\|x\|)\cdot\frac{x}{\|x\|} \qquad (0<\|x\|\leqslant 1)$$

and $\varphi(0)=0$. □

Let us say that an open subset V of an n-dimensional manifold Ω is a *cube* if there exists a homeomorphism between $\overline{V}$ (the closure of V) and $\overline{\mathbb{B}}^n$ mapping V onto $\mathbb{B}^n$. One immediate consequence of Lemma 4 is that if $U\subset V$ is open and $K\subset V$ is compact, then there is a $\varphi\in\Gamma(\Omega)$ such that $\varphi(K)\subset U$ and $\varphi(x)=x$ for $x\notin V$.

Lemma 5. *Let C be a compact set in an n-dimensional manifold Ω. There is a finite system $(V_i,p_i)_{i=1}^m$ such that:*

(i) *Each V_i is a cube containing p_i and $p_i\notin\overline{V_j}$ if $i\neq j$.*

(j) *$C\subset V_1\cup\ldots\cup V_m$.*

Proof. It is clear that each point of Ω belongs to some cube. By compactness, we may cover C by a finite number of cubes, which we label $V_1^0, \ldots, V_m^0$. Removing some sets if necessary, we may assume that no proper subclass of $\{V_1^0, \ldots, V_m^0\}$ covers C. Thus, for each $1 \leqslant i \leqslant m$, we can pick

$$p_i \in \left(V_i^0 \cap C\right) \setminus \bigcup_{j \neq i} V_j^0$$

and neighbourhoods U_i of p_i in such a way that $p_j \notin U_i$ for $i \neq j$.

Finally, define sets

$$V_j^k \qquad (1 \leqslant j \leqslant m;\ 0 \leqslant k \leqslant m)$$

inductively (on k) starting with V_j^0 $(1 \leqslant j \leqslant m)$ as follows. Assuming that V_j^k has been defined for some $0 \leqslant k < m$ and all $1 \leqslant j \leqslant m$, let us define V_j^{k+1} according to the position of p_{k+1} relative to V_j^k: if $p_{k+1} \notin \partial V_j^k$, then $V_j^{k+1} = V_j^k$; if $p_{k+1} \in \partial V_j^k$, then

$$V_j^{k+1} = V_j(k) \backslash \overline{V},$$

where $V \subset U_j$ is a cube containing p_{k+1} chosen in such a way that V_j^{k+1} is still a cube.

The mth 'output' system $(V_i^m, p_i)_{i=1}^m$ fulfils the conditions of the lemma. □

Proof of Theorem 1. We show that each connected manifold Ω satisfies condition (g) of Lemma 3. Fix a probability $\mu \in M(\Omega)$, a non-empty open $U \subset \Omega$ and $\varepsilon > 0$. By regularity, there is a compact $C \subset \Omega$ such that $\mu(C) > 1 - \varepsilon$.

Let $(V_i, p_i)_{i=1}^m$ be a system as in Lemma 5. Choose $\varphi \in \Gamma(\Omega)$ such that $\varphi(p_i) \in U$ for all i. At this point, connectedness is necessary! For each i, let U_i be an open neighbourhood of p_i in V_i such that

$$\varphi(U_i) \subset U \quad \text{and} \quad U_i \cap V_j = \varnothing \qquad (i \neq j).$$

Now, take compact $K_i \subset V_i$ such that

$$\mu(V_i \backslash K_i) < \frac{\varepsilon}{m}.$$

Clearly,

$$\mu\left(\bigcup_{i=1}^m K_i\right) \geqslant 1 - 2\varepsilon.$$

To finish, for each $1 \leqslant i \leqslant m$, let us take $\varphi_i \in \Gamma(\Omega)$ so that

$$\varphi_i(K_i) \subset U_i \quad \text{and} \quad \varphi_i(x) = x \qquad (x \notin V_i);$$

such a φ_i does exist, by the remark after Lemma 4. Put $\Psi = \varphi_m \circ \varphi_{m-1} \circ \ldots \circ \varphi_2 \circ \varphi_1$. Clearly,

$$\Psi\left(K_1\right) \subset U_1, \quad \Psi\left(K_2 \backslash V_1\right) \subset U_2, \quad \Psi\left(K_3 \backslash \left(V_1 \cup V_2\right)\right) \subset U_3, \quad \ldots$$

and, in general,

$$\Psi\left(K_j \Big\backslash \left(\bigcup_{i=1}^{j-1} V_i\right)\right) \subset U_j \qquad (1 \leqslant j \leqslant m).$$

Consider the disjoint union

$$K = K_1 \oplus (K_2 \backslash V_1) \oplus (K_3 \backslash (V_1 \cup V_2)) \oplus \ldots \oplus \left(K_m \Big\backslash \left(\bigcup_{i=1}^{m-1} V_i \right) \right).$$

This is clearly a compact set with $\mu(K) \geqslant 1 - 2\varepsilon$. Moreover, the composition $\Phi = \varphi \circ \Psi$ sends K into U, which completes the proof. $\square$

3. *Real functions*

In this section we study the isometry group of certain norms on the real space $C_0(\Omega)$. First of all, let us remark that Wood proved in [**22**] that $C_0^{\mathbb{R}}(\Omega)$ is convex-transitive in its usual (supremum) norm if and only if Ω is basically disconnected and satisfies condition (h) of Lemma 3. However, both (g) and (h) involve only real functions (or measures), and actually it is easily seen that they are equivalent, so we have the following extension of [**22**, Theorem 3.4].

THEOREM 2. *The supremum norm is convex-transitive on $C_0^{\mathbb{R}}(\Omega)$ if and only if Ω is basically disconnected and satisfies condition* (g) *of Lemma* 3.

In particular, $C_0^{\mathbb{C}}(\Omega)$ is convex-transitive if $C_0^{\mathbb{R}}(\Omega)$ is. As for concrete examples, Pełczýnski and Rolewicz proved that both $C^{\mathbb{R}}(\Delta)$ and $L_\infty^{\mathbb{R}}[0,1]$ are convex-transitive, where $\Delta = \{0,1\}^{\mathbb{N}}$ is the Cantor set. Although announced in 1962 (at the Stockholm International Congress of Mathematics), these results were not published until Rolewicz's [**20**] appeared. Unfortunately, all the results in [**21**] (the second edition of [**20**]) concerning the convex-transitivity of spaces of continuous functions are wrong, and so no proof of the convex-transitivity of the above spaces is easily available. (Actually, ℓ_∞^n is clearly a counterexample for [**21**, Theorem 9.7.3, Corollary 9.7.4 and Theorem 9.7.7]; a less trivial example is the space $c_0 = C_0(\mathbb{N})$: by the results in [**1**, **3**] this space cannot be renormed with a convex-transitive norm. See [**4**] for further explanations.) A simple proof based on the 'divisibility' of the underlying compact spaces follows. The symbol '$\approx$' indicates homeomorphism.

COROLLARY 1. *Let Ω be a locally compact space whose topology has a base $\mathcal{S}$ of clopen sets such that $C \approx \Omega\backslash C \approx \Omega$ for all $C \in \mathcal{S}$. Then the supremum norm is convex-transitive on $C_0^{\mathbb{R}}(\Omega)$. In particular, the real spaces $C(\Delta)$, $L_\infty[0,1]$ and ℓ_∞/c_0 all have convex-transitive norm.*

Proof. Clearly, Ω is basically disconnected. Let us verify condition (g) of Lemma 3. Let μ be a probability on Ω, U an open set and $\varepsilon > 0$. We show there is a subset K of Ω with $\mu(K) \geqslant 1 - \varepsilon$ and $\varphi \in \Gamma(\Omega)$ such that $\varphi(K) \subset U$.

We may assume that $U \in \mathcal{S}$, so $U \approx \Omega\backslash U \approx \Omega$. Since $\Omega \approx \Omega \oplus \Omega$, it is clear that

$$\Omega \approx \bigoplus_{i=1}^{n} \Omega \qquad (n = 1, 2 \ldots).$$

Take n so that $1/n < \varepsilon$. Then Ω can be decomposed as $\Omega = C_1 \oplus \cdots \oplus C_n$, with $C_i \in \mathcal{S}$ and $\mu(C_n) < \varepsilon$. Put $K = C_1 \oplus \cdots \oplus C_{n-1}$; clearly $\mu(K) > 1 - \varepsilon$. Since $\Omega \approx U \approx \Omega\backslash U$ it is obvious that there is a $\varphi \in \Gamma(\Omega)$ such that $\varphi(K) \subset U$. $\square$

We now study the space $C_0^{\mathbb{R}}(\Omega)$ for connected Ω. In this case it is clear that the usual norm cannot be convex-transitive unless Ω is a singleton. In [**5**] we proved (among other things) that there is an 'unusual' norm (the diameter norm) on $C_0(\mathbb{R}^n)$ whose isometry group is strictly larger than that of the sup norm. The main results of [**5**] were about the maximality of the diameter norm. In this section we study the convex-transitivity of that norm, thus obtaining simpler proofs of some results that are stronger than those of [**5**].

Given $f \in C_0(\Omega)$, put

$$\rho(f) = \sup\{|f(s) - f(t)| : s, t \in \Omega\};$$

that is, $\rho(f)$ is the diameter of the range of f. Clearly, ρ is a semi-norm on $C_0(\Omega)$ whose kernel consists of the constant functions that belong to $C_0(\Omega)$. Thus, ρ is a norm if (and only if) Ω is locally compact but not compact. If so, then

$$\|f\|_\infty \leqslant \rho(f) \leqslant 2\|f\|_\infty \qquad (f \in C_0(\Omega)).$$

In any case, we can define a norm on $C_0(\Omega)/\ker\rho$ taking

$$\rho[f] = \rho(f),$$

where $[f]$ stands for the class of f in $C_0(\Omega)/\ker\rho$.

Actually, there is no loss of (isometric types of) diameter norms in restricting our considerations to the compact case: it was noted in [**7**, Section 3] that

$$(C_0(\Omega), \rho(\cdot)) = (C(\alpha\Omega)/\ker\rho, \rho[\cdot]),$$

up to an (obvious) isometric isomorphism – here, $\alpha\Omega$ stands for the one-point compactification of Ω. Thus, the behaviour of the diameter norm on $C_0(\Omega)$ depends only on $\alpha\Omega$. Since $\alpha\Omega$ often has more symmetry than Ω (the group $\Gamma(\alpha\Omega)$ always contains $\Gamma(\Omega)$ and the inclusion may be proper), this explains the possible gain of symmetry when passing from the usual norm to the diameter norm.

Lemma 6. *For a compact space Ω, the following are equivalent.*

(l) *The diameter norm is convex-transitive on $C^{\mathbb{R}}(\Omega)/\ker\rho$.*

(m) *Given a real measure $\mu \in M(\Omega)$ with $|\mu|(\Omega) = 2$ and $\mu(\Omega) = 0$, two disjoint non-empty open sets $U^+, U^- \subset \Omega$ and $\varepsilon > 0$, there are disjoint (compact) sets $K^+, K^- \subset \Omega$, with $\mu(K^+) > 1 - \varepsilon$, $\mu(K^-) < \varepsilon - 1$, and $\varphi \in \Gamma(\Omega)$ such that $\varphi(K^+) \subset U^+$ and $\varphi(K^-) \subset U^-$.*

Before going into the proof, let us note some further properties of the Banach space $C_\rho(\Omega) = (C(\Omega)/\ker\rho, \rho[\cdot])$.

First of all, it is clear that if φ is a homeomorphism of Ω and τ is a number of modulus one, then the map

$$T[f] = [\tau f \circ \varphi] \qquad (f \in C(\Omega))$$

is a surjective isometry of $C_\rho(\Omega)$. Actually, every isometry of $C_\rho(\Omega)$ has the above form. See [**7**, Theorem 2].

Secondly, it will be convenient to have a concrete representation of the conjugate space of $C_\rho(\Omega)$. Note that when $\mathbb{K} = \mathbb{R}$, one has

$$\rho(f) = 2 \inf_{\lambda \in \mathbb{K}} \|f - \lambda 1_\Omega\|_\infty.$$

(This is not true when $\mathbb{K}=\mathbb{C}$!) Therefore $C_\rho^{\mathbb{R}}(\Omega)$ is isometric, up to a constant factor 2, to the quotient of (supremum norm) space $C^{\mathbb{R}}(\Omega)$ by the subspace of constant functions. Hence the conjugate space $C_\rho^{\mathbb{R}}(\Omega)^*$ is (up to a constant factor 1/2) isometric to a subspace of $M^{\mathbb{R}}(\Omega)$. More precisely,

$$C_\rho^{\mathbb{R}}(\Omega)^* = \{\mu \in M^{\mathbb{R}}(\Omega) : \mu(\Omega) = 0\},$$

isometrically.

Finally, a measure μ represents an extreme point of the unit ball of $C_\rho(\Omega)^*$ if and only if (see [**7**] again)

$$\mu = \tau(\delta_s - \delta_t) \qquad (|\tau| = 1;\ s, t \in \Omega;\ s \neq t).$$

Proof of Lemma 6. We prove the implication (m) $\Rightarrow$ (l). In view of Lemma 1 and the preceding remarks, it suffices to see that, given $\mu \in M^{\mathbb{R}}(\Omega)$, with $|\mu|(\Omega) = 2$ and $\mu(\Omega) = 0$, two points $p, q \in \Omega$, $f \in C^{\mathbb{R}}(\Omega)$ and $\varepsilon > 0$, there is a $\varphi \in \Gamma(\Omega)$ such that

$$|\langle \delta_p - \delta_q - \mu \circ \varphi, f\rangle| < \varepsilon.$$

We may assume that f is non-negative, with $\|f\|_\infty \leqslant 1$.

Let U^+ be a neighbourhood of p such that $|f(p) - f(t)| < \varepsilon$ for $t \in U^+$. Similarly, let U^- be a neighbourhood of q such that $|f(q) - f(s)| < \varepsilon$ for $s \in U^-$. Now, if K^+, K^- and φ are as in Lemma 6(m), then

$$\begin{aligned} |\langle \delta_p - \delta_q - \mu \circ \varphi^{-1}, f\rangle| &= \left| f(p) - f(q) - \int_\Omega f \circ \varphi \, d\mu \right| \\ &\leqslant \left| f(p) - \int_\Omega f \circ \varphi \, d\mu^+ \right| + \left| f(q) - \int_\Omega f \circ \varphi \, d\mu^- \right|, \end{aligned}$$

where $\mu = \mu^+ - \mu^-$ is the Hahn–Jordan decomposition of μ. Since $|\mu| = |\mu^+| + |\mu^-|$, from $\mu(\Omega) = 0$ and $|\mu|(\Omega) = 2$, we infer that both μ^+ and μ^- are probabilities, with $\mu^+(K^+)$ and $\mu^-(K^-)$ greater than $1 - \varepsilon$. Reasoning as in the proof of Lemma 3, we find that each summand in the right-hand side of the above inequality is at most 3ε, which proves the first implication.

As for the converse, let us fix $\mu \in M^{\mathbb{R}}(\Omega)$ such that $|\mu|(\Omega) = 2$ and $\mu(\Omega) = 0$. Fix also disjoint (non-empty) open sets $U^+, U^- \subset \Omega$ and $\varepsilon > 0$.

Take $p \in U^+$ and $q \in U^-$ and construct a continuous $f : \Omega \longrightarrow [-1, 1]$ so that

$$f(p) = 1, \quad f(q) = -1, \quad f_{|U^+} \geqslant 0, \quad f_{|U^-} \leqslant 0 \quad \text{and} \quad f \equiv 0 \qquad \text{outside } U^+ \oplus U^-.$$

By convex-transitivity, there is an isometry T of $C_\rho^{\mathbb{R}}(\Omega)$ for which

$$|\langle T^*(\mu) - (\delta_p - \delta_q), [f]\rangle| < \varepsilon.$$

Hence $\langle \mu, T[f]\rangle > 2 - \varepsilon$. By the form of the isometries of the diameter norm we have $T[f] = [\tau f \circ \varphi]$, where $\tau = \pm 1$ and $\varphi \in \Gamma(\Omega)$. Assuming that $\tau = 1$ (the case $\tau = -1$

is similar), we obtain

$$\begin{aligned}
2-\varepsilon &\leqslant \int_{\Omega} f\circ\varphi\,d\mu \\
&= \int_{\varphi^{-1}(U^+\oplus U^-)} f\circ\varphi\,d\mu^+ - \int_{\varphi^{-1}(U^+\oplus U^-)} f\circ\varphi\,d\mu^- \\
&= \int_{\varphi^{-1}(U^+)} f\circ\varphi\,d\mu^+ + \int_{\varphi^{-1}(U^-)} f\circ\varphi\,d\mu^+ - \int_{\varphi^{-1}(U^+)} f\circ\varphi\,d\mu^- - \int_{\varphi^{-1}(U^-)} f\circ\varphi\,d\mu^- \\
&\leqslant \int_{\varphi^{-1}(U^+)} f\circ\varphi\,d\mu^+ - \int_{\varphi^{-1}(U^-)} f\circ\varphi\,d\mu^- \\
&\leqslant \mu^+(\varphi^{-1}(U^+)) + \mu^-(\varphi^{-1}(U^-)).
\end{aligned}$$

Bearing in mind that $\mu^\sigma(\Omega)=1$ for $\sigma=\pm$, we see that $\mu^\sigma(\varphi^{-1}(U^\sigma))>1-\varepsilon$ and, *a fortiori*,

$$\max\{\mu^+(\varphi^{-1}(U^-)),\mu^-(\varphi^{-1}(U^+))\}<\varepsilon.$$

Therefore,

$$\mu(\varphi^{-1}(U^+))>1-2\varepsilon \quad\text{and}\quad \mu(\varphi^{-1}(U^-))<2\varepsilon-1.$$

The proof is now complete. □

THEOREM 3. *Let Ω be a connected, compact manifold of dimension at least* 2. *Then $C_\rho^{\mathbb{R}}(\Omega)$ is convex-transitive.*

Proof. We will verify condition (m) of Lemma 6. Suppose that μ, U^+, U^- and ε are as in Lemma 6.

First, there are Borel sets Ω^σ $(\sigma=\pm)$ such that

$$\mu^+(\Omega^+)=\mu^-(\Omega^-)=1 \quad\text{with}\quad \Omega^+\cap\Omega^-=\varnothing.$$

By regularity, there exist compact sets $C^\sigma\subset\Omega^\sigma$ such that $\mu^\sigma(C^\sigma)>1-\varepsilon$. By compactness of Ω, we can find disjoint open sets O^σ such that $K^\sigma\subset O^\sigma$. Since the O^σ are manifolds, we can apply Lemma 5 to obtain two finite systems $(V_i^\sigma,p_i^\sigma)_{i=1}^{m_\sigma}$ such that:

(i) each V_i^σ is a cube containing p_i and $p_i^\sigma\notin\overline{V}_j^\sigma$ if $i\neq j$;

(ii) $C^\sigma\subset V_1^\sigma\cup\ldots\cup V_{m_\sigma}^\sigma$ for $\sigma=\pm$.

Let us take $\varphi\in\Gamma(\Omega)$ so that $\varphi(p_i^\sigma)\in U^\sigma$ for all i and each σ. This is the only point where the hypothesis on the dimension enters!

Also, for each (σ,i), let $U_i^\sigma\subset V_i^\sigma$ be a neighbourhood of p_i^σ such that

$$\varphi\left(U_i^\sigma\right)\subset U^\sigma \quad\text{with}\quad U_i^\sigma\cap V_j^\sigma=\varnothing \qquad (i\neq j).$$

Finally, take compact sets K_i^σ such that

$$K_i^\sigma\subset V_i^\sigma \quad\text{and}\quad |\mu|\left(V_i^\sigma\backslash K_i^\sigma\right)<\varepsilon/m_\sigma$$

and homeomorphisms φ_i^σ such that

$$\varphi_i^\sigma\left(K_i^\sigma\right)\subset U_i^\sigma \quad\text{and}\quad \varphi_i^\sigma(x)=x \qquad (x\notin V_i^\sigma).$$

Putting $\Psi^\sigma=\varphi_{m_\sigma}^\sigma\circ\ldots\circ\varphi_1^\sigma$ and $\Psi=\Psi^+\circ\Psi^-$, we have

$$\Psi\left(K_j^\sigma\Big\backslash\left(\bigcup_{i=1}^{j-1}V_i^\sigma\right)\right)\subset U_j^\sigma \qquad (\sigma=\pm,\ 1\leqslant j\leqslant m_\sigma).$$

Finally, set

$$K^\sigma = \bigoplus_{j=1}^{m_\sigma} \left(K_j^\sigma \Big\backslash \left(\bigcup_{i=1}^{j-1} V_i^\sigma \right) \right).$$

Then $\mu(K^+) = \mu^+(K^+) > 1 - 2\varepsilon$ and $\mu(K^-) = -\mu^-(K^-) < 2\varepsilon - 1$ and the composition $\Phi = \varphi \circ \Psi$ maps each K^σ into the corresponding U^σ. This completes the proof. □

COROLLARY 2. *Let Ω be a non-compact, connected manifold of dimension at least 2. Then the diameter norm is convex-transitive on $C_0^{\mathbb{R}}(\Omega)$ if and only if $\alpha\Omega$ is also a manifold.*

Proof. The 'if' part follows from Theorem 3 – when applied to $\alpha\Omega$. As for the converse, if $\alpha\Omega$ were not a manifold, then the infinity would be fixed by $\Gamma(\alpha\Omega)$ and Lemma 6(m) would fail. □

In particular, the diameter norm is convex-transitive on $C_0^{\mathbb{R}}(\Omega)$ if $\Omega = \mathbb{R}^n$ with $n \geqslant 2$, or if Ω is, for instance, the Möbius strip, but not if $\Omega = \mathbb{T} \times \mathbb{R}$ is a cylinder.

In the setting of Corollary 2, we see that the diameter norm is convex-transitive on $C_0^{\mathbb{R}}(\Omega)$ precisely when the sup norm is not maximal. It is a little ironic that the failure of maximality for the sup norm on spaces of real functions turns out to be a ubiquitous phenomenon.

PROPOSITION 1. *If Ω is a locally compact, non-compact, connected space* (*not necessarily a manifold*), *then the sup norm fails to be maximal at least on one of the spaces $C_0^{\mathbb{R}}(\Omega)$ or $C^{\mathbb{R}}(\alpha\Omega)$.*

Proof. If $\Gamma(\alpha\Omega) \neq \Gamma(\Omega)$, then the diameter norm has more symmetry than the sup norm on $C_0^{\mathbb{R}}(\Omega)$. Otherwise, each homeomorphism of $\alpha\Omega$ fixes the infinity and we can obtain a larger bounded group of automorphisms in $C^{\mathbb{R}}(\alpha\Omega)$ adding the 'reflexion' given by

$$I_\infty(f) = f - 2f(\infty)1_{\alpha\Omega}.$$

The idea originates with Partington [**17**]: if $\Gamma(K)$ leaves invariant some non-zero measure in $M(K)$, then $C^{\mathbb{R}}(K)$ is not maximal under the sup norm. □

We close with two remarks. First, the hypothesis on the dimension is necessary in Theorem 3: the space $C_\rho^{\mathbb{R}}(\mathbb{T}) = (C_0^{\mathbb{R}}(\mathbb{R}), \rho(\cdot))$ is not convex transitive; indeed, condition (m) fails for all $\varepsilon \leqslant 1/2$ taking

$$\mu = \frac{\delta_1 - \delta_i + \delta_{-1} - \delta_{-i}}{2}$$

with $U^+ = \{z \in \mathbb{T} : \Re z > 0\}$ and $U^- = \{z \in \mathbb{T} : \Re z < 0\}$. As far as I know, the problem about the maximality of $C_\rho^{\mathbb{R}}(\mathbb{T})$ and $C^{\mathbb{R}}(\mathbb{T})$ is still open.

Recently, F. Rambla [**18**] has shown that there exists a (metrizable) locally compact space Ω such that $C_0^{\mathbb{C}}(\Omega)$ is almost isotropic, thus solving a long-standing problem posed by Wood in [**22**]. It is worth noting that if $C_0^{\mathbb{C}}(\Omega)$ is almost isotropic and $\Gamma(\alpha\Omega)$ does not fix the infinity (which is the case in Rambla's example), then

$C_0^{\mathbb{R}}(\Omega)$ is almost isotropic under the diameter norm. Note that (in view of [**9**]) the space Ω has dimension one in the (very weak) sense of covering.

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