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Local isometries on spaces of continuous functions*

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Introduction

Let X and Y be Banach spaces and $\mathfrak{S}$ a subset of the space of (linear, continuous) operators from X to Y. We say that an operator T belongs locally to $\mathfrak{S}$ if for every $x \in X$ there is $S \in \mathfrak{S}$, possibly depending on x, such that $Tx = Sx$. 'Pointwise' should be better than 'locally', but we have followed tradition. If each operator that belongs locally to $\mathfrak{S}$ belongs in fact to $\mathfrak{S}$ we say that $\mathfrak{S}$ is algebraically reflexive.

When $Y = X$ and $\mathfrak{S} = \mathrm{Iso}(X)$ is the group of isometries of X we say that T is a local isometry of X. (In this paper 'isometry' means linear surjective isometry.) Similarly, a local automorphism of a Banach algebra is an operator which agrees at every point with some automorphism. Also, we will consider approximate local isometries and automorphisms. These are operators having the following property: given $x \in X$ and $\varepsilon > 0$, there is an isometry (respectively, an automorphism) S of X such that $\|Tx - Sx\| < \varepsilon$.

The study of local isometries and automorphisms of Banach algebras spurred a considerable interest in recent years (see the bibliography of the dissertation [16]). In this paper we deal with local isometries and automorphisms of the algebras $C_0(L)$. As usual, we write $C_0(L)$ or $C_0^{\mathbb{K}}(L)$ for the Banach algebra of all continuous $\mathbb{K}$-valued functions on the locally compact space L vanishing at infinity, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. If L is compact the subscript will be omitted. By the Banach-Stone theorem, if T is local isometry of $C_0(L)$, then for each f there are a homeomorphism φ of L and a continuous unitary $u : L \to \mathbb{K}$ such that

$$Tf = u(f \circ \varphi).$$

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If T is an approximate local isometry, then

$$Tf = \lim_{n\to\infty} u_n(f \circ \varphi_n)$$

for suitable sequences φ_n and u_n (depending on f). Local and approximate local automorphisms have the same form, but with $u = u_n = 1$.

Two basic results in the complex case are:

($\natural$) If K is compact, every approximate local isometry of $C^{\mathbb{C}}(K)$ is a unital endomorphism followed by multiplication by some unitary [5, Theorem 5].

($\sharp$) If the one-point compactification of L is metrizable, then every local isometry of $C_0^{\mathbb{C}}(L)$ is surjective (see [14, Theorem 4] and the Appendix of the present paper; the compact case was obtained earlier in [18, Theorem 2.2]).

On the other hand, very little is known about spaces of real functions. We do not even know if local automorphisms of $C^{\mathbb{R}}(K)$ are in fact automorphisms when K is compact metric (as it is the case for complex functions). It is worth noting that the proofs of ($\natural$) and ($\sharp$) strongly depend on the Gleason-Kahane-Żelazko theorem (or on some of its generalizations), a result which applies only to complex algebras (see [13]).

This paper consists of three Sections and one Appendix. In Section 1 we give a representation of approximate local isometries on $C_0(L)$ when L is totally disconnected (roughly that ($\natural$) remains true for real and complex functions even in the non-unital case). As a consequence we obtain that Iso $C_0^{\mathbb{R}}(L)$ is algebraically reflexive if L is a totally disconnected locally compact space whose one-point compactification is metrizable.

In Section 2 we revisit a result by Molnár and Győry and we obtain the algebraic reflexivity of Iso $C_0^{\mathbb{R}}(L)$ when L is a manifold with boundary.

In Section 3 we investigate the structure of local operators above, both in the real and complex cases. We obtain counterexamples for most conceivable variations of ($\natural$). A sample: local automorphisms of $C^{\mathbb{R}}(K)$ algebras need not be separating, in particular they are not endomorphisms and their adjoints do not preserve extreme points; this solves a problem raised by Jarosz and Rao in [14].

Finally, the Appendix contains an amendment of the proof of ($\sharp$).

The notation is standard. The group of automorphisms of the Banach algebra $\mathcal{A}$ is denoted by Aut $\mathcal{A}$. All topological spaces are assumed to be Hausdorff. If f is a function defined on the topological space S, then $\operatorname{supp}_S f$ denotes the support of f, that is, the closure of the set $\{x \in S : f(x) \neq 0\}$. If L is a locally compact space, then αL stands for its one-point compactification. If D is a closed subset of L, then $C_0(L\|D)$ will denote the subalgebra of functions in $C_0(L)$ vanishing on D. The characteristic function of the set A is denoted by 1_A. Given a map $\psi : E \to F$ we write ψ^* for the map $\mathbb{K}^F \to \mathbb{K}^E$ given by composition on the right with ψ, that is, $\psi^*(f) = f \circ \psi$. This applies in particular to the adjoint of an operator acting between Banach spaces.

1. Totally disconnected spaces

We begin with a representation of local isometries of $C_0(L)$ for totally disconnected L. Let us recall that a topological space is said to be totally disconnected if its only nonempty connected subsets are the one-point sets [23, § 29]. A locally compact space is totally disconnected if and only if it is 0-dimensional (i.e., each point has a neighborhood base of closed-open sets [23, Theorem 29.7]).

Proposition 1. *Let T be an approximate local isometry of $C_0(L)$, where L is a totally disconnected locally compact space. Then there is an open $A \subset L$, a continuous surjection $\psi : A \to L$ and a continuous unitary $u : A \to \mathbb{K}$ such that the following square is commutative:*

$$\begin{array}{ccc} C_0(L) & \xrightarrow{\ T\ } & C_0(L) \\ {\scriptstyle \psi^*}\downarrow & & \uparrow{\scriptstyle E} \\ C_0(A) & \xrightarrow{\ M_u\ } & C_0(A) \end{array}$$

where $\psi^* f = f \circ \psi$, M_u denotes multiplication by u and E extends functions in $C_0(A)$ as zero outside A. If T is an approximate local automorphism, then $u = 1_A$ and so T is a homomorphism. If L is compact, then $A = L$.

Proof. We write the proof for approximate local isometries and $\mathbb{K} = \mathbb{C}$. The other cases are easier. The hypothesis on L implies that $C_0(L)$ is the closure of the linear span of its idempotents (they are the characteristic functions of the compact-open sets). Let $K \subset L$ be a compact-open set. Then there are unitary functions u_n and homeomorphisms φ_n such that

$$T(1_K) = \lim_{n\to\infty} u_n(1_K \circ \varphi_n) = \lim_{n\to\infty} u_n 1_{\varphi_n^{-1}(K)}.$$

It is clear that the sequence of sets $(\varphi_n^{-1}(K))$ is eventually constant. Thus there is a (nonempty) compact-open set we denote $\Lambda(K)$ such that $\varphi_n^{-1}(K) = \Lambda(K)$ for n large enough. It is evident that (u_n) converges uniformly on $\Lambda(K)$ to some unitary u_K defined and continuous on $\Lambda(K)$. Thus, we have a partial representation

$$T(1_K) = u_K \cdot 1_{\Lambda(K)}. \tag{1}$$

We claim that $K \longmapsto \Lambda(K)$ preserves disjoint unions, that is,

$$\Lambda(K \oplus H) = \Lambda(K) \oplus \Lambda(H)$$

whenever K and H are disjoint compact-open subsets of L. To see this, note that in view of (1) and the linearity of T, we have

$$u_K \cdot 1_{\Lambda(K)} + u_H \cdot 1_{\Lambda(H)} = u_{K\oplus H} \cdot 1_{\Lambda(K\oplus H)}.$$

Thus the point is to check that $\Lambda(K)$ and $\Lambda(H)$ are disjoint. Suppose $x \in \Lambda(K) \cap \Lambda(H)$.

- If $x \notin \Lambda(K \oplus H)$, then we have $u_K(x) + u_H(x) = 0$, with $|u_K(x)| = |u_H(x)|$=1. It follows that

$$\|T\,(1_K - 1_H)\,\| = \|u_K \cdot 1_{\Lambda(K)} - u_H \cdot 1_{\Lambda(H)}\| \geq |u_K(x) - u_H(x)| = 2,$$

which is absurd since $\|1_K - 1_H\| = 1$ and T is norm-preserving.

- In case $x \in \Lambda(K \oplus H)$ we have $|u_K(x) + u_H(x)| = 1$, with $|u_K(x)| = |u_H(x)| = 1$. Let us assume for a moment $u_K(x) + u_H(x) = 1$. After a moment's reflection we realize that $\{u_K(x), u_H(x)\} = 1/2 \pm i\sqrt{3}/2$. In general, we have $\{u_K(x), u_H(x)\} = (1/2 \pm i\sqrt{3}/2)(u_K(x) + u_H(x))$. As before,

$$\|T(1_K - 1_H)\| \geq |u_K(x) - u_H(x)| = \sqrt{3} > 1,$$

a contradiction.

This proves our claim and shows that, in fact, Λ preserves the Boolean operations on the algebra of compact-open sets, in particular it preserves finite unions and intersections, as well as inclusions.

Next, we show that the unitary function appearing in (1) is independent on K. To this end, define

$$A = \bigcup_K \Lambda(K),$$

where K runs over all compact open-subsets of L. Then A is open and we can define $u : A \to \mathbb{T}$ taking

$$u(x) = u_K(x) \qquad (x \in \Lambda(K)).$$

The definition makes sense because if $x \in \Lambda(K) \cap \Lambda(H) = \Lambda(K \cap H)$, then $u_K(x) = u_H(x) = u_{K \cap L}(x)$. It is moreover clear that u is continuous on A and also that we have a representation

$$(T1_K)(x) = \begin{cases} u(x)1_{\Lambda(K)}(x) & (x \in A) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Finally, let $\tilde{T} : C_0(L) \to C_0(A)$ be given by $\tilde{T}1_K = 1_{\Lambda(K)}$. It is clear that $\tilde{T}$ extends to an injective homomorphism (that we do not relabel) and therefore

$$\tilde{T}f = f \circ \psi \qquad (f \in C_0(L)),$$

where $\psi : A \to L$ is a continuous surjection (given by $\tilde{T}^*\delta_a = \delta_{\psi(a)}$, where $\tilde{T}^*$ is the Banach space adjoint of $\tilde{T}$). In particular we have $1_{\Lambda(K)} = 1_K \circ \psi$ and so (2) implies that $Tf = E(u \cdot (f \circ \psi))$ when f is the characteristic function of a compact-open set and therefore for all $f \in C_0(L)$. □

Corollary 1. *Let L be a totally disconnected locally compact space whose one-point compactification is metrizable. Then* Iso $C_0(L)$ *and* Aut $C_0(L)$ *are algebraically reflexive.*

Proof. Suppose T is either a local isometry or a local automorphism of $C_0(L)$. Take $f \in C_0(L)$ such that $f(x) \neq 0$ for all $x \in L$. Then Tf does not vanish on L and we have $A = L$ in Proposition 1. It remains to see that ψ is injective. We give the argument appearing in [18, Theorem 2.2] for the sake of completeness and for future reference. Suppose $x_1, x_2 \in L$ are such that $\psi(x_1) = \psi(x_2) = y$. Take a nonnegative $f \in C_0(L)$ such that $f(y) = 1$ and $f(z) < 1$ for $z \neq y$. Then for $i = 1, 2$, we have

$$1 = f(y) = f(\psi(x_i)) = |(Tf)(x_i)| = f(\varphi(x_i)),$$

where φ is any homeomorphism of L such that $Tf = \sigma(f \circ \varphi)$, with σ unitary. Hence $y = \varphi(x_i)$ and so $x_1 = x_2$, which completes the proof. □

Let T be a local isometry of $C_0(L)$, with L totally disconnected and let $T_+ = E \circ \psi^*$. It is clear that T_+ is an injective homomorphism. It seems very likely that T_+ should be a local automorphism of $C_0(L)$, but I have been unable even to prove that T_+ is a local isometry without imposing additional conditions on L (e.g., it is extremally disconnected). It is clear, however, that T_+ behaves as a local automorphism on the positive cone of $C_0(L)$: if g is nonnegative then $T_+(g) = |T(g)|$ and there is an automorphism U of $C_0(L)$ such that $T_+(g) = U(g)$. Also, it is evident that T is onto if and only if T_+ is since T is onto precisely when $A = L$ and ψ is a homeomorphism.

If L is not compact we can consider $C(\alpha L)$ as the unitization of $C_0(L) = C_0(\alpha L \| \infty)$. Every $f \in C(\alpha L)$ admits a unique decomposition

$$f = \lambda 1_{\alpha L} + g$$

with $\lambda \in \mathbb{K}$ and $g \in C_0(L)$ —just take $\lambda = f(\infty)$ and $g = f - \lambda 1_{\alpha L}$. Thus, we may extend T_+ to a unital endomorphism of $C(\alpha L)$ that we do not relabel. It is clear that T_+ is an automorphism of $C(\alpha L)$ if and only if T is an isometry of $C_0(L)$. Moreover, since automorphisms of $C_0(L)$ extend to automorphisms of $C(\alpha L)$, we see that if $f \in C(\alpha L)$ admits a decomposition as above with $g \geq 0$, then there is an automorphism U of $C(\alpha L)$ such that $T_+(f) = U(f)$.

We turn next to maximal ideals of algebras of bounded measurable functions. Let μ be a countably additive measure on the measure space (Ω, Σ). As usual we write $L_\infty(\mu)$ or $L_\infty^{\mathbb{K}}(\mu)$ for the Banach algebra of all essentially bounded measurable functions $f : \Omega \to \mathbb{K}$ with the essential supremum norm, "pointwise operations", and the traditional convention about identifying functions equal almost everywhere. When μ is Lebesgue measure on $[0, 1]$ we simply write L_∞. When μ is the counting measure on the set Γ we write $\ell_\infty(\Gamma)$ instead of $L_\infty(\mu)$.

Declare two sets $A, B \in \Sigma$ equivalent if $\mu(A \triangle B) = 0$. Identifying equivalent sets we obtain a Boolean algebra denoted Σ/μ in what follows. The Boolean structure of Σ/μ comes from that of Σ by the rules

$$[A] \cup [B] = [A \cup B], \quad [A] \cap [B] = [A \cap B], \quad [A]^c = [A^c],$$

where $[A]$ denotes the class of $A \in \Sigma$ in Σ/μ and A^c is the complement of A in Ω. Countable operations are defined on Σ/μ in the obvious way. It is worth noting that 1_A and 1_B are the same function in $L_\infty(\mu)$ if and only if A and B are equivalent and so the notation $1_{[A]}$ makes perfect and clear sense. It is evident that every idempotent of $L_\infty(\mu)$ is of the form $1_{[A]}$ for some measurable A. Thus Σ/μ is Boolean isomorphic to the algebra of idempotents of $L_\infty(\mu)$. Hence every linear and unital ring homomorphism $L : L_\infty(\mu) \to L_\infty(\mu')$ induces (and comes from) a Boolean homomorphism $\Lambda : \Sigma/\mu \to \Sigma'/\mu'$ through the formula

$$L(1_{[A]}) = 1_{\Lambda[A]}.$$

Regarding $\mathbb{K}$ as the algebra of functions on a single point of mass one, we see that characters $\chi : L_\infty(\mu) \to \mathbb{K}$ correspond to finitely additive measures defined on Σ/μ that take the values zero and one, only.

By general representation theorems every $L_\infty(\mu)$ is isomorphic to the algebra of continuous functions on a compact space, say K (actually, the set of all continuous characters with the relative weak* topology). Quite clearly K is totally disconnected and so Proposition 1 applies to maximal ideals of $L_\infty(\mu)$.

Lemma 1. *Let $\mathcal{A}$ be a maximal ideal of $L_\infty(\mu)$ and let T be a local isometry of $\mathcal{A}$. Then the Boolean endomorphism of Σ/μ induced by T_+ preserves countable operations.*

Proof. Let Λ denote the Boolean endomorphism induced by T_+. It clearly suffices to prove that

$$[\Omega] = \bigcup_{n=1}^{\infty} \Lambda[A_n] \tag{3}$$

whenever (A_n) is a countable partition of Ω. A moment's reflection suffices to realize that there is at most one n for which 1_{A_n} does not belong to $\mathcal{A}$, and so we may assume $1_{A_n} \in \mathcal{A}$ for all $n \geq 2$. Take a sequence $\lambda_n \to 0$, with $\lambda_1 = 1$ and $0 < \lambda_n < 1$ for $n \geq 2$. Let

$$f = \sum_{n=1}^{\infty} \lambda_n 1_{A_n}$$

(summation in $L_\infty(\mu)$). Now, if $1_{A_1} \in \mathcal{A}$, then $f \in \mathcal{A}$ and since $f \geq 0$ we have

$$T_+(f) = \sum_{n=1}^{\infty} \lambda_n 1_{\Lambda[A_n]} = U(f),$$

for some $U \in \operatorname{Aut} L_\infty(\mu)$. Hence $T_+(f)$ vanishes on no set of positive measure, which proves (3). If $1_{A_1} \notin \mathcal{A}$ we have a decomposition

$$f = 1_\Omega + (f - 1_\Omega)$$

and since $f - 1_\Omega \leq 0$ we get the same conclusion. □

Corollary 2 (See [14,4] for the unital case). *Suppose $\mathcal{A}$ is a maximal ideal either of L_∞ or of $\ell_\infty(\Gamma)$, where Γ is of non-measurable cardinal. Then* Iso $\mathcal{A}$ *is algebraically reflexive.*

Proof. First suppose $\mathcal{A}$ is a maximal ideal of $\ell_\infty(\Gamma)$, where Γ is of non-measurable cardinal. This means that every countably additive zero-one measure on the power set of Γ is concentrated at some point of Γ. As explained in the second half of the proof of theorem 3 of [4], Lemma 1 implies that T_+ is induced by a mapping $\sigma : \Gamma \to \Gamma$, in the sense that $T_+ = \sigma^*$. This map is onto because T_+ is injective. To see that σ is also injective, take $\gamma \in \Gamma$ and observe that

$$T_+(1_\gamma) = 1_\gamma \circ \sigma = 1_{\sigma^{-1}(\gamma)}.$$

By the comments made after Corollary 1 there is $U \in \operatorname{Aut} \ell_\infty(\Gamma)$ such that $T_+(1_\gamma) = U(1_\gamma)$. Hence $\sigma^{-1}(\gamma)$ has only one point and so σ is bijective and T_+ is an automorphism.

We pass to the nonatomic case. It is well-known that $L_\infty(\mu)$ algebras built from nonatomic Borel measures on compact metric spaces are all isomorphic. Thus there is no loss of generality in assuming that μ is Haar measure on the Cantor group $\Delta = \{-1, 1\}^{\mathbb{N}}$. It is convenient to regard the points of Δ as functions $x : \mathbb{N} \to \{-1, 1\}$. As a preparation, consider the function $\kappa : \Delta \to [0, 1]$ given by

$$\kappa(x) = \sum_{n=1}^{\infty} \frac{1 - x(n)}{2^{n+1}}.$$

This is clearly a continuous injective function vanishing at the unit of Δ. Hence $\kappa_y(x) = \kappa(x \cdot y^{-1})$ defines a continuous injection $\Delta \to [0, 1]$ vanishing only at y. It follows from the Stone-Weierstrass theorem [11, Theorems 7.31 and 7.34] that, given $y \in \Delta$, the polynomials in κ_y with coefficients in $\mathbb{K}$ are uniformly dense in $C(\Delta)$ and therefore they are weak* dense in $L_\infty(\Delta)$ (here, we consider $L_\infty(\Delta)$ as the conjugate space of $L_1(\Delta)$).

Now let T be a local isometry of $\mathcal{A}$ and let T_+ be the endomorphism of $L_\infty(\Delta)$ given by Proposition 1. By Lemma 1 and [4, Lemma 2] T_+ is weak* continuous. Next, observe that $\mathcal{A} \cap C(\Delta)$ is a maximal ideal of $C(\Delta)$ and so

$$\mathcal{A} \cap C(\Delta) = C_0(\Delta \| y)$$

for some $y \in \Delta$. Hence $\kappa_y \in \mathcal{A}$ and since $\kappa_y \geq 0$ there is $U \in \operatorname{Aut} L_\infty(\Delta)$ such that $T_+(\kappa_y) = U(\kappa_y)$. If follows that T_+ and U agree at every polynomial in κ_y and since both T_+ and U are weak* continuous we conclude that $T_+ = U$. □

Although all the results in this paper are independent on the ground field there is some evidence that the real-valued case is harder to deal with. Let us mention the following implication: if $\operatorname{Aut} C_0^{\mathbb{R}}(L)$ is algebraically reflexive, then so is $\operatorname{Aut} C_0^{\mathbb{C}}(L)$. Indeed, suppose T is a local automorphism of $C_0^{\mathbb{C}}(L)$. Since each $f \in C_0^{\mathbb{C}}(L)$ can be written as $f = u + iv$, with $u, v \in C_0^{\mathbb{R}}(L)$ we see that if R denotes the restriction of T to $C_0^{\mathbb{R}}(L)$, then R is a local automorphism and $Tf = Ru + iRv$. Thus, if $\operatorname{Aut} C_0^{\mathbb{R}}(L)$ is algebraically reflexive then R is an automorphism and so is T. But actually R is not only a local automorphism of $C_0^{\mathbb{R}}(L)$, but even a bi-local automorphism: given $u, v \in C_0^{\mathbb{R}}(L)$ there is one automorphism S such that $Ru = Su$ and $Rv = Sv$. We shall show in Section 3 that not every local automorphism is bi-local. See the paper [9] for some interesting results on (not necessarily linear) bi-local isometries.

2. Manifolds

In this Section we study local isometries of $C_0(L)$ when L is a manifold. In view of (♯) we may consider only real functions, but our reasonings also work in the complex case. Let us recall that a manifold (respectively, a manifold with boundary) is a Hausdorff topological space with countable base in which every point has a neighborhood homeomorphic to an open set of $\mathbb{R}^n$ (respectively, of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$) for some fixed n, called the dimension of L.

The following Lemma is a slight improvement of [17, Lemma 2.7] with a different proof.

Lemma 2. *Let T be a local isometry of $C_0(L)$, where L is a locally compact (possibly compact) space whose one-point compactification is metrizable. Then there is a closed subset S of L, a homeomorphism $\psi : S \to L$ and a continuous unitary $u : S \to \mathbb{K}$ such that*

$$Tf(s) = u(s) f(\psi(s)) \qquad (f \in C_0(L), s \in S). \tag{4}$$

Proof. Since T preserves the norm of $C_0(L)$, its adjoint T^* maps the closed unit ball of $C_0(L)^*$ onto itself. The Kreĭn-Milman theorem implies that for each $x \in L$ there exists $s \in L$ and a number u of modulus one such that $T^*(u\delta_s) = \delta_x$. Let S be the set of those $s \in L$ such that $T^*\delta_s$ is an extreme point of the unit dual ball. Then we have a partial representation (in fact, a rewording of (4))

$$T^*\delta_s = u(s)\delta_{\psi(s)} \qquad (s \in S), \tag{5}$$

with u continuous on S and $\psi : S \to L$ continuous and onto. Now, the metrizability of αL goes at work: first, note that $T^*\delta_y$ is non zero for all $y \in L$. Indeed, if $f \in C_0(L)$ is a nonvanishing function, then so is Tf and we have

$$\langle T^*\delta_y, f\rangle = Tf(y) \neq 0.$$

We prove that S is closed in L. Suppose t is a cluster point of S in L. Then we have

$$T^*\delta_t = \text{weak*-}\lim_{s\to t} T^*\delta_s = \text{weak*-}\lim_{s\to t} u(s)\delta_{\psi(s)} \qquad (s \in S).$$

Since $T^*\delta_t \neq 0$ we see that the above limit must be an extreme point of the dual ball. Hence t belongs to S. That ψ is injective is proved as in Corollary 1 and so, if L is compact we have done.

If L is not compact, then neither S is and therefore no compact subset of L contains it. Hence the infinity is a cluster point of S in αL and we can extend ψ to a continuous bijection $\alpha S \to \alpha L$ leaving the infinity fixed.

Thus in any case the map ψ appearing in (5) is a homeomorphism. This completes the proof. □

As remarked in [17], an easy consequence of Lemma 2 is that if $C_0(L)$ contains a nonnegative injective function, then it is algebraically reflexive: indeed, if f is such a function, then $|Tf|$ is also injective and (4) implies that $L\backslash S$ is empty. This is the case if, for instance, αL is first countable and scattered, since (being in fact a countable ordinal) it is homeomorphic to some subset of the real line (the homeomorphism mapping the infinity to zero). Nevertheless compact scattered spaces are totally disconnected and they are covered by Corollary 1. A more inviting consequence is:

Corollary 3. *Let L be a topological manifold without boundary. Then the isometry group of $C_0(L)$ is algebraically reflexive.*

Proof. The key point is an standard result in algebraic topology known as the invariance of domain: if M and N are manifolds without boundary of the same dimension, then every injective continuous mapping $M \to N$ is automatically open (see, e.g., [7, Proposition 7.4]).

We show that the set S appearing in Lemma 2 must be the whole of L.

Since S is homeomorphic with L, the invariance of domain applies to the inclusion map $S \to L$ and so S is open in L. But according to the Lemma it is also closed. Hence, if L is connected we have the equality $S = L$, as required.

If L has finitely many components, it is clear that $S = L$ (since S has the same number of components).

Finally, if there are infinitely many components, we can write (recall that manifolds are assumed to have countable bases)

$$L = \bigoplus_n L_n,$$

where L_n are clopen subsets of L. Define $f : L \to \mathbb{R}$ as $f(x) = 1/n$ for $x \in L_n$. Then from (5) we see that $|Tf|$ attains all its values at points of S, which gives no room for points of $L\backslash S$, that must be empty. This completes the proof. □

Thus, to get a non-surjective local isometry we should consider a topological space homeomorphic to some of its closed proper sets. Looking for connected spaces with this property, the first ones that come to mind are manifolds with boundary (e.g., closed balls). We have, however, the following.

Corollary 4. *Let L be a manifold with boundary. Then the isometry group of $C_0(L)$ is algebraically reflexive.*

Proof. Let ∂L denote the boundary of L. The complement $L\backslash\partial L$ is often called the interior of L. Both $L\backslash\partial L$ and ∂L are manifolds without boundary, with $\dim \partial L = \dim L - 1$. By Tiezte's extension theorem we have an exact sequence

$$0 \longrightarrow C_0(L\|\partial L) \longrightarrow C_0(L) \xrightarrow{R} C_0(\partial L) \longrightarrow 0,$$

where R is the restriction map and $C_0(L\|\partial L) = \ker R$.

Now let T be a local isometry of $C_0(L)$. Every homeomorphism of a manifold with boundary preserves the boundary and the interior, hence T maps $C_0(L\|\partial L)$ into itself and we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(L\|\partial L) & \longrightarrow & C_0(L) & \xrightarrow{R} & C_0(\partial L) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T_2} & & \\
0 & \longrightarrow & C_0(L\|\partial L) & \longrightarrow & C_0(L) & \xrightarrow{R} & C_0(\partial L) & \longrightarrow & 0
\end{array}$$

where T_1 is the restriction of T to $C_0(L\|\partial L)$ and T_2 is given by $T_2Rf = RTf$. The operators T_1 and T_2 are local isometries: this is evident for T_1; as for T_2, let $g \in C_0(\partial L)$ and take $f \in C_0(L)$ such that $g = Rf$. Now, if T is given by $u(f \circ \varphi)$ at f, then $T_2g = (Ru) \cdot (g \circ (R\varphi))$.

By Corollary 3, T_1 and T_2 are surjective. By the five-lemma (e.g., [7, p. 8]) the middle operator T must be surjective, too. □

A consequence for unbounded functions is the following.

Corollary 5. *Let L be a (possibly noncompact) manifold with boundary and let $C'(L)$ be the Fréchet algebra of all continuous functions on L equipped with the compact-open topology. Then every continuous local automorphism of $C'(L)$ is an automorphism.*

Proof. See [8] for unexplained terms. Every manifold is realcompact and so automorphisms of $C'(L)$ are induced by homeomorphisms of L. Hence every local automorphism T of $C'(L)$ leaves $C_0(L)$ invariant. By Corollary 4 there is a homeomorphism φ of L such that

$$Tf = f \circ \varphi \qquad (f \in C_0(L)). \tag{6}$$

But $C_0(L)$ is dense in $C'(L)$ and (6) holds true for all $f \in C'(L)$. □

3. Counterexamples

In this Section we give some examples of local operators showing that the hypotheses of $(\natural)$ are really necessary. Suppose T is a local isometry (or a local automorphism) of $C_0(L)$. Must $T^*\delta_x$ be either an extreme point of the unit dual ball or zero? (The second possibility arises when L is not compact.) Of course this would imply a representation of T similar to that given in Proposition 1 and would force T to be separating: if $f \cdot g = 0$, then $Tf \cdot Tg = 0$.

We will present examples of (approximate) local isometries and automorphisms which fail to be separating. We need the notion of an isotropic Banach space.

A Banach space X is said to be isotropic if, given norm-one $x, y \in X$, there exists $T \in \operatorname{Iso} X$ such that $y = Tx$ (i.e., the isometry group acts transitively on the unit sphere). Also, X is said to be almost isotropic if, given norm-one $x, y \in X$ and $\varepsilon > 0$, there is $T \in \operatorname{Iso} X$ such that $\|y - Tx\| < \varepsilon$. See [20, 3, 1] for background on this topic. It is clear that every norm-preserving operator on an isotropic (respectively, almost isotropic) space is a local isometry (respectively, an approximate local isometry).

Very recently, Fernando Rambla [19] has shown that there are locally compact spaces L for which the complex space $C_0(L)$ is almost isotropic, thus solving a long-standing problem posed by Wood in [24]. Rambla uses the pseudoarc, a certain continuum constructed by Bing in [2], to solve the problem. We describe the pseudoarc as follows:

- A chain is a finite sequence $\mathfrak{D} = (D_1, \dots, D_n)$ of open, connected, bounded subsets of $\mathbb{R}^2$ such that $D_i \cap D_j \neq \varnothing$ if and only if $|i - j| \leq 1$. Each D_i is called a link and n is the length of $\mathfrak{D}$, written $\ell(\mathfrak{D})$. Also, we write $\delta(\mathfrak{D})$ for the maximum of the diameters of the links of $\mathfrak{D}$ and $\mathfrak{D}^*$ for the union of the links of $\mathfrak{D}$. Given $a, b \in \mathbb{R}^2$, we say that $\mathfrak{D}$ is a chain from a to b if $a \in D_i$ only for $i = 1$ and $b \in D_i$ only for $i = n$.
- Given two chains $\mathfrak{D}$ and $\mathfrak{E}$, we say that $\mathfrak{D}$ is contained in $\mathfrak{E}$ if the closure of every link of $\mathfrak{D}$ is a subset of a link of $\mathfrak{E}$. If so, $\mathfrak{D}$ is said to be crooked in $\mathfrak{E}$ if, given links E_h, E_k, with $k > h + 2$, and $D_i \subset E_h$; $D_j \subset E_k$, with $i < j$, there are $i < r < s < j$ such that D_r is contained in E_{k-1} or E_{k+1} and D_s in E_{h-1} or E_{h+1}. See the figure on page 730 of Bing's [2].
- Let $a, b \in \mathbb{R}^2$ be two different points and let $(\mathfrak{D}_n)$ be a sequence of chains from a to b, with $\mathfrak{D}_n$ contained and crooked in $\mathfrak{D}_{n+1}$ and such that $\delta(\mathfrak{D}_n) \to 0$. Then the set

$$P = \bigcap_{n=1}^{\infty} \mathfrak{D}_n^*$$

is called a pseudoarc from a to b.

It is proved in [2, Theorem 11] that all the topological spaces that follow the above construction are homeomorphic. Thus, from now on we speak of the pseudoarc instead of a pseudoarc. It is worth noting that P is homogeneous (in the sense that, given $s, t \in P$ there is a homeomorphism of P sending s to t; see [2, Theorem 13]). The property of being an endpoint of P merely reflects a peculiarity of the embedding of P into $\mathbb{R}^2$. The pseudoarc is homeomorphic to each of its nondegenerate subcontinua [15]. For our purposes, it suffices to know that if P is a pseudoarc from a to b there exists a subset $Q \subset P$ homeomorphic to P and containing a but not b. Indeed, let c be another point in P. For each n, let $\mathfrak{C}_n$ be the subchain of $\mathfrak{D}_n$ that goes from a to c and let Q be the pseudoarc from a to c constructed with the sequence $(\mathfrak{C}_n)$.

Rambla shows in [19, Corollary 4.5] that the complex space $C_0(P\|p)$ is almost isotropic for all $p \in P$. We are now ready for the counterexamples. Let us begin with the "approximate" case.

Example 1. There are

(a) approximate local isometries of $C_0^{\mathbb{C}}(P\|p)$,
(b) approximate local automorphisms of $C_0^{\mathbb{R}}(P\|p)$ and
(c) approximate local automorphisms of $C^{\mathbb{R}}(P)$

that are not separating.

Proof. Let P be the pseudoarc from a to b, with $a, b \in \mathbb{R}^2$ and let $Q \subset P$ be a subset of P containing a but not b and homeomorphic to P itself. Also, let K be a compact neighbourhood of b in P such that $K \cap Q = \varnothing$.

By the Borsuk-Dugundji theorem (as presented in [21, Theorem 21.1.4]) there exists a norm preserving extension operator $\Lambda : C(Q \oplus K) \to C(P)$ with the additional property that $\Lambda(f)$ takes values in the convex hull of the range of $f \in C(Q \oplus K)$. Set

$$\mathcal{A} = \{f \in C(Q \oplus K) : f|_K = 0, f(a) = 0\}.$$

There is an obvious isometric multiplicative isomorphism between $C_0(P\|a)$ and $\mathcal{A}$ we denote Φ. Now, let $I : C_0(P\|a) \to C_0(P\|a)$ denote the composition $\Lambda \circ \Phi$. Let h be any norm-one, non-negative function vanishing outside K and pick two points $x, y \in P\backslash\{a\}$. Finally, define $T : C_0(P\|a) \to C_0(P\|a)$ by

$$Tf = If + \frac{f(x) + f(y)}{2} \cdot h. \tag{7}$$

It is clear that T is not separating: if $f, g \in C(P\|a)$ are functions with disjoint supports such that $f(x) = g(y) = 1$, but $f(y) = g(x) = 0$, then

$$Tf \cdot Tg = \left(If + \frac{h}{2}\right) \cdot \left(Ig + \frac{h}{2}\right) = \frac{h^2}{4} \neq 0.$$

(a) That T is an approximate local isometry of $C_0^{\mathbb{C}}(P\|a)$ follows from the facts that T is norm-preserving (obvious) and that $C_0^{\mathbb{C}}(P\|a)$ is almost isotropic.

(b) We claim that Tf takes the same values as f when the ground field is $\mathbb{R}$. This is completely obvious for If in view of the connectedness of the pseudoarc and the form of the Borsuk-Dugundji theorem we have used. We only have to check that the range of the second summand in the right hand side of (7) is contained in that of f. But $(f(x) + f(y))/2 = f(z)$ for some $z \in P$ (by connectedness) and the range of $f(z) \cdot h$ lies between 0 and $f(z)$, which proves our claim.

By the main result of [19] (Theorem 4.4) this implies that, given $f \in C_0^{\mathbb{R}}(P\|a)$ and $\varepsilon > 0$, there is a homeomorphism φ of P leaving a fixed and such that

$$\|T(f) - f \circ \varphi\|_\infty < \varepsilon.$$

Hence, T is an approximate local automorphism of $C_0^{\mathbb{R}}(P\|a)$, but it fails to be separating, which proves (b).

To verify (c), simply extend the above T to all of $C^{\mathbb{R}}(P)$ sending the unit into itself. □

We now present the technique of ultraproducts that shifts approximate local isometries (or automorphisms) to local isometries (or automorphisms) on a larger space. We refer the reader to [22, 10] or [6, Chapter 8] for general information on ultraproducts. We use $\mathbb{N}$ as index set, for simplicity. Let (X_n) be a sequence of Banach spaces and consider the (Banach space) product

$$\prod X_n = \left\{(x_n) : x_n \in X_n \text{ for all } n \in \mathbb{N}, \text{ with } \sup_n \|x_n\| < \infty\right\}$$

endowed with the supremum norm. Let U be a nonprincipal ultrafilter on $\mathbb{N}$ and put $N_U = \{(x_n) : \lim_U \|x_n\| = 0\}$. The Banach space $\prod X_n/N_U$ with the quotient norm is called the ultraproduct of the family (X_n) with respect to U and it is denoted $(X_n)_U$. The class of (x_n) in $(X_n)_U$ will be denoted by $(x_n)_U$. The norm of an ultraproduct has the nice property that

$$\|(x_n)_U\| = \lim_U \|x_n\|.$$

When all X_n coincide with some Banach space X we speak of the ultrapower of X and we write X_U instead of $(X_n)_U$. There is a canonical isometric embedding $X \to X_U$ given by $x \mapsto (x)_U$.

Suppose $T_n : X_n \to Y_n$ is a uniformly bounded sequence of operators. Then we can define the ultraproduct operator

$$(T_n)_U : (X_n)_U \to (Y_n)_U$$

by the rule $(T_n)_U(x_n)_U = (T_n(x_n))_U$. Note that $\|(T_n)_U\| = \lim_U \|T_n\|$. In particular, $(T_n)_U$ is a (surjective) isometry if all T_n are, from where it follows that ultraproducts preserve local isometries. Actually we have the following stronger result:

Lemma 3. *If, for each n, T_n is an approximate local isometry of X_n, then $(T_n)_U$ is a local isometry of $(X_n)_U$.*

Proof. Fix $(x_n)_U$. Then, for every n there exists an isometry I_n of X_n such that $\|T_n(x_n) - I_n(x_n)\| < 1/n$. It is clear that $(T_n)_U(x_n)_U = (I_n)_U(x_n)_U$. □

Now, suppose X_n are Banach algebras. Then $(X_n)_U$ is also a Banach algebra under the product $(x_n)_U \cdot (y_n)_U = (x_n y_n)_U$. By general representation theorems if L_n are (locally) compact spaces, the algebra $(C_0(L_n))_U$ is representable as $C_0(L)$ for a suitable (locally) compact space. The following result is now clear:

Lemma 4. *If, for each n, T_n is an approximate local automorphism of X_n, then $(T_n)_U$ is a local automorphism of $(X_n)_U$.* □

Thus, passing to the ultrapowers of the operators of Example 1, we obtain:

Example 2. There are local isometries of $C_0^{\mathbb{C}}(L)$ and local automorphisms of $C_0^{\mathbb{R}}(L)$ and $C^{\mathbb{R}}(K)$ which are not separating. □

The above examples show that ($\natural$) fails if the ground field is $\mathbb{R}$ and/or if we allow noncompact spaces. Also, it follows that local automorphisms of $C^{\mathbb{R}}(K)$ algebras and local isometries of $C_0^{\mathbb{C}}(L)$ spaces need not be bi-local: every bi-local isometry of $C_0^{\mathbb{K}}(L)$ is separating.

4. Appendix

We close with a correction of the proof of the following result.

Theorem 1 (Jarosz-Rao). *Let L be a locally compact space whose one-point compactification is metrizable. Then* Iso $C_0^{\mathbb{C}}(L)$ *is algebraically reflexive.*

Of course, this would be a specialization of [14, Theorem 4]. The argument given there uses the statement labeled as Theorem 3. Unfortunately, not every isometry of a finite-codimensional subspace $\mathcal{A}$ of $C(K)$ can be extended to the whole space: consider the isometry of $c_0 = C_0(\mathbb{N})$ given by

$$(Tf)(n) = (-1)^n f(n).$$

It is impossible to extend T to an isometry of $c = C(\alpha\mathbb{N})$ since $(-1)^n$ does not converge as $n \to \infty$. It is unclear to me whether the whole statement of Theorem 4 in [14] remains true or not.

Proof. Let us regard $C_0^{\mathbb{C}}(L)$ as a maximal ideal of $C^{\mathbb{C}}(\alpha L)$. Assume T is a local isometry of $C_0^{\mathbb{C}}(L)$ and, for each $x \in L$, put

$$\mathcal{A}_x = \ker T^*\delta_x = \{f \in C_0^{\mathbb{C}}(L) : (Tf)(x) = 0\}.$$

Note that $\mathcal{A}_x$ is a hyperplane of $C_0^{\mathbb{C}}(L)$ since $T^*(\delta_x) \neq 0$ as $C^{\mathbb{C}}(\alpha L)$ contains functions vanishing only at ∞.

By the form of the isometries of $C_0^{\mathbb{C}}(L)$, we see that each function $f \in \mathcal{A}_x$ must vanish at least at two different points of αL (one of them is ∞) possibly depending on f. By a result of Jarosz ([12, Theorem 1] or [14, Theorem 1]), there is $y \in \alpha L$ such that

$$\mathcal{A}_x \subset \{f \in C(\alpha L) : f(y) = f(\infty) = 0\}. \tag{8}$$

But $\mathcal{A}_x$ has codimension one in $C_0^{\mathbb{C}}(L)$, hence y lies in L and we have the equality in (8). So, $\mathcal{A}_x = C_0^{\mathbb{C}}(L\|y)$ and $\ker T^*(\delta_x) = \ker \delta_y$, from where it follows that

$$T^*(\delta_x) = u(x)\delta_{\varphi(x)} \qquad (x \in L),$$

where $u : L \to \mathbb{C}$ and $\varphi : L \to L$ are continuous and we have written $y = \varphi(x)$. Thus, we have

$$Tf = u \cdot (f \circ \varphi) \qquad (f \in C_0^{\mathbb{C}}(L)).$$

The rest is straightforward. The above representation forces u to be unitary and φ onto. That φ is also injective can be verified as in the Proof of Corollary 1. □

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Note added in proof

Jarosz and Rao gave a new, correct proof of Theorem 4 in [14]. This and much more is available at Jarosz's homepage http://www.siue.edu/~kjarosz