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Contribution to the classification of minimal extensions[☆]

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Abstract

We show that certain quasi-Banach spaces arising from quasi-linear functions on ℓ_1 are not isomorphic. We also show that if Γ is a large set, then every quasi-linear function on $\ell_1(\Gamma)$ is near to a true linear map on some infinite dimensional subspace.

1. Introduction

The study of extensions of (quasi) Banach spaces frequently leads to interesting creatures: non-Hilbert Banach spaces $\mathfrak{X}$ containing a Hilbert space $\mathfrak{Y}$ such that also $\mathfrak{X}/\mathfrak{Y}$ is a Hilbert space [10], non-locally convex (quasi-Banach) spaces $\mathfrak{X}$ with a line L such that $\mathfrak{X}/L \approx \ell_1$ (see [5,11,12]), Banach spaces not isomorphic to their complex conjugates [8], (quasi-Banach) spaces without basic sequences [9], etc.

Let us recall here that, given quasi-Banach spaces $\mathfrak{Z}$ and $\mathfrak{Y}$, an extension of $\mathfrak{Z}$ by $\mathfrak{Y}$ is a short exact sequence

$$0 \longrightarrow \mathfrak{Y} \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{Z} \longrightarrow 0 \tag{1.1}$$

in which $\mathfrak{X}$ is another quasi-Banach space and the arrows represent (linear, continuous) operators. Less technically, we can regard $\mathfrak{X}$ as space containing $\mathfrak{Y}$ as a closed subspace in such a way that $\mathfrak{X}/\mathfrak{Y}$ is (isomorphic to) $\mathfrak{Z}$. The space $\mathfrak{X}$ itself is often called a twisted sum of $\mathfrak{Y}$ and $\mathfrak{Z}$ (in that order!).

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Regarding the sequence (1.1) as a whole is one of the basic principles of the theory. Accordingly, two extensions

$$0 \to \mathfrak{Y} \to \mathfrak{X}_i \to \mathfrak{Z} \to 0 \quad (i = 1, 2)$$

are called equivalent if there is an operator T making commute the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathfrak{X}_1 & \longrightarrow & \mathfrak{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathfrak{X}_2 & \longrightarrow & \mathfrak{Z} & \longrightarrow & 0. \end{array}$$

Such a T must be an isomorphism by the five lemma and the open mapping theorem. An extension is said to be trivial if it is equivalent to the trivial sequence

$$0 \longrightarrow \mathfrak{Y} \longrightarrow \mathfrak{Y} \oplus \mathfrak{Z} \longrightarrow \mathfrak{Z} \longrightarrow 0. \tag{1.2}$$

The set of all possible extensions of $\mathfrak{Z}$ by $\mathfrak{Y}$ (modulo equivalence) is denoted $\mathrm{Ext}(\mathfrak{Z}, \mathfrak{Y})$. It admits a natural linear structure in such a way that (1.2) corresponds to zero and so on, see [1].

While the theory of extensions emphasizes on the equivalence of extensions, very little is known about the isomorphic types of the resulting twisted sums.

In this note, we analyze some minimal extensions of ℓ_1, that is middle spaces $\mathfrak{X}$ in non-trivial sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{X} \longrightarrow \ell_1 \longrightarrow 0.$$

Precisely, we show that Kalton extension in [5] is not isomorphic to Ribe's one in [11] (Kalton proved in [6] that the corresponding exact sequences are not equivalent, even in a weaker sense). Actually, we shall prove (even more) that Ribe space contains no isomorphic copy of that of Kalton, thus answering a question explicitly raised by Castillo and Moreno in [2] and implicit in [6,7]. This is the main result of Section 2. In Section 3, we exploit some preliminary results to obtain a continuum of mutually non-isomorphic (Ribe like) minimal extensions of ℓ_1. In Section 4, we solve another problem posed by Castillo and Moreno in [3].

Curiously enough, a great portion of the theory of extensions is highly nonlinear. It was recognized from the very beginning of the theory that extensions are in correspondence with quasi-linear maps. In fact, both Kalton and Ribe constructions use them. A quasi-linear map $\Omega : \mathfrak{Z} \to \mathfrak{Y}$ is one which, in addition of being homogeneous ($\Omega(\lambda z) = \lambda \Omega(z)$), satisfies an estimate

$$\|\Omega(x + y) - \Omega(x) - \Omega(y)\|_{\mathfrak{Y}} \leqslant \Delta(\|x\|_{\mathfrak{Z}} + \|y\|_{\mathfrak{Z}}) \quad (x, y \in \mathfrak{Z})$$

for some constant Δ. Such a map induces a quasi-norm on $\mathfrak{Y} \times \mathfrak{Z}$ by

$$\|(y, z)\|_{\Omega} \stackrel{\text{def}}{=} \|y - \Omega(z)\|_{\mathfrak{Y}} + \|z\|_{\mathfrak{Z}}.$$

The product $\mathfrak{Y} \times \mathfrak{Z}$ equipped with the quasi-norm induced by Ω is often denoted $\mathfrak{Y} \oplus_{\Omega} \mathfrak{Z}$. The diagram

$$0 \longrightarrow \mathfrak{Y} \xrightarrow{\ i\ } \mathfrak{Y} \oplus_{\Omega} \mathfrak{Z} \xrightarrow{\ \pi\ } \mathfrak{Z} \longrightarrow 0,$$

where $i(y) = (y, 0)$ and $\pi(y,z) = z$, is clearly an extension of quasi-Banach spaces. Actually defining Ω on a dense subspace of $\mathfrak{Z}$ would suffice [5,10].

And, conversely, every extension (1.1) comes (modulo equivalence) from some quasi-linear map $\Omega : \mathfrak{Z} \to \mathfrak{Y}$. It turns out that two quasi-linear maps $\Omega_i : \mathfrak{Z} \to \mathfrak{Y}$ induce equivalent extensions if and only if their difference is near to a linear (but not necessarily continuous!) map $\ell : \mathfrak{Z} \to \mathfrak{Y}$ in the sense that

$$\|\Omega_2(z) - \Omega_1(z) - \ell(z)\|_{\mathfrak{Y}} \leqslant M\|z\|_{\mathfrak{Z}}$$

for some M and all $z \in \mathfrak{Z}$. In this case, we shall write $\Omega_2 \sim \Omega_1$. We say that Ω is trivial if $\Omega \sim 0$ (that is, the induced extension is trivial).

Ribe space is obtained from the quasi-linear map $R : \ell_1 \to \mathbb{R}$ defined on the space $\ell_1 \cap c_{00}$ of all finitely supported sequences of ℓ_1 by

$$R(x) = \sum_i x_i \log |x_i| - \left(\sum_i x_i\right) \log \left|\sum_i x_i\right|,$$

where $x = \sum_i x_i \mathrm{e}_i$ and assuming $0 \cdot \log 0 = 0$. We write $\mathfrak{R} = \mathbb{R} \oplus_R \ell_1$.

Kalton's map is given by

$$K(x) = \sum_n \tilde{x}_n \log n \quad (x \geqslant 0),$$

where $\tilde{x}$ is the decreasing arrangement of x and then extended to $\ell_1 \cap c_{00}$ by

$$K(x) = K(x^+) - K(x^-) \quad (x = x^+ - x^-; x^+, x^- \geqslant 0; x^+ \cdot x^- = 0).$$

We write $\mathfrak{K}$ instead of $\mathbb{R} \oplus_K \ell_1$.

We assume some acquaintance with the basics of homology and, in particular, with the pull-back construction [1,4].

2. The spaces of Kalton and Ribe

Theorem 1. *$\mathfrak{R}$ contains no subspace isomorphic to $\mathfrak{K}$.*

Suppose $\tau : \mathfrak{K} \to \mathfrak{R}$ is an isomorphic embedding. Since the line $L_{\mathfrak{K}} = \{(t,0) : t \in \mathfrak{R}\}$ is the only one-dimensional subspace of $\mathfrak{K}$ that cannot be separated of the origin (by elements of $\mathfrak{K}^*$) it is clear that τ must sent $L_{\mathfrak{K}}$ into the line $L_{\mathfrak{R}}$ spanned by $(1,0) \in \mathfrak{R}$. There is no loss of generality in assuming that τ maps $(1,0) \in \mathfrak{K}$ to $(1,0) \in \mathfrak{R}$. Thus, we have a commutative square

$$\begin{array}{ccc} L_{\mathfrak{R}} & \longrightarrow & \mathfrak{R} \\ \uparrow & & \uparrow \tau \\ L_{\mathfrak{K}} & \longrightarrow & \mathfrak{K} \end{array}$$

Drawing the complete diagram we get another commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{R} & \longrightarrow & \ell_1 & \longrightarrow & 0 \\ & & \| & & \uparrow \tau & & \uparrow T & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{K} & \longrightarrow & \ell_1 & \longrightarrow & 0. \end{array} \tag{2.1}$$

By the universal property of the pull-back construction, (2.1) must be a pull-back diagram (in the sense that the lower row is equivalent to the pull-back extension induced by T). Thus Theorem 1 will follow from the following:

Proposition 1. *Kalton extension cannot be obtained from that of Ribe via pull-back.*

The next two Lemmas establish that a perturbation of T with a small enough operator does not change the lower extension of (2.1).

Lemma 1. *Let L be an operator on ℓ_1 such that $\|L(e_n)\| \leqslant 1/n$ for all n. Then L lifts to minimal extensions of ℓ_1. That is, the lower row in any pull-back diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{X} & \longrightarrow & \ell_1 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow L & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & PB & \longrightarrow & \ell_1 & \longrightarrow & 0 \end{array} \tag{2.2}$$

splits.

Proof. Notice that a quasi linear map $\Omega: \ell_1 \to \mathbb{R}$ is trivial if (and only if) there is a constant K such that

$$\left| \Omega(x) - \sum_i x_i \Omega(\mathrm{e}_i) \right| \leqslant K\|x\|_1 \tag{2.3}$$

for all $x = \sum_i x_i \mathrm{e}_i \in \ell_1 \cap c_{00}$. Indeed, if (2.3) holds, then the linear map $\ell(x) = \sum_i x_i \Omega(\mathrm{e}_i)$ approximates Ω on $\ell_1 \cap c_{00}$. We will also use the following estimate [5]: if $\Omega: \mathfrak{Z} \to \mathbb{R}$ is quasi linear, then

$$\left| \Omega\left(\sum_{i=1}^n z_i \right) - \sum_{i=1}^n \Omega(z_i) \right| \leqslant \Delta_\Omega \sum_{i=1}^n i\|z_i\|$$

for all n and all $z_i \in \mathfrak{Z}$.

Now, let $\Omega: \ell_1 \to \mathbb{R}$ be a quasi linear map defining the upper row of (2.2). The lower extension comes from the composition $\Omega \circ L$ (see [1]). We show that $\Omega \circ L$ is trivial. Take $x \in \ell_1 \cap c_{00}$. One has

$$\left| \Omega(L(x)) - \sum_i x_i \Omega(L(\mathrm{e}_i)) \right| \leqslant \Delta_\Omega \sum_i i|x_i| \, \|L(\mathrm{e}_i)\|_1 \leqslant \Delta_\Omega \|x\|_1,$$

as desired. □

Lemma 2. *Let* $0 \to \mathfrak{Y} \to \mathfrak{X} \to \mathfrak{Z} \to 0$ *be an extension of quasi-Banach spaces and* $T: \mathfrak{Z}' \to \mathfrak{Z}$ *an operator. Then the lower row in the pull-back diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{Z} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow T & & \\ 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & PB & \longrightarrow & \mathfrak{Z}' & \longrightarrow & 0 \end{array} \tag{2.4}$$

does not vary when T *is perturbed by a liftable operator* $L: \mathfrak{Z}' \to \mathfrak{Z}$ (*explanations follow*).

Proof. We want to see that if $L: \mathfrak{Z}' \to \mathfrak{Z}$ lifts to $\mathfrak{X}$ then the lower row in the new pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{Y} & \xrightarrow{i} & \mathfrak{X} & \xrightarrow{\pi} & \mathfrak{Z} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow T+L & & \\ 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & PB' & \longrightarrow & \mathfrak{Z}' & \longrightarrow & 0 \end{array}$$

remains equivalent to that of (2.4). Needless to say, this implies that PB and PB' are isomorphic.

It is proved in [1] that

$$0 \longrightarrow L(\mathfrak{Z}', \mathfrak{Y}) \xrightarrow{i_*} L(\mathfrak{Z}', \mathfrak{X}) \xrightarrow{\pi_*} L(\mathfrak{Z}', \mathfrak{Z}) \xrightarrow{\alpha} \operatorname{Ext}(\mathfrak{Z}', \mathfrak{Y})$$

is an exact sequence of linear spaces and maps. In fact, this is nothing but the (not so) long homology sequence in the second variable, truncated at the fourth term. Here, i_* and π_* are plain composition (on the left) and α acts taking pull-backs. Thus the lemma is (even less than) a restatement of the exactness of the above sequence at $L(\mathfrak{Z}', \mathfrak{Z})$. □

Before going into the proof of Proposition 1 let us remark that both $\mathfrak{K}$ and $\mathfrak{R}$ have certain symmetry that we will exploit as follows:

Lemma 3. *Let* $\Omega: \ell_1 \cap c_{00} \to \mathbb{R}$ *be either Kalton's map or Ribe's map. Let* $\sigma: \mathbb{N} \to \mathbb{N}$ *be any injection and let* Σ *be the operator sending* e_n *into* $\mathrm{e}_{\sigma(n)}$. *Then* $\Omega \circ \Sigma = \Omega$.

Beginning of the Proof of Proposition 1. In this step, we prove that if $\mathfrak{K}$ were a pull-back of $\mathfrak{R}$ then the operator T appearing in (2.1) could be replaced by another one sending the usual basis into a block sequence whose sums are constant.

Assume that drawing (2.1) is possible. Then $K \sim R \circ T$. Since $T(\mathrm{e}_n)$ contains a weakly* convergent subsequence (in the $\sigma(\ell_1, c_0)$-topology) an obvious application of Lemma 3 yields that we may and do assume that $T(\mathrm{e}_n)$ has a weak* limit, say f, in ℓ_1. Let $\tilde{T}(x) = T(x) - S(x)f$, where $S(x) = \sum_i x_i$. Since the map $x \mapsto S(x)f$ is liftable (it has one-dimensional range!) there is no loss of generality in assuming that T maps the unit basis of ℓ_1 into a weakly* null sequence.

It is clear that there is a further subsequence $(e_{\sigma(n)})$ of (e_n) and a block sequence (f_n) in ℓ_1 such that $\|T(e_{\sigma(n)}) - f_n\|_1 \leqslant 1/n$. Applying Lemma 3 and then Lemmas 1 and 2, we see that $Te_n = f_n$ can be assumed to be a block sequence.

Passing to another subsequence if necessary, we may suppose that

$$k = \lim_{n\to\infty} S(f_n)$$

exists (it is, of course, finite) and also that $|S(f_n) - k| \leqslant 1/n$. Therefore, there are f_n^* with $\mathrm{supp}(f_n^*) \subset \mathrm{supp}(f_n)$, $\|f_n^* - f_n\| \leqslant 1/n$ and $S(f_n^*) = k$. Applying Lemmas 1 and 2 we conclude that $K \sim R \circ T$, where T is an operator on ℓ_1 sending the unit basis into a block sequence (f_n) such that $S(f_n) = k$, where k is some constant.

The following result reveals a further, hidden, symmetry of Ribe's creature.

Lemma 4. *If (f_i) are blocks of sum k, then*

$$R\left(\sum_i \lambda_i f_i\right) = kR\left(\sum_i \lambda_i e_i\right) + \sum_i \lambda_i R(f_i). \tag{2.5}$$

Proof. Write $f_i = \sum_j x_{ij} e_j$. Since $R(x) = S(x \cdot \log|x|) - S(x)\log|S(x)|$, we have

$$\begin{aligned}
R\left(\sum_i \lambda_i f_i\right) &= \sum_{i,j} \lambda_i x_{ij} \log|\lambda_i x_{ij}| - S\left(\sum_i \lambda_i f_i\right) \log\left|S\left(\sum_i \lambda_i f_i\right)\right| \\
&= \sum_i \left(\lambda_i \log|\lambda_i| \sum_j x_{ij}\right) + \sum_i \lambda_i \left(\sum_j x_{ij}\log|x_{ij}|\right) \\
&\quad -k\left(\log|k|\sum_i \lambda_i - \left(\sum_i \lambda_i\right) \log\left|\sum_i \lambda_i\right|\right) \\
&= k\sum_i \lambda_i \log|\lambda_i| + \sum_i \lambda_i (R(f_i) + k\log|k|) \\
&\quad -k\log|k|\sum_i \lambda_i - k\left(\sum_i \lambda_i\right)\log\left|\sum_i \lambda_i\right| \\
&= kR\left(\sum \lambda_i e_i\right) + \sum \lambda_i R(f_i).
\end{aligned}$$

End of the Proof of Proposition 1. Since the second summand in the right-hand side of (2.5) is linear in (λ_i) our assumption that $\mathfrak{K}$ is a pull-back of $\mathfrak{R}$ now implies that $K \sim k \cdot R$. Any linear map approximating $K - k \cdot R$ must be bounded on $\ell_1 \cap c_{00}$ and so one would have

$$|K(x) - kR(x)| \leqslant M\|x\| \quad (x \in \ell_1 \cap c_{00}) \tag{2.6}$$

(that is, $\mathfrak{K}$ and $\mathfrak{R}$ are projectively equivalent extensions). The remainder of the proof was given by Kalton in [6, p. 275], but we write the details for the sake of completeness.

Take $x = \sum_{i=1}^{n} e_i$. Then $K(x) = \sum_{i=1}^{n} \log i = \log n! \sim n \log n + O(n)$, while $R(x) = -n \log n$ and (2.6) implies that

$$n \log n + kn \log n = O(n).$$

We see that $k = -1$.

In only remains to show that

$$\sup_{x \in \ell_1 \cap c_{00}} \frac{|K(x) + R(x)|}{\|x\|} = +\infty. \tag{2.7}$$

This is an amusing exercise in elementary calculus: the maximum value of $|K + R|$ on the positive part of the unit sphere of $\ell_1^n \subset \ell_1$ is attained at some point $x = \sum_{i=1}^{n} x_i e_i$, with $x_1 > x_2 > \cdots > x_n > 0$, and such that all partial derivatives $\partial/\partial x_i (K+R)$ coincide. Hence $\log i + \log x_i$ is constant and x is proportional to

$$x^* = \sum_{i=1}^{n} \frac{e_i}{i}.$$

But $|K(x^*) + R(x^*)| = \|x^*\| \log \|x^*\|$ and since $\log \|x^*\| \sim \log \log n \to \infty$ as $n \to \infty$ we have (2.7) and the proof is complete. □

3. Many non-isomorphic minimal extensions

In this section we construct a continuum of mutually non-isomorphic (Ribe-like) extensions.

Let $\theta : [0, \infty) \to \mathbb{R}$ be a Lipschitz function vanishing at zero, with Lipschitz constant $\mathrm{Lip}(\theta)$. The map defined on the finitely supported sequences of ℓ_1 by

$$R_\theta(x) = \sum_i x_i \theta\left(-\log \frac{|x_i|}{\|x\|}\right)$$

(with $0 \cdot \theta(-\log 0) = 0$) is quasi-linear (with constant $\mathrm{Lip}(\theta) \log 2$). The induced extension shall be denoted by $\mathfrak{R}(\theta)$ in the sequel. Ribe space corresponds to the choice $\theta(t) = t$. Note that "our" spaces $\mathfrak{R}(\theta)$ are nothing but push-outs of the spaces $\ell_1(\theta)$ introduced by Kalton and Peck in [10]. Actually, each construction can be obtained from the other in a natural way (that is, by a natural equivalence of functors). It is not hard to see that two extensions

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{R}(\theta_i) \longrightarrow \ell_1 \longrightarrow 0 \quad (i = 1, 2)$$

are equivalent if and only if

$$\sup_{0 \leqslant t < \infty} |\theta_1(t) - \theta_2(t)| < \infty.$$

Also, the above extensions are projectively equivalent if and only if there is a constant a for which $\theta_1(t) - a\theta_2(t)$ remains bounded. Although it seems very likely that $\mathfrak{R}(\theta_i)$ are isomorphic if and only if the corresponding extensions are projectively equivalent, I have not been able to prove this for arbitrary θ_i's.

We have, however, the following partial result for a restricted class of Lipschitz maps.

Theorem 2. *Let ϕ and θ be Lipschitz functions. Suppose that θ is non-negative and subadditive. If*

$$\liminf_{t\to\infty} \frac{\theta(t)}{\phi(t)} = 0, \tag{3.1}$$

then $\mathfrak{R}(\phi)$ cannot be obtained from $\mathfrak{R}(\theta)$ via pull-back. In particular $\mathfrak{R}(\theta)$ does not contain isomorphs of $\mathfrak{R}(\phi)$.

Proof. Suppose there is a commutative (necessarily a pull-back) diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{R}(\theta) & \longrightarrow & \ell_1 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow T & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{R}(\phi) & \longrightarrow & \ell_1 & \longrightarrow & 0. \end{array} \tag{3.2}$$

Reasoning as in Section 2, we may suppose that $f_n = T(\mathrm{e}_n)$ is a block sequence with $\|f_n\| = k$ for all $n \in \mathbb{N}$. Hence $R_\phi \sim R_\theta \circ T$ and we have

$$\left| (R_\phi - R_\theta \circ T)\left(\sum_{i=1}^n \mathrm{e}_i\right) - \sum_{i=1}^n (R_\phi - R_\theta \circ T)(\mathrm{e}_i) \right| = O(n).$$

Writing $f_i = \sum_j x_{ij}\mathrm{e}_j$, we obtain

$$\begin{aligned} n\phi(\log n) + O(n) &\leqslant \left| \sum_{i,j} \theta\left(\log \frac{kn}{|x_{ij}|}\right) - \sum_{i,j} x_{ij}\theta\left(\log \frac{k}{|x_{ij}|}\right) \right| \\ &\leqslant \sum_{i,j} |x_{ij}| \left| \theta\left(\log n + \log \frac{k}{|x_{ij}|}\right) - \theta\left(\log \frac{k}{|x_{ij}|}\right) \right| \\ &\leqslant \sum_{i,j} |x_{ij}| \theta(\log n) \\ &= k \cdot n \cdot \theta(\log n) \end{aligned}$$

in contradiction to (3.1). □

Thus, the family of functions

$$\theta_\alpha(t) = \min\{t, t^\alpha\} \quad (0 < \alpha \leqslant 1)$$

produces a continuum of mutually non-isomorphic spaces $\mathfrak{R}(\theta_\alpha)$.

4. On strictly singular maps

We close with a remark on paper [3]. There, a quasi-linear map $\Omega : \mathfrak{Z} \to \mathfrak{Y}$ is called strictly singular if its restrictions to infinite dimensional subspaces of $\mathfrak{Z}$ are never trivial. This means that the natural quotient map $\mathfrak{Y} \oplus_\Omega \mathfrak{Z} \to \mathfrak{Z}$ is strictly singular, that is, its restrictions to infinite dimensional subspaces of $\mathfrak{Y} \oplus_\Omega \mathfrak{Z}$ are never isomorphisms.

In [9], Kalton succeeds in constructing a strictly singular map $\ell_1 \to \mathbb{R}$—this is not the map K treated in Section 2! Castillo and Moreno conjectured that, if Γ is a "large" index set, then no strictly singular quasi-linear map $\ell_1(\Gamma) \to \mathbb{R}$ exists. The following result proves their conjecture. We denote by $\mathfrak{c}$ the cardinality of the continuum and by $\#\Gamma$ the cardinal of the set Γ.

Proposition 2. *If $\#\Gamma > \mathfrak{c}$, then no strictly singular quasi-linear map $\Omega : \ell_1(\Gamma) \to \mathbb{R}$ exists.*

Proof. Let us first note that $\#\mathrm{Ext}(\ell_1, \mathbb{R}) = \mathfrak{c}$. That $\#\mathrm{Ext}(\ell_1, \mathbb{R}) \geqslant \mathfrak{c}$ is obvious: actually there are $\mathfrak{c}$ non-isomorphic twisted sums of $\mathbb{R}$ with ℓ_1, in view of the results in the preceding section. As for the other inequality, let $\pi : \ell_p \to \ell_1$ be any quotient map, with $0 < p < 1$. Put $K = \ker \pi$, so that we have an extension

$$0 \longrightarrow K \xrightarrow{\ i\ } \ell_p \xrightarrow{\ \pi\ } \ell_1 \longrightarrow 0.$$

Taking homology in the first variable with respect to $\mathbb{R}$ we obtain the exact sequence

$$0 \longrightarrow \ell_1^* \xrightarrow{\ \pi^*\ } \ell_p^* \xrightarrow{\ i^*\ } K^* \longrightarrow \mathrm{Ext}(\ell_1, \mathbb{R}) \longrightarrow \mathrm{Ext}(\ell_p, \mathbb{R}).$$

Since $\mathrm{Ext}(\ell_p, \mathbb{R}) = 0$ (proved by Kalton in [5]) we have

$$\mathrm{Ext}(\ell_1, \mathbb{K}) = K^*/i^*(\ell_p^*)$$

(in the pure algebraic sense) and so $\#\mathrm{Ext}(\ell_1, \mathbb{R}) \leqslant \#K^*$. But K is separable and, therefore, $\#K^* = \mathfrak{c}$. Thus $\mathrm{Ext}(\ell_1, \mathbb{R})$ has the power of continuum.

Let now $\Omega : \ell_1(\Gamma) \to \mathbb{R}$ be a quasi-linear map, where $\#\Gamma > \mathfrak{c}$. Partition Γ into a family of disjoint (infinite) countable subsets Γ_α, so that

$$\Gamma = \bigoplus_{\alpha \in I} \Gamma_\alpha.$$

Clearly, $\#I > \mathfrak{c}$. For each $\alpha \in I$, let $\sigma_\alpha : \mathbb{N} \to \Gamma_\alpha$ be a fixed bijection and define $\Sigma_\alpha : \ell_1 = \ell_1(\mathbb{N}) \to \ell_1(\Gamma)$ as $\Sigma_\alpha(x) = \sum_{n=1}^\infty x_n \mathrm{e}_{\sigma_\alpha(n)}$. Since $\#I > \#\mathrm{Ext}(\ell_1, \mathbb{R})$ there are $\alpha, \beta \in I$ such that $\Omega \circ \Sigma_\alpha \sim \Omega \circ \Sigma_\beta$, with $\alpha \neq \beta$. Obviously, the restriction of Ω to the range of $\Sigma_\alpha - \Sigma_\beta : \ell_1 \to \ell_1(\Gamma)$ is trivial. But that subspace is spanned by the sequence $\mathrm{e}_{\sigma_\alpha(n)} - \mathrm{e}_{\sigma_\beta(n)}$ and it is isometric to ℓ_1. This completes the proof. □

5. Concluding remarks

Lemma 1 is not best possible: it can be proved that, with the same notations, if L is such that $\|L(e_n)\| \to 0$ and $d_n \log n$ is bounded, where d_n is the decreasing arrangement of $\|L(e_n)\|$, then L lifts to minimal extensions of ℓ_1. And, conversely, if $d_n \log n$ is unbounded, then the operator $L(x) = \sum_n d_n x_n e_n$ cannot be lifted to $\mathfrak{R}$ (nor to $\mathfrak{K}$). This is essentially in [6, Section 8].

It can be proved that the Banach–Mazur distance between $\mathbb{R} \oplus_R \ell_1^n$ and $\mathbb{R} \oplus_K \ell_1^n$ goes to infinity as n increases. This is not immediate from Theorem 1, but requires a kind of "reduced" ultra-technique. Details will appear elsewhere.

A "localization" of the proof of Proposition 2 shows that there is a function $n : \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{N}$, with $n(m, \varepsilon, \Delta) \to \infty$ as $m \to \infty$ for each fixed $\varepsilon, \Delta > 0$, and such that whenever $\Omega : \ell_1^m \to \mathbb{R}$ is quasi linear with constant Δ there exists an n-dimensional block subspace B of ℓ_1^m and a linear map $\ell : B \to \mathbb{R}$ such that $|\Omega(x) - \ell(x)| \leqslant \varepsilon \|x\|$ for all $x \in B$. So the arguments of [9] cannot be localized. A version of Proposition 2 for extensions of (quasi) Banach spaces would be welcome.

I do not know if Proposition 2 is true for $\#\Gamma = \mathfrak{c}$. A possible counterexample could be obtained as follows. Let $\{\theta_\alpha : \alpha \in \mathfrak{c}\}$ be a continuum of 1-Lipschitz functions such that

$$\sup_t |\theta_\alpha(t) - a\theta_\beta(t)| = \infty \quad (a \in \mathbb{R})$$

whenever $\alpha \neq \beta$—such a family do exist! Then define $\Omega : \ell_1(\mathfrak{c}) \to \mathbb{R}$ by

$$\Omega(x) = \sum_\alpha x_\alpha \theta_\alpha \left(\log \frac{\|x\|}{|x_\alpha|} \right).$$

I do not know even if one can construct a strictly singular quasi-linear map $\ell_1 \to \mathbb{R}$ in this way, although Kalton thinks it is a too-simple-to-work construction (personal communication).

References

[1] F. Cabello Sánchez, J.M.F. Castillo, The long homology sequence for quasi-Banach spaces, with applications, Positivity, to appear.

[2] J.M.F. Castillo, Y. Moreno Salguero, On isomorphically equivalent extensions of quasi-Banach spaces. Recent progress in functional analysis (Valencia, 2000), North-Holland Mathematical Studies, Vol. 189, North-Holland, Amsterdam, 2001, pp. 263–272.

[3] J.M.F. Castillo, Y. Moreno Salguero, Strictly singular quasi-linear maps, Nonlinear Anal. 49 (2002) 897–903.

[4] W.B. Johnson, Extensions of c_0, Positivity 1 (1997) 55–74.

[5] N.J. Kalton, The three space problem for F-spaces, Compositio Math. 37 (1978) 243–276.

[6] N.J. Kalton, Convexity, type and the three space problem, Stud. Math. 69 (1981) 247–287.

[7] N.J. Kalton, Sequences of random variables in L_p for $p < 1$, J. Reine Angew. Math. 329 (1981) 204–214.

[8] N.J. Kalton, An elementary example of a Banach space not isomorphic to its complex conjugate, Can. Math. Bull. 38 (2) (1995) 218–222.

[9] N.J. Kalton, The basic sequence problem, Stud. Math. 116 (2) (1995) 167–187.

[10] N.J. Kalton, N.T. Peck, Twisted sums of sequence spaces and the three space problem, Trans. Am. Math. Soc. 255 (1979) 1–30.
[11] M. Ribe, Examples for the nonlocally convex three space problem, Proc. Am. Math. Soc. 237 (1979) 351–355.
[12] J.W. Roberts, A nonlocally convex F-space with the Hahn–Banach approximation property, Banach spaces of analytic functions (Proceedings of the Pelczynski Conference, Kent State University, Kent, Ohio, 1976), Lecture Notes in Mathematics, Vol. 604, Springer, Berlin, 1977, pp. 76–81.