Extending operators into $\mathcal{L}_\infty$ spaces under a twisted light

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Abstract

We present a homological approach to the problem of extending operators which take values in $C(K)$ or $\mathcal{L}_\infty$ spaces. In this way we obtain unified simpler proofs of results of Lindenstrauss-Pełczyński and Johnson-Zippin.

Introduction and preliminaries

The classical problem of extension of linear operators is: when can a given operator $Y \longrightarrow B$ from a subspace Y of a Banach space X be extended to an operator $X \longrightarrow B$? Equivalently, when is the restriction operator $\mathfrak{L}(X, B) \longrightarrow \mathfrak{L}(Y, B)$ surjective?

Linear continuous operators cannot, as a rule, be extended; but it is possible in many interesting situations. For instance, when either **i)** Y is complemented in X; or **ii)** B is injective; or **iii)** the operator is 2-summing (regardless of the spaces involved; see [10, Theorem 4.15]). We see that there are three variables involved here: the pair $Y \longrightarrow X$, the space B, and the class of operators $\mathfrak{A}$ we want to extend; and, accordingly, there are three different extension problems

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depending on which two of these three elements are chosen as the data.

Our main concern will be generalizations of **ii)** due to Lindenstrauss-Pełczyński and Johnson-Zippin. The fact that an injective Banach space has to be an $\mathcal{L}_\infty$-space attracts one's attention to $\mathcal{L}_\infty$-spaces. The fact that the identity operator $c_0 \longrightarrow c_0$ cannot be extended to ℓ_∞ sets the limits of what can be done: in general, even $C(K)$-valued operators cannot be extended.

The problem admits an algebraic formulation and treatment, and this is precisely the point of view and the language we adopt in this paper. For general information about $\mathcal{L}_\infty$-spaces, see [10, Chapter 3] or [2]. In section 3 we assume from the reader familiarity with the basic theory of exact sequences and extensions; in section 4, only, we moreover assume familiarity with the theory of quasi and zero-linear mappings. All this background can be found in [1] or [8]. Here is a brief description.

A short exact sequence (also called an extension of Z by Y) is a diagram

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0 \qquad (1)$$

composed of Banach spaces and operators in which the kernel of each arrow coincides with the image of the preceding one. Two exact sequences $0 \longrightarrow Y_i \longrightarrow X_i \longrightarrow Z_i \longrightarrow 0 \quad (i = 1, 2)$ are said to be **isomorphically equivalent** if there exist isomorphisms α, β, γ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & X_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \end{array}$$

commutative. This notion was introduced in [5,9] and clearly generalizes the older notion of equivalent sequences (where it is assumed that $Y_2 = Y_1$, $Z_2 = Z_1$ and α and γ are the identity). An exact sequence (1) is said to split if it is equivalent to the trivial sequence $0 \longrightarrow Y \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow 0$; a common shorthand for this is $\mathrm{Ext}(Z, Y) = 0$.

There is a correspondence between exact sequences of Banach spaces (1) and the so-called **zero-linear** maps, which are homogeneous maps $F : Z \longrightarrow Y$ satisfying an estimate $\|F(\sum_{i=1}^n x_i) - \sum_{i=1}^n F(x_i)\| \leq \Delta_F \sum \|x_i\|$ for some constant Δ_F, each $n \in \mathbb{N}$ and all $x_i \in Z$.

With these tools let us return to the extension problem and consider it in full generality, replacing $\mathfrak{L}$ by other operator ideals. Since it is well known that given a surjective operator ideal $\mathfrak{A}$ (see [8, §2.5]), an exact sequence (1) and a fixed Banach space B the induced sequence

$$0 \longrightarrow \mathfrak{A}(Z, B) \xrightarrow{\pi^*} \mathfrak{A}(X, B) \xrightarrow{\imath^*} \mathfrak{A}(Y, B) \longrightarrow 0 \qquad (2)$$

is exact **up to** $\mathfrak{A}(X, B)$, the extension problem consists in deciding whether the whole diagram (2) is exact. If so, we shall say that the functor $\mathfrak{A}(\cdot, B)$ is exact at (1). A functor is called exact (see [8]) when it is exact at every exact sequence. For instance, the functor $\mathfrak{L}(\cdot, B)$ is exact if and only if the space B is injective.

A brief reminder of extension results

When $\mathfrak{A} = \mathfrak{K}$, the ideal of compact operators, an affirmative answer to the extension problem characterizes $\mathcal{L}_\infty$-spaces. Precisely:

Proposition 1 (Lindenstrauss [15]) *A Banach space B is an $\mathcal{L}_\infty$ space if and only if the functor $\mathfrak{K}(\cdot, B)$ is exact.*

Proof Let B be an $\mathcal{L}_\infty$-space and let $T : Y \longrightarrow B$ be a finite rank operator. Since $T(Y)$ is λ-isomorphic to a subspace of some ℓ_∞^n-space, the operator T admits a finite rank extension $\tau : X \longrightarrow B$ with $\|\tau\| \leq \lambda\|T\|$. But $\mathcal{L}_\infty$ spaces have the (bounded) approximation property and so the finite rank operators are dense in $\mathfrak{K}(Y, B)$. Thus, the restriction $\mathfrak{K}(X, B) \longrightarrow \mathfrak{K}(Y, B)$ is almost open and therefore (open and) surjective.

To prove the converse just observe that the canonical inclusion $\delta : B \longrightarrow B^{**}$ extends to any superspace X by: $\Delta(x)(b^*) = \lim_{\mathfrak{U}}(\Delta_E x)(b^*)$, where E denotes a typical finite dimensional subspace of B, $\mathfrak{U}$ is an ultrafilter refining the Fréchet filter on the directed set of all such subspaces, and Δ_E is an extension (with norm at most λ) to X of the restriction of δ to E. It follows easily that B^{**} is λ-complemented in X^{**}, and by a well known characterization of $\mathcal{L}_\infty$-spaces, that is all that we need. $\square$

The choice $\mathfrak{A} = \mathfrak{W}$, the ideal of weakly compact operators, was first published by Bourgain and Delbaen [3, Theorem 2.4]. Their real contribution was to give examples of spaces showing that the following result is not vacuous.

Proposition 2 *A Banach space B is an $\mathcal{L}_\infty$ space with the Schur property if and only if the functor $\mathfrak{W}(\cdot, B)$ is exact.*

The proof is as before taking into account that weakly compact operators into a Schur space are compact (only if); to get the if part, take a weakly null sequence in B. The sequence is contained in the image of some reflexive space; when extended to all the $C(K)$-space this weakly compact operator should be completely continuous.

Straeuli considered in [19, 3.1] the extension problem for different operator ideals. We quote:

Proposition 3 *Let $\mathfrak{A}$ be an idempotent operator ideal with $\mathfrak{K} \subset \mathfrak{A} \subset \mathfrak{R}$, where $\mathfrak{R}$ denotes the Rosenthal operators. Then the functor $\mathfrak{A}(\cdot, B)$ is exact if and* *only if B is an $\mathcal{L}_\infty$-space such that $\mathfrak{A}(\cdot, B) = \mathfrak{K}(\cdot, B)$*

The extension problem for the choices $\mathfrak{A} = \mathfrak{K}$ or $\mathfrak{A} = \mathfrak{W}$ but fixing the exact sequence was solved by Fakhoury [11, Théorème 3.1]:

Proposition 4 *The functors $\mathfrak{K}(\cdot, B)$ —or $\mathfrak{W}(\cdot, B)$— are exact at (1) if and only if the dual sequence $0 \longrightarrow Z^* \longrightarrow X^* \longrightarrow Y^* \longrightarrow 0$ splits.*

One then says that (1) **locally splits** or that Y is **locally complemented** in X. (Other authors use different terminologies.)

Returning to the general problem, there are two distinguished situations in which $C(K)$-valued operators extend: the Lindenstrauss-Pełczyński theorem [16] for subspaces of c_0 and the Johnson-Zippin theorem [12] for weakly* closed subspaces of ℓ_1.

Proposition 5 (Lindenstrauss-Pełczyński) *Every $C(K)$-valued operator defined on a subspace of c_0 can be extended to all of c_0.*

Their proof is of Hahn-Banach style, showing that if $H \subset c_0$, $T : H \longrightarrow C(K)$ is an operator, $\varepsilon > 0$ and $p \notin H$, then there exists an extension $T_p : H + [p] \longrightarrow C(K)$ with norm $\|T_p\| \leq \|T\| + \varepsilon$. Therefore, an operator $T : H \longrightarrow C(K)$ defined on a subspace H of c_0 can, for every $\varepsilon > 0$, be extended to an operator $\tau : c_0 \longrightarrow C(K)$ with $\|\tau\| \leq (1+\varepsilon)\|T\|$. An equal norm extension (i.e., $\varepsilon = 0$) cannot be achieved as an example of Johnson and Zippin [13] shows. In this same paper Johnson and Zippin extend the validity of the extension result to subspaces of $c_0(\Gamma)$ using a different approach, which we will discuss now, that allows one to circumvent the ε in the original proof.

Given an inclusion $\imath : Y \longrightarrow X$, Lindenstrauss [15] approached the extension problem for $C(K)$-valued operators in different ways, one of them being the existence of a certain selection for the dual quotient map $\imath^*$. This was precisely formulated by Zippin in [20]. There, Y is said to be **λ-almost complemented** in X if each operator $T : Y \longrightarrow C(K)$ extends to X with norm at most $\lambda\|T\|$:

Lemma 1 *Let $\imath : Y \longrightarrow X$ be an isomorphic embedding. Then Y is λ-almost complemented in X if and only if there exists a weakly* continuous function $\omega : \mathrm{Ball}\, Y^* \longrightarrow \lambda\, \mathrm{Ball}\, X^*$ such that $\imath^*\omega = \mathrm{id}$. (We shall call such a function a selector for $\imath^*$.)*

Proof As usual, we identify K with the carrier space of $C(K)$. If a weakly* continuous selector $\omega : \mathrm{Ball}\, Y^* \longrightarrow \lambda\, \mathrm{Ball}\, X^*$ exists for $\imath^*$ then $\tau(x)(k) = \omega(T^*k)(x)$ defines an operator $\tau : X \longrightarrow C(K)$ with $\|\tau\| \leq \lambda\|T\|$ and such that $\tau(\imath(y))(k) = \omega(T^*k)(\imath y) = \imath^*\omega(T^*k)(y) = T^*k(y) = Ty(k)$.

Conversely, if every $C(K)$-valued operator admits such an extension then consider the canonical operator $\delta : Y \longrightarrow C(\mathrm{Ball}\, Y^*, \mathrm{weak}^*)$ and let D be its extension to X with norm at most λ. The weakly* continuous selector for $\imath^*$ is $\omega(y^*)(x) = D(x)(y^*)$, since $\omega(y^*)(\imath(y)) = D(\imath(y))(y^*) = y^*(y)$. $\square$

Let us say that (1) λ-almost splits if Y is λ-almost complemented in X through $\imath$. Observe that the notions of almost split and locally split sequences are independent. On one hand, the sequence

$$0 \longrightarrow X \xrightarrow{\ \delta\ } C(\operatorname{Ball} X^*) \longrightarrow C(\operatorname{Ball} X^*)/\delta(X) \longrightarrow 0$$

almost splits, as follows from the existence of the weakly* continuous selector $\operatorname{Ball} X^* \longrightarrow \operatorname{Ball} C(\operatorname{Ball} X^*)^*$ for δ^* given by $D(x^*)(f) = f(x^*)$. Since $\|D(x^*)\| = \|x^*\|$ the sequence 1-almost splits. This sequence locally splits if and only if X is an $\mathcal{L}_\infty$-space since in that case X^{**} would be complemented in $C(\operatorname{Ball} X^*)^{**}$. On the other hand, the sequence $0 \longrightarrow X \longrightarrow X^{**} \longrightarrow H(X) \longrightarrow 0$ locally splits, while it does not almost split when X is a non-injective $C(K)$-space.

With the aid of this approach Zippin obtained in [21,22] different proofs for the Lindenstrauss-Pełczyński theorem. At the end of the paper [21] Zippin poses three questions, one of which connects with the nature of the Ext functor: When does an exact sequence $0 \longrightarrow E \longrightarrow \ell_1 \longrightarrow Z \longrightarrow 0$ almost split? By, e.g. [7, Theorem 1.1], this is the same as asking: When is $\operatorname{Ext}(Z, C(K)) = 0$? We will use this equivalence several times in the sequel. A substantial answer to this problem was obtained by Johnson and Zippin in [12]. (A partial converse to this in [14] shows that the hypothesis on the subspace cannot be weakened much.)

Proposition 6 (Johnson-Zippin) *Every $\mathcal{L}_\infty$-valued operator defined on a $\sigma(\ell_1, c_0)$-closed subspace of ℓ_1 can be extended to ℓ_1.*

The proof of Johnson and Zippin is rather technical and long and at first sight seems to have no common point with that of Lindenstrauss and Pełczyński. Nevertheless, the remainder of this paper will show that the two results share a common homological nature. The basic idea of both our proofs is the same: each result will first be established under the additional assumption of a finite dimensional decomposition, and then we appeal to the result that any separable Banach space X admits a short exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ with Y and Z both having FDDs. Of course the details differ a bit.

The Lindenstrauss-Pełczyński theorem

We reformulate Proposition 5.

Theorem 1 *Every exact sequence $0 \longrightarrow H \longrightarrow c_0 \longrightarrow c_0/H \longrightarrow 0$ almost splits.*

Proof The first ingredient of our proof is a decomposition result of Johnson, Rosenthal and Zippin (see [18, Theorems 1.g.2 and 2.d.1]) asserting that every subspace H of c_0 admits an exact sequence $0 \longrightarrow c_0(A_n) \longrightarrow H \longrightarrow c_0(B_n) \longrightarrow 0$ in which A_n and B_n are finite dimensional spaces. Crossing the new sequence with the starting one and completing the diagram we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & c_0(A_n) \arrow[r, equal] \arrow[d] & c_0(A_n) \arrow[d] & & \\
0 \arrow[r] & H \arrow[r] \arrow[d] & c_0 \arrow[r] \arrow[d] & c_0/H \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0(B_n) \arrow[r, "w"'] \arrow[d] & c_0/c_0(A_n) \arrow[r] \arrow[d] & c_0/H \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

To control the vertical sequence centered at c_0 we use a theorem of Lindenstrauss and Rosenthal [17] which implies that any such sequence is isomorphically equivalent to a sequence

$$0 \longrightarrow c_0(A_n) \xrightarrow{\ \kappa\ } c_0(\ell_\infty^{m(n)}) \longrightarrow c_0(\ell_\infty^{m(n)}/A_n) \longrightarrow 0,$$

where κ is defined by $\kappa((a_n)_n) = (\jmath_n(a_n))_n$, for suitable $(1+\varepsilon)$-into isomorphisms $\jmath_n : A_n \longrightarrow \ell_\infty^{m(n)}$. The Bartle-Graves selection principle (we use the stronger version appearing in [1, Proposition 1.19 (ii)]) yields a continuous selector $\omega_n : \operatorname{Ball} A_n^* \longrightarrow (1+\varepsilon)\operatorname{Ball}\ell_1^{m(n)}$ for $\jmath_n^*$. Putting together all these maps, we obtain a selector $\omega : \operatorname{Ball}(\ell_1(A_n^*)) \longrightarrow (1+\varepsilon)\operatorname{Ball}(\ell_1(\ell_1^{m(n)}))$ for κ^* defined by $\omega((a_n^*)_n) = (\omega_n(a_n^*))_n$. This selector turns out to be weakly* continuous since its domain is metrizable and a bounded sequence in a space $\ell_1(F_n)$ with F_n finite-dimensional is weakly* null if and only if the norms of its projections into F_n are convergent to 0. In conclusion, every exact sequence $0 \longrightarrow c_0(A_n) \longrightarrow c_0 \longrightarrow Q \longrightarrow 0$ almost-splits.

We just finish the proof with a kind of 3-space property for the extension of operators. (Actually, one should be a bit more careful because the Lindenstrauss-Rosenthal result used above only gives an isomorphically equivalent sequence; we leave the reader to worry about these details.)

Proposition 7 *If the functor $\mathfrak{L}(\cdot,\spadesuit)$ is exact at the sequences $0 \longrightarrow A \longrightarrow X \longrightarrow X/A \longrightarrow 0$ and $0 \longrightarrow Y/A \longrightarrow X/A \longrightarrow X/Y \longrightarrow 0$ then it is also exact at $0 \longrightarrow Y \longrightarrow X \longrightarrow X/Y \longrightarrow 0$.*

Proof This can be seen by drawing homology sequences but this time it is easier to go back to basics. So let $T \in \mathfrak{L}(Y, \spadesuit)$ be any operator. By our first assumption, $T|_A$ admits an extension $S : X \longrightarrow \spadesuit$. Since $S - T$ vanishes on A, its restriction to Y factorizes as RQ where $Q : Y \longrightarrow Y/A$ is the quotient map and $R \in \mathfrak{L}(Y/A, \spadesuit)$. Our second assumption guarantees that R has an extension (for which we use the same symbol) in $\mathfrak{L}(X/A, \spadesuit)$. Then $x \mapsto Sx - R(x + A)$ is the required extension of T. $\square$

Our homological approach has some advantages and, of course, some disadvantages. The worst of these is the absence of a good estimate for the norm of the extension: we obtain $(1 + \varepsilon)$-for subspaces of c_0 of the form $c_0(A_n)$ but the estimate for general subspaces blows up during the last part of the proof. This is precisely the benefit of Zippin's careful work [22]: he obtains $1 + \varepsilon$. The advantages, on the other hand, are clear: our proof is clean and, as we shall see, by duality it provides a proof for the Johnson-Zippin theorem. We close this section with an example showing that the hypotheses of the Lindenstrauss-Pełczyński theorem are optimal, in the sense that $C(K)$-spaces cannot be replaced by arbitrary $\mathcal{L}_\infty$-spaces. Since we will refer to the following construction of Bourgain and Pisier [4] later on, we quote it at this stage.

Proposition 8 *Any separable Banach space Y can be embedded in a separable $\mathcal{L}_\infty$ space X such that X/Y has the Schur property.*

Example Let $0 \longrightarrow H \longrightarrow c_0 \longrightarrow Q \longrightarrow 0$ be an exact sequence with $Q \neq c_0$. Then there is an operator from H into an $\mathcal{L}_\infty$-space that cannot be extended to c_0.

Proof Let $0 \longrightarrow H \longrightarrow \mathcal{L}_\infty(H) \longrightarrow S \longrightarrow 0$ be the corresponding Bourgain-Pisier sequence. Pushing out and completing the diagram [8] we obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H \arrow[r] \arrow[d] & c_0 \arrow[r] \arrow[d] & c_0/H \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}_\infty(H) \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r] \arrow[d] & c_0/H \arrow[r] & 0 \\
 & S \arrow[r, equal] \arrow[d] & \mathrm{PO}/c_0 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Now, if the inclusion $H \longrightarrow \mathcal{L}_\infty(H)$ extends to c_0 then the lower row splits and $\mathrm{PO} = \mathcal{L}_\infty(H) \oplus c_0/H$. However, the middle vertical sequence certainly splits by Sobczyk's theorem, so that $\mathrm{PO} = c_0 \oplus S$. Therefore $\mathcal{L}_\infty(H) \oplus c_0/H = c_0 \oplus S$ thus making c_0/H a complemented subspace of $c_0 \oplus S$. Since S is Schur, it is totally incomparable with c_0 and thus by the Edelstein-Wojtasczyk theorem (see [18, Theorem 2.c.13]) c_0/H decomposes as $A \oplus B$ where A is a complemented subspace of c_0 and B a complemented subspace of S. Since c_0/H is a subspace of c_0 then B has to be finite dimensional, and therefore $c_0/H = c_0$, which we assumed not to happen. $\square$

The Johnson-Zippin theorem

We reformulate Proposition 6 via almost splitting sequences; of course the extension result is also valid for $\mathcal{L}_\infty$-valued operators and not only for $C(K)$-valued ones.

Theorem 2 *Let H be a subspace of c_0. Every exact sequence $0 \longrightarrow H^\perp \longrightarrow \ell_1 \longrightarrow H^* \longrightarrow 0$ almost splits.*

Proof First of all, a fundamental difference from other extension results is our access to the tool which we mentioned earlier: a sequence $0 \longrightarrow W \longrightarrow \ell_1 \longrightarrow X \longrightarrow 0$ almost splits if and only if $\mathrm{Ext}(X, C(K)) = 0$ for all $C(K)$-spaces. Thus the result asserts that whenever H is a subspace of c_0 then $\mathrm{Ext}(H^*, C(K)) = 0$ for all $C(K)$-spaces.

Now, entering into the proof, since every subspace H of c_0 admits a decomposition $0 \longrightarrow c_0(A_n) \longrightarrow H \longrightarrow c_0(B_n) \longrightarrow 0$, every dual H^* of a subspace of c_0 admits a decomposition $0 \longrightarrow \ell_1(B_n^*) \longrightarrow H^* \longrightarrow \ell_1(A_n^*) \longrightarrow 0$. The property $\mathrm{Ext}(\cdot, \spadesuit) = 0$ is a 3-space property (e.g. [7, Corollary 1.2]), so it is enough to prove that $\mathrm{Ext}(\ell_1(F_n), \mathcal{L}_\infty) = 0$.

Every exact sequence $0 \longrightarrow \mathcal{L}_\infty \longrightarrow X \longrightarrow Z \longrightarrow 0$ locally splits, so there is a constant C such that every exact sequence $0 \longrightarrow \mathcal{L}_\infty \longrightarrow X_F \longrightarrow F \longrightarrow 0$ with F finite dimensional splits and there exists a projection $X_F \longrightarrow \mathcal{L}_\infty$ with norm at most C (see [6]); i.e., $\mathrm{Ext}(F, \mathcal{L}_\infty) = 0$ uniformly on F. So, $\mathrm{Ext}(\ell_1(F_n), \mathcal{L}_\infty)) = \prod \mathrm{Ext}(F_n, \mathcal{L}_\infty) = 0$. $\square$

An estimate of the norm of the extension can be obtained through a non-linear argument:

Lemma 2 *Let $0 \longrightarrow Y \longrightarrow X \xrightarrow{\pi} F \longrightarrow 0$ be an exact sequence with F finite-dimensional. For every $\varepsilon > 0$ there exists a homogeneous lifting $\varrho : F \longrightarrow X$ for π with finite-dimensional range, and $\|\varrho(x)\| \le (1+\varepsilon)\|x\|$ for all $x \in F$. If furthermore Y is an $\mathcal{L}_{\infty,\lambda}$ space then there is a projection $P : X \longrightarrow Y$ with norm $\|P\| \le \lambda + 2 + \varepsilon$.*

Proof Let A be a finite (symmetric) subset of the unit sphere of F whose convex hull contains the ball of radius $1 - \varepsilon$. For each $a \in A$ choose a point

$\varrho(a) \in a+Y$ with $\|\varrho(a)\| \leq 1+\varepsilon$, and put $\varrho(0)=0$. Any $x \in F$ with norm $1-\varepsilon$ is a finite convex combination of elements of A; define $\varrho(x)$ as the corresponding convex combination from $\varrho(A)$. Extending ϱ to all of F by homogeneity gives us a selection for the quotient mapping with finite dimensional range. It also satisfies the inequality $\|\varrho(x)\| \leq (1+\varepsilon)(1-\varepsilon)^{-1}\|x\|$ for all $x \in F$, which is near enough.

Now suppose that Y is an $\mathcal{L}_{\infty,\lambda}$ space. Let $\ell : F \longrightarrow X$ be any linear lifting for the quotient mapping, and define $\Omega : F \longrightarrow Y$ by $\Omega(z) = \ell(z) - \varrho(z)$. This is a zero-linear map with constant $\Delta_\Omega \leq 1+\varepsilon$. Since $\Omega(F)$ is finite dimensional, it is contained in some finite-dimensional subspace E of Y which is within Banach-Mazur distance λ of a finite dimensional ℓ_∞^n space. Now we apply the so-called nonlinear Hahn-Banach Theorem [5, Lemma 1] coordinatewise to obtain a linear mapping $L : F \longrightarrow Y$ satisfying the inequality $\|L(z) - \Omega(z)\| \leq \lambda\|z\|$.

The operator $(\ell - L)\circ\pi : X \longrightarrow X$ is a projection with kernel Y (one has $((\ell - L)\circ\pi)^2 = (\ell - L)\circ\pi$ since $\pi\circ L = 0$ and $\pi\circ\ell = \mathrm{id}$) and verifies, for any x,

$$\|(\ell - L)\pi x\| \leq \|\ell\pi x - \Omega\pi x\| + \|\Omega\pi x - L\pi x\| \leq (1+\varepsilon+\lambda)\|\pi x\|.$$

Hence $\mathrm{id} - (\ell - L)\pi$ is the desired projection onto Y with norm $\leq 2+\lambda+\varepsilon$.$\square$

Since $C(K)$-spaces are $\mathcal{L}_{\infty,\lambda}$-spaces for every $\lambda > 1$, the decomposition technique of the previous section gives us an estimate of $3+\varepsilon$ for the norm of a projection in an exact sequence $0 \longrightarrow C(K) \longrightarrow \ell_1(F_n) \longrightarrow H^* \longrightarrow 0$ when H is a subspace of $c_0(F_n^*)$. This is the same estimate that Johnson and Zippin obtain in [12], although our proof is considerably simpler. However, under the additional assumption that E has the approximation property, they were able to obtain the stronger conclusion that the extension operator $\tilde{T}$ could be chosen so that $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$. They ask if this estimate could be true without the approximation property; this remains unknown.

Further results and open problems

Let's begin this section with another especially interesting example of almost splitting obtained by Zippin in [20,21]:

Proposition 9 *Every sequence* $0 \longrightarrow W \xrightarrow{\imath} \ell_p \longrightarrow \ell_p/W \longrightarrow 0$ *(with* $1 < p < \infty$*) almost-splits.*

Since the proof does not explicitly appear in the literature, let us present here a sketch: the dual space $\ell_p^* = \ell_{p^*}$ is strictly convex, so there exists a unique Hahn-Banach extension operator $\omega : W^* \longrightarrow \ell_{p^*}$. We show that its restriction

ω : Ball $W^* \longrightarrow$ Ball ℓ_{p^*} is weak*-to-weak* continuous. The map ω is the composition of three maps

$$\begin{array}{ccc} W^* & \xrightarrow{\omega} & l_{p^*} \\ {\scriptstyle J_W^{-1}}\downarrow & & \uparrow{\scriptstyle J_p} \\ W & \xrightarrow{\imath} & l_p \end{array}$$

where J_p and J_W are the support mappings (also called the duality maps). Since $J_p(x) = \|x\|^{2-p}|x|^{p-1}\mathrm{sgn}\, x$, it is clear that J_p is weakly sequentially continuous, as is the inclusion $\imath$. The support mapping J_W is just $J_p \circ \imath$. Its restriction to Ball W endowed with the weak topology is a continuous map from a compact space to a Hausdorff space, hence a homeomorphism; this makes its inverse weakly continuous.

In general, however, the extension of operators on a reflexive subspace is not to be expected. Suppose for example that a sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ does not locally split (as will be the case when Y is complemented in its bidual but not in X). Then there exists, by Proposition 4 and the well known factorization of weakly compact operators, an operator $T : Y \longrightarrow R$ into some reflexive space R that cannot be extended to X. This gives rise to the question of what happens when R is embedded into some $C(K)$ space. Does the resulting composition $Y \longrightarrow R \longrightarrow C(K)$ extend to X? In particular

Question 1 Does every sequence $0 \longrightarrow \ell_2 \longrightarrow C[0,1] \longrightarrow C[0,1]/\ell_2 \longrightarrow 0$ almost split?

We remark that, in general, $\mathcal{L}_\infty$-valued operators cannot be extended: consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \xrightarrow{\imath} & C[0,1] & \longrightarrow & C[0,1]/\ell_2 & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & \ell_2 & \xrightarrow{\jmath} & \mathcal{L}_\infty(\ell_2) & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

in which $\mathcal{L}_\infty(\ell_2)$ is the Bourgain-Pisier space associated to ℓ_2 (Proposition 8). Since S is Schur, $\mathcal{L}_\infty(\ell_2)$ does not contain c_0 and therefore any possible extension $J : C(K) \longrightarrow \mathcal{L}_\infty(\ell_2)$ of $\jmath$ would be weakly compact, hence completely continuous (by the Dunford-Pettis property of $C(K)$-spaces) and therefore $J \circ \imath$ could not be the identity on ℓ_2.

A simpler argument (using the fact that every operator $\ell_\infty \longrightarrow C[0,1]$ is weakly compact) shows that if X is a separable non-Schur space, then no sequence $0 \longrightarrow X \longrightarrow \ell_\infty \longrightarrow \ell_\infty/X \longrightarrow 0$ almost-splits.

The notion of almost-complementation puts all $C(K)$-spaces at the same level. We could, however, try to distinguish between different $C(K)$-spaces regarding extension properties of operators into them. As a token that such finer classification is possible, we consider the sequence $0 \longrightarrow W_T \longrightarrow \ell_1 \longrightarrow T \longrightarrow 0$

where T denotes Tsirelson's space. Since Tsirelson's space emphatically fails the Schur property, it follows from [7,14] that this sequence admits operators $W_T \longrightarrow C[0,1]$ which do not extend to ℓ_1, yet every operator $W_T \longrightarrow C(\omega^\omega)$ does extend to ℓ_1. This suggests

Question 2 Given a fixed compact space K, characterize the sequences $0 \longrightarrow Y \longrightarrow \ell_1 \longrightarrow Z \longrightarrow 0$ such that all operators $Y \longrightarrow C(K)$ extend to ℓ_1.

The problem of distinguishing between $C(K)$ and $\mathcal{L}_\infty$-spaces pervades the paper, so another natural question is:

Question 3 If $\mathrm{Ext}(Z, C(K)) = 0$ for all compact K, does $\mathrm{Ext}(Z, \mathcal{L}_\infty) = 0$ for all $\mathcal{L}_\infty$-spaces?

And, more generally and not only for subspaces of ℓ_1,

Question 4 Let $\mathcal{H}$ be the class of all subspaces of c_0, and let $\mathcal{L}_\infty(\mathcal{H})$ be the class of all $\mathcal{L}_\infty$ spaces with the property that every operator $T : H \longrightarrow L$ from $H \in \mathcal{H}$ into $L \in \mathcal{L}_\infty(\mathcal{H})$ can be extended to c_0. Characterize the class $\mathcal{L}_\infty(\mathcal{H})$.

The same question for the class of subspaces of ℓ_p, of subspaces of L_p, and for Hilbert subspaces of $C[0,1]$ does not lack interest. Or difficulty: although many Banach spaces (e.g. complemented subspaces of $C(K)$ spaces) belong to these classes, any characterization seems to be quite elusive. Now let $0 \longrightarrow \ell_2 \longrightarrow \mathcal{L}_\infty(\ell_2) \longrightarrow S \longrightarrow 0$ be a Bourgain-Pisier sequence. If H is a subspace of c_0 then $\mathfrak{L}(H, \ell_2) = \mathfrak{K}(H, \ell_2)$ and also $\mathfrak{L}(H, S) = \mathfrak{K}(H, S)$ since S is a Schur space. It is not hard to verify that $\mathfrak{L}(H, \spadesuit) = \mathfrak{K}(H, \spadesuit)$ is a 3-space property (see [8, §6.1 and §6.7]). In particular $\mathfrak{L}(H, \mathcal{L}_\infty(\ell_2)) = \mathfrak{K}(H, \mathcal{L}_\infty(\ell_2))$ and thus, by Proposition 1, operators $H \longrightarrow \mathcal{L}_\infty(\ell_2)$ extend to c_0. Since $\mathcal{L}_\infty(\ell_2)$ does not contain c_0 (another 3-space property, see [8]) it cannot be a complemented subspace of a $C(K)$-space.

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