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SEVEN VIEWS ON APPROXIMATE CONVEXITY AND THE GEOMETRY OF K-SPACES

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Abstract

The interplay between the behaviour of approximately convex (and approximately affine) functions on the unit ball of a Banach space and the geometry of Banach K-spaces is studied.

Introduction

This paper deals with the local stability of convexity, affinity and Jensen functional equation on infinite dimensional Banach spaces. Recall that a function $f : D \to \mathbb{R}$ is said to be ε-convex if it satisfies

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \varepsilon$$

for all $x, y \in D, t \in [0,1]$. If no specific ε is required we speak of an approximately convex function. Of course, any arbitrary function which is uniformly close to a true convex function is approximately convex. These will be called trivial or approximable. It may happen that there are no more; Hyers and Ulam proved in [**17**] that if D is a convex set in $\mathbb{R}^n$, then for every ε-convex function $f : D \to \mathbb{R}$ there exists a convex $a : D \to \mathbb{R}$ such that

$$\sup_{x \in D} |f(x) - a(x)| \overset{\text{def}}{=} d_D(f,a) \leqslant C \cdot \varepsilon,$$

where $C = C_n$ is a constant depending only on n. It is apparent that the papers [**3**, **7**–**9**, **11**–**14**, **16**, **17**, **28**] contain the complete story of C_n.

As far as we know, the first connections between approximately convex functions and the geometry of infinite dimensional Banach spaces appear in [**3**, **7**]. In [**7**] it was proved that Lipschitz ε-convex functions are approximable on bounded sets of B-convex spaces, with the distance to the approximating convex function depending only on ε. (Recall that B-convexity means 'having non-trivial type $p > 1$'.) That paper contains some counterexamples based on the fact that ℓ_1 is the Banach envelope of the spaces ℓ_p for all $0 < p < 1$. In [**3**] it was remarked that every Banach space which is not a K-space (see Section 1 for precise definitions) admits a 'bad' (that is, non-approximable) ε-convex function defined on its unit ball. To be a K-space is a homological property of Banach spaces which is closely related to the behaviour of quasi-linear maps. It suffices to recall here that the spaces ℓ_p and L_p are K-spaces if and only if $p \neq 1$.

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With this background, let us explain the contents and the organization of the paper.

The first section is preliminary; we use the fact that ℓ_1 is not a K-space to obtain explicit examples of 'bad' approximately convex functions on infinite dimensional simplexes (the examples in [**3**, **11**] are not explicit). This leads to the question of whether K-spaces admit 'bad' ε-convex functions on their unit balls. The (affirmative) answer comes in Section 2, where we exhibit a non-trivial approximately convex function on the infinite dimensional cube (the unit ball of ℓ_∞). This solves the main problem raised in [**3**].

Having seen that the local stability of convexity does not hold in K-spaces, we prove in Section 3 that the local stability of affinity is equivalent to being a K-space; precisely, a Banach space X is a K-space if and only if for every ε-affine function f defined on its unit ball B_X there exists a true affine $a : B_X \to \mathbb{R}$ such that

$$d_{B_X}(f, a) \leqslant A \cdot \varepsilon,$$

where A is a constant depending only on X. In Section 4 we prove a similar result for Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

We conclude Sections 3 and 4 by showing that there is a universal constant $A_D \leqslant 224$ such that if f is an ε-affine function defined on the n-dimensional (euclidean) ball D then there exists an affine function $a : D \to \mathbb{R}$ with $d_D(f, a) \leqslant A_D$. A similar result is proved for n-cubes. This solves a problem posed by Laczkovich in [**28**].

Section 5 deals with the question of the uniform approximation. Under rather mild assumptions on the convex set D we show that if every ε-affine (respectively, ε-Jensen) $f : D \to \mathbb{R}$ is approximable by an affine (respectively, Jensen) function a, then this can be done with $d_D(f, a) \leqslant M \cdot \varepsilon$, where M is a constant depending only on D.

Finally, Sections 6 and 7 deal with Banach envelopes. In some sense, the Banach envelope $\operatorname{co} X$ is the nearest Banach space to a given quasi-Banach space X. Here, we regard X as a topological vector space whose topology is 'approximately convex' while that of $\operatorname{co} X$ is truly convex. Our contribution complements previous results by Kalton. Precisely, we show that c_0 is not isometric to the Banach envelope of a non-locally convex space (with separating dual) although there are non-locally convex spaces X whose Banach envelopes are arbitrarily close (in the Banach–Mazur distance) to c_0.

1. *Quasi-linear and approximately convex maps*

Let X and Y be (real) Banach spaces. A map $f : X \to Y$ is said to be quasi-linear if the following hold.

(1) It is homogeneous: $f(tx) = tf(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

(2) It is quasi-additive: $\|f(x+y)-f(x)-f(y)\| \leqslant Q(\|x\|+\|y\|)$ for some constant Q independent of $x, y \in X$.

The least possible constant in the above inequality is denoted $Q(f)$ and referred to as the quasi-additivity constant of the map f. When f is homogeneous we also speak of $Q(f)$ as the quasi-linearity constant of f.

Although the original notion of a K-space refers to the possibility of lifting operators (see [**25**] for background), it will be convenient for our purposes to give the following definition.

DEFINITION 1. A Banach space is a *K-space* if for every quasi-linear map $f : X \to \mathbb{R}$ there is a linear (although not necessarily continuous!) functional $\ell : X \to \mathbb{R}$ such that

$$|f(x) - \ell(x)| \leqslant M\|x\|$$

for some M and all $x \in X$.

Thus K-spaces are closely related to the stability of linear functionals, but that stability is 'asymptotic' rather than 'epsilonic'. It will be convenient to introduce the following asymptotic distance for functions acting between Banach spaces.

$$\operatorname{dist}(f, g) = \inf\{M : \|f(x) - g(x)\| \leqslant M\|x\| \text{ for all } x\},$$

where the infimum of the empty set is treated as infinity. In this way, a K-space is a Banach space in which every quasi-linear functional f is at finite distance from some linear functional ℓ. In fact ℓ can be chosen in such a way that $\operatorname{dist}(f, \ell) \leqslant \kappa \cdot Q(f)$, where κ is a constant depending only on X; see [**19**, Proposition 3.3].

There are two main types of K-spaces: B-convex spaces [**19**, **20**, **25**] and $\mathcal{L}_\infty$-spaces [**24**]. Thus, for instance, the classical spaces ℓ_p and L_p are K-spaces for all $1 < p \leqslant \infty$ as well as c_0 and all $C(K)$ spaces.

On the other hand, ℓ_1 (and also every infinite dimensional $\mathcal{L}_1$-space) is not a K-space. This was proved by Kalton [**19**, **22**], Ribe [**31**] and Roberts [**32**]. In fact Kalton [**19**] and Ribe [**31**] give (more or less) explicit examples of quasi-linear maps $f : \ell \to \mathbb{R}$ with $\operatorname{dist}(f, \ell) = \infty$ for all linear maps $\ell : \ell_1 \to \mathbb{R}$.

Ribe's map is given by

$$R(x) = \sum_i x_i \log_2|x_i| - \left(\sum_i x_i\right) \log_2 \left|\sum_i x_i\right|,$$

where $x = \sum_i x_i e_i$ and assuming that $0 \log 0 = 0$. It is quasi-linear with constant 2. Kalton's map is defined as

$$K(x) = \sum_i \tilde{x}_i \log i \qquad (x \geqslant 0),$$

where $\tilde{x}$ is the decreasing arrangement of x and then extended to the finitely supported sequences of ℓ_1 by

$$K(x) = K(x^+) - K(x^-).$$

It should be noted that the above formulæ have sense only for finitely supported sequences. Nevertheless, quasi-linear maps can be extended from dense subspaces to the whole space (preserving quasi-linearity).

One of the basic observations in [**3**] was that the restriction of a quasi-linear map to a bounded set of a Banach space is an approximately convex function which is uniformly close to a convex function if and only if the starting quasi-linear map is asymptotically close to a linear map. Since in this paper we shall deal with convexity, affinity and Jensen functional equation, let us state a slightly stronger result for midconvex functions. Recall that a midconvex function is one satisfying the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}.$$

LEMMA 1. *Let $f: X \to \mathbb{R}$ be a quasi-linear function. Suppose there is a midconvex function $a: B_X \to \mathbb{R}$ such that $d_{B_X}(f,a) < \infty$. Then there is a linear map $\ell: X \to \mathbb{R}$ such that* $\mathrm{dist}(f,\ell) < \infty$.

Proof. This follows from [**3**, Proof of theorem 2] (in which the result was proved for convex a) and [**18**, Lemma 8.8] (asserting that the midconvex function a is actually convex). □

In this way, every (non-trivial) quasi-linear map produces a 'bad' approximately convex function on the ball of the corresponding Banach space. Even if nobody knows the values of Ribe's function on points x for which the series $\sum_n x_n \log_2|x_n|$ diverges, we can use it to produce an explicit counterexample on the 'infinite dimensional simplex'

$$\Delta^\infty = \left\{ x \in \ell_1 : x_i \geqslant 0 \text{ for all } i \text{ and } \sum_{i=1}^{\infty} x_i = 1 \right\}.$$

Indeed, if Δ^∞_{00} denotes the subset of all finitely supported sequences in Δ^∞, then Δ^∞_{00} is convex and $\Delta^\infty \backslash \Delta^\infty_{00}$ acts as an ideal: if $x \in \Delta^\infty \backslash \Delta^\infty_{00}$ and $y \in \Delta^\infty$ then every non-trivial convex combination $tx + (1-t)y$ belongs to $\Delta^\infty \backslash \Delta^\infty_{00}$. Therefore the function

$$f(x) = \begin{cases} -\sum_{i=1}^{\infty} x_i \log_2 x_i & x \in \Delta^\infty_{00} \\ 0 & x \notin \Delta^\infty_{00} \end{cases} \tag{1}$$

is 1-convex on Δ^∞, but $d_{\Delta^\infty}(f,g) = \infty$ for any convex g. Kalton's function leads to another explicit counterexample taking $f(x) = \sum_{i=1}^{\infty} \tilde{x}_i \log_2 i$ for $x \in \Delta^\infty_{00}$ and $f(x) = 0$ for $x \notin \Delta^\infty_{00}$.

This yields explicit counterexamples for the Hyers–Ulam stability of convexity in infinite dimensions.

PROPOSITION 1 (compare with [**3**, **12**]). *Every infinite dimensional Banach space contains a compact convex set $D \subset X$ and an approximately convex map $h: D \to \mathbb{R}$ such that $d_D(h,g) = \infty$ for every convex g.*

Proof. Let $(e_n)_n$ be a (normalized) basic sequence in X. The map $\Phi: \Delta^\infty \to X$ sending $(t_n) \in \Delta^\infty$ into $\sum_{n=1}^{\infty} t_n(e_n/n)$ defines a one-to-one affine map between Δ^∞ and a certain compact convex set $D \subset X$. Now, if f is the function given in (1), then $h = f \circ \Phi^{-1}$ is a non-approximable 1-convex function on D. □

Let $f: D \to \mathbb{R}$ be an arbitrary function defined on a convex set. The function

$$\operatorname{co} f(x) = \inf \left\{ \sum_{i=1}^{n} t_i f(x_i) : x = \sum_{i=1}^{n} t_i x_i \right\}$$

represents the greatest convex minorant of f. It is clear that f is uniformly close to some convex function on D if and only if ($\operatorname{co} f$ takes only finite values and) $d_D(f, \operatorname{co} f) < \infty$. Actually the distance from f to the convex functions is

$\frac{1}{2}d_D(f, \operatorname{co} f)$ and it is attained at $g = \operatorname{co} f + \frac{1}{2}d_D(f, \operatorname{co} f)$. This will be used without further mention.

2. *Bad ε-convex functions on the unit ball of ℓ_∞*

All non-trivial ε-convex functions presented so far depend on the fact that ℓ_1 is not a K-space. Thus at this moment the question is if there are bad ε-convex functions on the ball of a K-space. To tackle this question one needs a completely different type of approximately convex function; since we will prove in Section 3 that the notion of affinity is stable on the ball of a K-space, one idea is to work with a function which is approximately convex, but not approximately concave.

Such an example is supplied by Cholewa and Kominek [**9**, **27**] (see also [**1**]) as follows. Let c_{00}^+ be the positive cone of the space of all finitely supported sequences; for $x \in c_{00}^+$ set $m(x) = \max_i x_i$ and then

$$\omega(x) = \min\{n \in \mathbb{N} : m(x) \geqslant 2^{-n}\}.$$

This function is 2-convex on c_{00}^+ [**9**, **27**]. Since $\omega(e_n) = 0$ but

$$\omega\left(\frac{1}{2^n}\sum_{i=1}^{2^n} e_i\right) = n,$$

the function ω is not uniformly close to any convex function on c_{00}^+. Since every infinite dimensional space (no topology is assumed) contains a subset affinely isomorphic to c_{00}^+ we obtain a non-trivial approximately convex function defined on 'some part' of it.

It remains to establish the connection with the normed structure of the space. To this end, assume that X is an ordered Banach space [**29**] and let $X^+ = \{x \in X : x \geqslant 0\}$ be its positive cone. The continuous analogue of the Cholewa–Kominek function is

$$x \longmapsto -\log_2\|x\|.$$

Lemma 2. *Let X be an ordered Banach space. Then $-\log_2\|\cdot\|$ is 1-convex on $X^+\backslash\{0\}$.*

Proof. If x and y are positive, then $\|x\| \leqslant \|x+y\|$. Thus, for $0 \leqslant t \leqslant 1$, one has $\|tx\| \leqslant \|tx + (1-t)y\|$ and so $\log_2 t + \log_2\|x\| \leqslant \log_2\|tx + (1-t)y\|$; also $\log_2(1-t) + \log_2\|y\| \leqslant \log_2\|tx + (1-t)y\|$. Therefore

$$t\log_2 t + (1-t)\log_2(1-t) + t\log_2\|x\| + (1-t)\log_2\|y\| \leqslant \log_2\|tx + (1-t)y\|.$$

Since $t\log_2 t + (1-t)\log_2(1-t) \geqslant -1$ for all $0 \leqslant t \leqslant 1$ (the minimum value is attained at $t = 1/2$) the function $-\log_2\|\cdot\|$ is 1-convex on $X^+\backslash\{0\}$ □

Lemma 3. *With the above notations, if X^+ contains a weakly null normalized sequence then $-\log_2\|\cdot\|$ cannot be uniformly approximated by any convex function on $B_X^+\backslash\{0\}$.*

Proof. Let (u_n) be a weakly null sequence in X^+ with $\|u_n\| = 1$. Then there exists a sequence (σ_n) of convex combinations of the u_n such that $\|\sigma_n\| \to 0$ and so

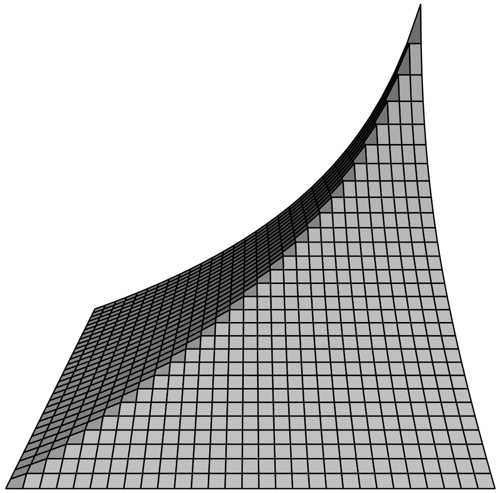

FIGURE 1. *The graph of the function* $-\log_2\|\cdot\|$ *on the positive part of the ball of* ℓ_∞^2.

$-\log_2\|(\sigma_n)\| \to \infty$. However, any convex function approximating $-\log_2\|\cdot\|$ should be bounded on convex combinations of the u_n since $-\log_2\|u_n\| = 0$ for all n. □

Thus we have defined a 'bad' approximately convex function on the unit ball of non-Schur cones. Observe that in ℓ_1 the preceding function becomes approximately convex. Note, however, that composing with the inclusion of ℓ_1 into c_0 one obtains another explicit 'bad' 1-convex function on the infinite dimensional simplex:

$$(t_n) \in \Delta^\infty \longmapsto -\log_2(\max_n t_n).$$

(see Figure 1.) To obtain the function defined on the whole unit ball requires some extra work using additional properties of the spaces.

PROPOSITION 2. *There exist non-trivial approximately convex functions defined on the* (*closed*) *unit ball of either* ℓ_∞ *or* c (*the space of convergent sequences*).

Proof. Let X denote one of the spaces ℓ_∞ or c. The key point is that the unit ball of X is affine-isomorphic to its positive part; consider the map $B_X \to B_X^+$ given by $x \longmapsto (1+x)/2$. Thus, it suffices to get a 'bad' approximately convex function on the positive part of the unit ball.

First of all, note that Lemma 3 asserts that $x \longmapsto -\log_2\|x\|$ is a 'bad' 1-convex map on $B_X^+\backslash\{0\}$.

It remains to avoid the singularity at the origin. Consider the 'involution' of B_X^+ given by $x \longmapsto 1-x$ and let us modify the Cholewa–Kominek function to make 1 the singular point instead of 0. This yields the amended function

$$F(x) = -\log_2\|1-x\|.$$

Now, for each $0<\theta<1$, we set the 1-convex functions

$$F_\theta(x) = -\log_2\|1-\theta x\|$$

defined on the whole B_X^+; they are increasingly far from convex functions as θ approaches 1. It remains to paste these pieces together.

Let P_n be the projection of X onto its first n coordinates, and let (θ_n) be a sequence converging to 1. We define the functions

$$F_n(x) = -\log_2 \|P_n(1) - \theta_n P_n(x)\|$$

which are non-negative and 1-convex on the positive part of the unit ball of X.

It is clear that if $x_j = 0$ for all $j \geqslant m$, then $F_j(x) = 0$ for all $j \geqslant m$. It follows that for every finitely supported $x \in B_X^+$ the sequence $F_n(x)$ is bounded.

Hence we can define a new 1-convex function on the finitely supported sequences of B_X^+ by taking

$$F^*(x) = \sup_n F_n(x).$$

(The pointwise supremum of ε-convex functions is ε-convex whenever it makes sense.) Finally, we extend F^* to all of B_X^+ by putting $F^*(x) = 0$ for $x \notin c_{00}$. Using the fact that the complement of c_{00} in B_X^+ acts as an ideal (with respect to convex combinations) and that F^* is non-negative on $c_{00} \cap B_X^+$ it is easily verified that the resulting map F^* is 1-convex on the whole of B_X^+.

The construction concludes showing that F^* is at infinite distance from any convex function on B_X^+. Actually, the ensuing argument shows that

$$d_{c_{00} \cap B_X^+}(F^*, g) = \infty$$

for all convex $g : c_{00} \cap B_X^+ \to \mathbb{R}$.

To see this, fix $n \geqslant 1$ and let $S_n = \sum_{i=1}^n e_i$. Consider the points $p_i = S_n - e_i$. We see that $F^*(p_i) = 0$. However,

$$\frac{1}{n} \sum_{i=1}^{n} p_i = \frac{n-1}{n} \cdot S_n$$

and so

$$\limsup_{n \to \infty} F^* \left(\frac{1}{n} \sum_{i=1}^{n} p_i \right) \geqslant \limsup_{n \to \infty} \left(-\log_2 \left(1 - \theta_n \frac{n-1}{n} \right) \right) = +\infty,$$

which already implies that F^* cannot be approximated by any convex function defined on a convex set containing $\{S_n - e_i : n \in \mathbb{N}, 1 \leqslant i \leqslant n\}$. □

We have thus solved the main problem in [**3**] showing that non-trivial ε-convex functions can occur in the unit ball of K-spaces.

As for the other type of K-spaces, the B-convex ones, the following question remains.

QUESTION 1. Do B-convex spaces admit bad ε-convex functions on their unit balls?

3. *The stability of affinity in K-spaces*

We already know that the stability of convexity does not hold on (the unit ball of) arbitrary K-spaces. However, under some homogeneity conditions, ε-convex functions defined on K-spaces can be approximated by convex ones (see, for example, [**3**, Theorems 2 and 4]). Thus one may wonder which other classes of functions could provide a characterization of K-spaces. Having in mind the behaviour of quasi-linear functions on balls, it is not too surprising that ε-affinity works. Recall that a mapping $f: D \to \mathbb{R}$ (here D is a convex subset of a linear space) is said to be ε-affine if it satisfies

$$|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \leqslant \varepsilon$$

for all $x, y \in D, t \in [0,1]$. That is, an ε-affine map is one which, in addition to being ε-convex, is also 'ε-concave'. Before going further, let us prove the following simple consequence of the Hahn–Banach separation theorem.

Proposition 3. *Let f be any function defined on a convex set. If f is approximable by a convex function and by a concave function, then it is approximable by an affine function.*

Proof. Put $\alpha = d_D(f,g), \beta = d_D(f,g)$, where D is the underlying convex set, g is convex and h is concave. Then $-\alpha \leqslant f(x) - g(x) \leqslant \alpha$ and $-\beta \leqslant f(x) - g(x) \leqslant \beta$; hence

$$-\beta + h(x) \leqslant f(x) \leqslant \alpha + g(x) \qquad (x \in D).$$

However $-\beta + h$ is concave and $\alpha + g$ is convex and so, if X is any linear space containing D, the sets

$$E = \{(x,t) : x \in D, t > \alpha + g(x)\} \quad \text{and} \quad F = \{(x,t) : X \in D, t < h(x) - \beta\}$$

are non-overlapping convex subsets of $X \times \mathbb{R}$. The Hahn–Banach theorem gives a hyperplane H separating E from F, that is, an $\mathbb{R}$-linear functional $a: X \times \mathbb{R} \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that

$$E \subset \{(x,t) : a(x,t) \geqslant \gamma\} \quad \text{and} \quad F \subset \{(x,t) : a(x,t) \leqslant \gamma\}.$$

Write $a(x,t) = b(x) + \lambda t$, where $b: X \to \mathbb{R}$ is $\mathbb{R}$-linear. It is clear that $\lambda \neq 0$. It follows that H is the graph of a certain affine function $a': X \to \mathbb{R}$ separating $-\beta + h$ from $\alpha + g$:

$$-\beta + h(x) \leqslant a'(x) \leqslant \alpha + g(x) \qquad (x \in D).$$

Hence $-2\beta \leqslant a'(x) - f(x) \leqslant 2\alpha$ and so $d_D(f, a') \leqslant 2\max\{\alpha, \beta\}$. □

In this way we obtain an alternative proof of the fact that approximately affine functions are uniformly approximable by affine functions on finite dimensional convex sets [**28**, Theorem 2]. On the other hand, it is clear that the restriction of a quasi-linear map to a bounded set is approximately affine, and so is Ribe's original function on Δ^∞. Thus, we obtain that the 'affine' version of Proposition 1 is also true: if X is an infinite dimensional Banach space, there is an approximately affine function defined on a compact subset of X which cannot be approximated by any affine function.

Our main result on ε-affine functions is the following theorem about the local stability of affinity for K-spaces.

Theorem 1. *A Banach space X is a K-space if and only if every ε-affine function $f: B_X \to \mathbb{R}$ is (uniformly) approximable by a true affine function $a: B_X \to \mathbb{R}$.*

Proof. The 'if' part is contained in Lemma 1, taking into account that if $f:X\to\mathbb{R}$ is quasi-linear with constant ε, the restriction on the unit ball is ε-affine.

The 'only if' part follows from the following slightly stronger result which gives us more flexibility on the choice of the domain D. □

PROPOSITION 4. *Let D be a convex bounded set with non-empty interior in a K-space X. Then to every ε-affine function $f:D\to\mathbb{R}$ there corresponds an affine function $a:D\to\mathbb{R}$ such that*

$$d_D(f,a)\leqslant A\cdot\varepsilon,$$

where $A=A_D$ is a constant depending only on D.

Proof. There is no loss of generality in assuming that the origin is interior to D and also that f is 1-affine, with $f(0)=0$. Let $\mathcal{L}$ denote the set of lines of X passing through 0 and, for each $\ell\in\mathcal{L}$, set $D_\ell=D\cap\ell$ and let f_ℓ be the restriction of f to D_ℓ.

Each f_ℓ is clearly ε-affine and since D_ℓ is one-dimensional, there is an affine function $a_\ell:D_\ell\to\mathbb{R}$ with

$$d_{D_\ell}(f_\ell,a_\ell)\leqslant 1.$$

(See [**28**].) In particular, we have $|a_\ell(0)|\leqslant 1$ and so we can extend $x\mapsto a_\ell(x)-a_\ell(0)$ to a linear map $L_\ell:\ell\to\mathbb{R}$ with

$$d_{D_\ell}(f_\ell,L_\ell)\leqslant 2.$$

Let us define $f^*:X\to\mathbb{R}$ by

$$f^*(x)=L_{[x]}(x),$$

where $[x]$ is the line spanned by x. It is clear that f^* is homogeneous on X and also that $d_D(f,f^*)\leqslant 2$, from which it follows that f^* is 3-affine on D.

We see that f^* is quasi-linear. Indeed, for $x,y\in D$ one has

$$|f^*(x+y)-f^*(x)-f^*(y)|=2\left|f^*\left(\frac{x+y}{2}\right)-\frac{f^*(x)+f^*(y)}{2}\right|\leqslant 6.$$

It follows that f^* is quasi-additive, with $Q(f^*)\leqslant 6/r_0$, where $r_0=\sup\{r>0:rB_X\subset D\}$. Now, since X is a K-space, there is a linear (hence affine) function $a:X\to R$ satisfying

$$|f^*(x)-a(x)|\leqslant\frac{6M}{r_0}\|x\|\quad(x\in X),$$

where M depends only on X. Therefore, if $R_0=\inf\{R>0:D\subset RB_X\}$, we have $d_D(f^*,a)\leqslant 6MR_0/r_0$ and so

$$d_D(f,a)\leqslant\frac{6MR_0}{r_0}+2,$$

which completes the proof. □

4. *Local stability of the Jensen equation*

Suppose D is a midpoint convex subset of a linear space. Jensen's functional equation is

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \qquad (x, y \in D).$$

Accordingly, we say that f is ε-Jensen if

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right| \leqslant \varepsilon \qquad (x, y \in D).$$

Let us prove the Jensen analogue of Theorem 1.

THEOREM 2. *A Banach space X is a K-space if and only if every ε-Jensen function $f : B_X \to \mathbb{R}$ is (uniformly) approximable by a function $a : B_X \to \mathbb{R}$ which satisfies the Jensen equation.*

The 'if' part follows from Lemma 1, while the converse is contained in the following result.

PROPOSITION 5. *Let D be a convex bounded set with non-empty interior in a K-space X. Then for every ε-Jensen function $f : D \to \mathbb{R}$ there is a true Jensen function $a : D \to \mathbb{R}$ such that $d_D(f, a) < \infty$.*

Proof. The proof is almost the same as before and we only give the main steps. Assume again that 0 is interior to D and that $f : D \to \mathbb{R}$ is 1-Jensen, with $f(0) = 0$. With the same notations, it is clear that every f_ℓ is 1-Jensen and since D_ℓ is one-dimensional and convex there is a Jensen function $a_\ell : D_\ell \to \mathbb{R}$ with

$$d_{D_\ell}(f_\ell, a_\ell) \leqslant 2.$$

(See [**28**, Theorem 3].) Putting $L_\ell(x) = a_\ell(x) - a_\ell(0)$ we obtain a 2-homogeneous Jensen function that clearly extends to a 2-homogeneous Jensen function on ℓ (which we do not relabel), with $d_{D_\ell}(f_\ell, L_\ell) \leqslant 4$. Setting

$$f^*(x) = L_{[x]}(x),$$

(where $[x]$ is the line spanned by x), it is clear that f^* is 2-homogeneous on X and also that $d_D(f, f^*) \leqslant 4$, from which it follows that f^* is 6-Jensen on D.

Thus, for $x, y \in D$, one has

$$|f^*(x+y) - f^*(x) - f^*(y)| = 2\left| f^*\left(\frac{x+y}{2}\right) - \frac{f^*(x)+f^*(y)}{2} \right| \leqslant 12.$$

Hence f^* is quasi-additive, with $Q(f^*) \leqslant 12/r_0$, where $r_0 = \sup\{r > 0 : rB_X \subset D\}$.

Now, since X is a K-space, we can use [**4**, Corollary 2] to get an additive (hence Jensen) function $a : X \to \mathbb{R}$ with $f^* - a$ continuous at the origin of X. Therefore, there is $\delta > 0$ so that

$$|f^*(x) - a(x)| \leqslant 1 \qquad (\|x\| \leqslant \delta).$$

Having in mind that both f^* and a are 2-homogeneous we see that, in fact,

$$|f^*(x) - a(x)| \leqslant \frac{\|x\|}{\delta} \qquad (x \in X).$$

In particular one has $d_D(f^*, a) \leqslant R_0/\delta$, where $R_0 = \inf\{R > 0 : D \subset RB_X\}$ and so

$$d_D(f, a) \leqslant 4 + \frac{R_0}{\delta}, \tag{2}$$

which completes the proof. □

Following Laczkovich [**28**], given a convex set D, we write C_D (respectively, A_D and J_D) for the least constant κ such that to every ε-convex (respectively, ε-affine and ε-Jensen) function $f : D \to R$ there corresponds a convex (respectively, affine and Jensen) function $g : D \to \mathbb{R}$ such that $d_D(f, g) \leqslant \kappa\varepsilon$.

One of the most surprising results in [**28**] is that (in the finite dimensional case) C_D is essentially independent of the shape of D. In fact $C_D \sim \log \dim D$, where $\dim D$ is the linear dimension of the least affine submanifold containing D.

Laczkovich also observed that both A_D and J_D are of the same order as $\log \dim D$ if D is either a simplex or an n-dimensional 'octahedron' (the unit ball of ℓ_1^n). He asks for the behaviour of the constants A_D and J_D for D either the n-dimensional cube (the unit ball of ℓ_∞^n) or the n-dimensional euclidean ball (the unit ball of ℓ_2^n). It turns out that these constants are uniformly bounded.

Corollary 1. *If D is the unit ball of one of the spaces ℓ_∞^n, c_0 or ℓ_∞ (or even of an $\mathcal{L}_{\infty,1}$-space), then $A_D \leqslant 6 \cdot 200 + 2$ and $J_D \leqslant 2 \cdot (6 \cdot 200 + 2)$.*

If D is the unit ball of a Hilbert space, then $A_D \leqslant 6 \cdot 37 + 2$ and $J_D \leqslant 2 \cdot (6 \cdot 37 + 2)$.

Proof. The statement concerning the constants A_D in the first case follows from the fact (proved by Kalton and Roberts [**24**]) that every $\mathcal{L}_{\infty,1}$-space is a K-space, with constant at most 200 (although 100 was announced in [**24**]; see [**30**]). The estimate for J_D follows from the fact that $J_D \leqslant 2A_D$ for finite dimensional D (see [**28**]) and local techniques; we give only a sketch. Let $\mathcal{F}$ be the set of all finite dimensional subspaces of X. Since X is an $\mathcal{L}_{\infty,1}$-space, given $F \in \mathcal{F}$ and $\varepsilon > 0$, there is $E \in \mathcal{F}$ containing F and such that the Banach–Mazur distance between E and $\ell_\infty^{\dim E}$ is at most $1 + \varepsilon$, which simply means that there is a linear isomorphism $T : E \to \ell_\infty^{\dim E}$ such that

$$B_{\ell_\infty^{\dim E}} \subset TB_E \subset (1 + \varepsilon)B_{\ell_\infty^{\dim E}}.$$

Thus, in view of (2), one has

$$A_{B_E} \leqslant 6 \cdot 200 \cdot (1 + \varepsilon) + 2$$

and therefore

$$J_{B_E} \leqslant 2 \cdot (6 \cdot 200 \cdot (1 + \varepsilon) + 2).$$

Since ε is arbitrary, we see that if f is 1-Jensen on B_X then for every $F \in \mathcal{F}$ there is a Jensen $a_F : B_F \to \mathbb{R}$ such that

$$d_{B_F}(f, a_F) < 2 \cdot (6 \cdot 200 + 2) + \frac{1}{\dim F}.$$

To end, let $\mathfrak{V}$ be an ultrafilter refining the Fréchet (= order) filter on $\mathcal{F}$ and define $a : B_X \to \mathbb{R}$ taking

$$a(x) = \lim_{\mathfrak{V}(F)} a_F(x).$$

It is easily seen that the above definition yields a Jensen function at distance at most $2\cdot(6\cdot 200+2)$ from f on B_X.

As for euclidean norms, the standard proofs that Hilbert spaces are K-spaces do not give any estimate for the corresponding constant [**5**, **19**]. There is however a recent paper by Šemrl [**33**] in which it is shown that if $f: H\to\mathbb{R}$ is quasi-linear and bounded on the unit ball, then there exists a linear map $\ell: H\to\mathbb{R}$ such that

$$\operatorname{dist}(f,\ell)\leqslant 37\cdot Q(f).$$

Since quasi-linear maps are always bounded on finite dimensional balls an obvious local argument shows that Hilbert spaces are K-spaces with constant 37. The result follows from this. □

It will be clear for those acquainted with twisted sums of Banach spaces that the results of Sections 3 and 4 cannot be extended to vector-valued maps. If X is any infinite dimensional Banach space there is another Banach space Y and a quasi-linear map $f: X\to Y$ such that $\operatorname{dist}(f,\ell)=\infty$ for all linear maps $\ell: X\to Y$.

(Indeed, if X contains a complemented subspace isomorphic to ℓ_1, set $Y=\mathbb{R}$ and use Ribe's map in the obvious way. Otherwise, take a quotient operator $\pi:\ell_1(\Gamma)\to X$ for a suitable set Γ, set $Y=\ker\pi$ and note that the exact sequence

$$0\to Y\to\ell_1(\Gamma)\to X\to 0$$

does not split. This implies the existence of a quasi-linear map $f: X\to Y$ at infinite distance from all linear maps $X\to Y$; see [**19**] or [**6**].)

It is clear that $f: B_X\to Y$ is approximately affine, but no Jensen map $a: B_X\to Y$ is uniformly close to f. For if a is a Jensen map such that $d_{B_X}(f,a)<\infty$ then a is actually affine and so f is asymptotically close to the linear map $\ell: X\to Y$ obtained by extending (by homogeneity) $x\to a(x)-a(0)$ to all of X.

5. *Uniform boundedness*

In this section we put the results of the preceding two sections in their proper setting by showing that, for many convex sets D, if every ε-affine (respectively, ε-Jensen) function $f: D\to\mathbb{R}$ is approximable by an affine (respectively, Jensen) $a: D\to\mathbb{R}$, then this can be achieved with $d_D(f,a)\leqslant C\varepsilon$, where C is a constant depending only on D.

Let D be a convex set, where no topology is assumed. A point $d\in D$ is said to be geometrically interior to D if for every v in the linear space spanned by D one has $d\pm tv\in D$ for $t>0$ small enough. This is equivalent to the following: D is a neighbourhood of d in the linear space spanned by D equipped with the strongest locally convex topology.

In the sequel, we say that D is thick if it has at least one geometrically interior point. Every set with non-empty interior in a normed space is thick, and so are all separable polytopes (a polytope is a closed bounded set in a Banach space whose finite dimensional sections all have finitely many extreme points; see [**15**, § 6]).

As for elementary simplices, let $\mathfrak{c}$ be an infinite cardinal (which we regard also as an index set). Then the $\mathfrak{c}$-dimensional simplex

$$\Delta^{\mathfrak{c}}=\left\{x\in\mathbb{R}^{\mathfrak{c}}: x_\alpha\geqslant 0 \text{ for all } \alpha\in\mathfrak{c} \text{ and } \sum_{\alpha\in\mathfrak{c}}x_\alpha=1\right\}$$

is thick if and only if $\mathfrak{c}$ is countable.

THEOREM 3. *Let D be a thick convex set. If every approximately Jensen function on D is uniformly approximable by some Jensen function, then there is a constant J, depending only on D, such that, to every ε-Jensen $f : D \to \mathbb{R}$ there corresponds a Jensen function $a : D \to \mathbb{R}$ with $d_D(f, a) \leqslant J \cdot \varepsilon$.*

This theorem applies to the sets appearing in Proposition 5, and thus J_D is finite (as we already know is A_D) if D is a bounded convex set with non-empty interior in a K-space.

The main step in the proof of Theorem 3 is the next lemma showing that if f is approximately Jensen, then a judicious choice of the affine function $a : D \to \mathbb{R}$ allows us to control $|f(x) - a(x)|$ 'pointwise' on x but 'uniformly' on f.

LEMMA 4. *Let D be a thick set. There is a function $\eta : D \to \mathbb{R}$ (depending only on D) such that, for every ε-Jensen function $f : D \to \mathbb{R}$ there is a Jensen $a : D \to \mathbb{R}$ satisfying the estimate*

$$|f(x) - a(x)| \leqslant \varepsilon\eta(x) \qquad (x \in D).$$

Proof. Without loss of generality we assume that the origin is geometrically interior to D. Let X be the linear space spanned by D and consider the gauge of D, that is, the function $\varrho : X \to [0, \infty]$ defined as

$$\varrho(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in D\}.$$

The hypothesis on D implies that ϱ is well-defined and also that it takes only finite values.

Now, let f be 1-Jensen on D with $f(0) = 0$. Just as in the first part of the proof of Proposition 5 we can find a function $f^* : X \to \mathbb{R}$ such that the following hold.

(1) $d_D(f, f^*) \leqslant 4$.

(2) f^* is homogeneous over the rationals: $f^*(qx) = qf^*(x)$ for all $x \in X$ and $q \in \mathbb{Q}$.

It follows that f^* is 6-Jensen on D and so

$$|f^*(x + y) - f^*(x) - f^*(y)| \leqslant 12 \quad (x, y \in D).$$

Using the $\mathbb{Q}$-homogeneity of f^* we obtain that f^* is quasi-additive with respect to ϱ:

$$|f^*(x + y) - f^*(x) - f^*(y)| \leqslant 12(\varrho(x) + \varrho(y)) \qquad (x, y \in X).$$

It follows (by induction on n; see [**19**]) that

$$\left| f^*\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f^*(x_i)\right| \leqslant 12 \sum_{i=1}^{n} i\varrho(x_i) \qquad (n \in \mathbb{N}, x_i \in X). \tag{3}$$

Finally, let $\mathcal{B}$ be a Hamel basis of X over the rationals, with $\pm b \in D$ for all $b \in \mathcal{B}$ and define a $\mathbb{Q}$-linear (hence Jensen) map $a : X \to \mathbb{R}$ taking $a(b) = f^*(b)$ for $b \in \mathcal{B}$ and extending linearly on the rest.

Fixing $x \in X$, let us estimate $f^*(x) - a(x)$. Write $x = \sum_{b \in \mathcal{B}} q_b b$ and let $n(x)$ be the number of non-zero summands in that decomposition, so that we can write

$x=\sum_{i=1}^{n(x)} q_{b(i)}b(i)$. We have

$$|f^*(x)-a(x)| = \left|f^*(x)-\sum_{b\in\mathcal{B}} q_b f^*(b)\right| = \left|f^*\left(\sum_{i=1}^{n(x)} q_{b(i)}b(i)\right)-\sum_{i=1}^{n(x)} f^*\big(q_{b(i)}b(i)\big)\right|$$
$$\leqslant 12\sum_{i=1}^{n(x)} i\varrho\big(q_{b(i)}b(i)\big) \leqslant 12n(x)\sum_{b\in\mathcal{B}}|q_b|.$$

Therefore, for $x\in D$ we get

$$|f(x)-a(x)|\leqslant 4+12n(x)\sum_b |q_b|,$$

and choosing $\eta(x)=4+12n(x)\sum_b|q_b|$ we conclude the proof. □

Let $\mathcal{AJ}(D)$ denote the linear space of all approximately Jensen functions on D and $\mathcal{J}(D)$ that of Jensen functions. We introduce two functionals on $\mathcal{AJ}(D)$ as follows:

$$\varepsilon_J(f)=\sup_{x,y\in D}\left|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right| = \inf\{\varepsilon : f \text{ is } \varepsilon\text{-Jensen}\}$$
$$\delta_J(f)=d_D(f,\mathcal{J}(D)).$$

With these notations our hypothesis is nothing but $\delta_J(f)<\infty$ for all $f\in\mathcal{AJ}(D)$ and we must prove that $\delta_J(f)\leqslant J\cdot\varepsilon_J(f)$ for some constant J independent on f. Obviously, $\varepsilon_J(\cdot)\leqslant 2\delta_J(\cdot)$.

It is clear that both ε_J and δ_J are seminorms. We have $\ker\varepsilon_J=\ker\delta_J=\mathcal{J}(D)$, so that both ε_J and δ_J are well-defined norms on the quotient space $\mathcal{AJ}(D)/\mathcal{J}(D)$. That $\mathcal{AJ}(D)/\mathcal{J}(D)$ is complete under δ_J is nearly obvious. Thus the following result finishes the proof of Theorem 3, thanks to the open mapping theorem.

Lemma 5. *The space $\mathcal{AJ}(D)/\mathcal{J}(D)$ is complete under the norm ε_J.*

Proof. It suffices to show that absolutely summable series converge in $\mathcal{AJ}(D)/\mathcal{J}(D)$. Let (f_n) be such that

$$\sum_{n=1}^{\infty}\varepsilon_J(f_n)<\infty.$$

By Lemma 4 there are Jensen functions $a_n : D\to\mathbb{R}$ such that $|f_n(x)-a_n(x)|\leqslant \varepsilon_J(f_n)\eta(x)$ for all $x\in D$. Hence we can define a function f pointwise as

$$f(x)=\sum_{n=1}^{\infty}(f_n(x)-a_n(x)).$$

It is straightforward that g is approximately Jensen, with $\varepsilon_J(f)\leqslant\sum_n\varepsilon_J(f_n)$ and also that $\sum_n[f_n]$ converges to $[f]$ in $\mathcal{AJ}(D)/\mathcal{J}(D)$ with the norm ε_J. □

The following 'affine' companion of Theorem 3 has a simpler proof which we leave to the reader.

Theorem 4. *Let D be a thick convex set. If every approximately affine function on D is uniformly approximable by some affine function, then there is a constant*

A_D, such that, to every ε-affine $f:D\to\mathbb{R}$ there corresponds an affine $a:D\to\mathbb{R}$ with $d_D(f,a)\leqslant A_D\cdot\varepsilon$.

We do not know if the hypothesis on D can be removed from the results in this section. Also, it would be interesting to get similar results for approximately convex functions. However we strongly believe that if every approximately convex function on D is approximable, then D has finite dimension. If so, the corresponding uniform boundedness result for approximate convexity would be a trivial tautology.

6. *c_0 is not isometric to a Banach envelope ...*

Let us recall from [**25**] the minimal background that one needs to understand what follows. A quasi-norm on a (real or complex) vector space X is a non-negative real-valued function on X satisfying the following.

(1) $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\|=|\lambda|\|x\|$ for all $x\in X$ and $\lambda\in\mathbb{K}$.
(3) $\|x+y\|\leqslant\Delta(\|x\|+\|y\|)$ for some fixed $\Delta\geqslant 1$ and all $x,y\in X$.

A quasi-normed space is a vector space X together with a specified quasi-norm. On such a space one has a (linear) topology defined as the smallest linear topology for which the set $B_X=\{x\in X:\|x\|\leqslant 1\}$ (the unit ball of X) is a neighbourhood of 0. In this way, X becomes a locally bounded space (that is, it has a bounded neighbourhood of 0); and, conversely, every locally bounded topology on a vector space comes from a quasi-norm. A quasi-Banach space is a complete quasi-normed space.

Needless to say, every Banach space is a quasi-Banach space, but there are important examples of quasi-Banach spaces which are not (isomorphic to) Banach spaces. Let us mention the ℓ_p and L_p spaces and the Hardy classes H^p for $0<p<1$.

Let X be a quasi-Banach space. The dual space X^* is always a Banach space under the norm

$$\|x^*\|=\sup_{\|x\|\leqslant 1}|x^*(x)|.$$

Consider the 'evaluation mapping' $\delta:X\to X^{**}$ given by $\delta(x)x^*=x^*(x)$. The Banach envelope $\operatorname{co}X$ of X is the closure of $\delta(X)$ in X^{**} equipped with the induced norm. Notice that δ is one to one if and only if X^* separates X in the sense that for every non-zero $x\in X$ there is x^* in X^* such that $x^*(x)\neq 0$. Thus, $\operatorname{co}X$ is a Banach space whose unit ball equals the closed convex hull of $\delta(B_X)$. In particular, when X is finite dimensional $\operatorname{co}X$ can be seen as a renorming of X itself. The map $\delta:X\to\operatorname{co}X$ has the following universal property: every bounded linear operator from X into a Banach space Y factorizes throughout δ with equal norm. This clearly implies that $\operatorname{co}X$ is the 'nearest' Banach space to X with respect to the Banach–Mazur distance: if T is an isomorphism from X into a Banach space Y, then $\delta:X\to\operatorname{co}X$ is an isomorphism and, in fact $\|T\|\|T^{-1}\|\leqslant\|\delta\|\|\delta^{-1}\|$. Note that $\|\delta\|=1$ and that $\|\delta^{-1}\|$ is the least constant K for which

$$\|\delta x\|_{\operatorname{co}X}\leqslant\|x\|_X\leqslant K\|\delta x\|_{\operatorname{co}X}$$

holds for all $x\in X$. Of course, X is locally convex (that is, isomorphic to a Banach space) if and only if $\|\cdot\|_{\operatorname{co}X}$ is equivalent to the original norm $\|\cdot\|_X$.

From the point of view we have adopted in this paper, it is worth noting that $\|\cdot\|_{\mathrm{co}\,X}$ equals $\mathrm{co}\,\|\cdot\|_X$ (on the common domain X) and that the Lipschitz counterexamples found in [**7**] depend on the fact that ℓ_1 is the Banach envelope of the non-locally convex spaces ℓ_p for $0 < p < 1$.

Which Banach spaces can be envelopes of non-locally convex spaces with separating dual? (The condition of having separating dual is to avoid trivial examples.) In [**19**], Kalton proved that the Banach envelope of a non-locally convex quasi-Banach space with separating dual is never B-convex. He then asks whether c_0 (or l_∞) can be the Banach envelope of a non-locally convex quasi-Banach space with separating dual [**19**]. Kalton himself solved the 'isomorphic' problem in the negative [**21**, Section 4] showing a rather pathological quasi-Banach space whose Banach envelope is isomorphic to c_0. Surprisingly one has the following.

Proposition 6. *c_0 is not isometric to the Banach envelope of a non-locally convex quasi-Banach space with separating dual.*

This straightforwardly follows after the following result.

Theorem 5. *Let X be a quasi-Banach space. If $T : X \to c_0$ is a bounded operator such that $\overline{\mathrm{co}}TB_X = B_{c_0}$ then T is an open mapping of X onto c_0.*

The proof is based on a few elementary observations that we put together in the following lemma. Let us say that $u \in B_{c_0}$ is a locally extreme point if for all k one has $|u_k| \in \{0, 1\}$.

Lemma 6. *With the same notations as in Theorem* 5, *the following hold.*

(a) *For every $y \in B_{c_{00}}$ there is a locally extreme point $u \in B_{c_0}$ such that $\|y - \frac{1}{2}u\| \leqslant \frac{1}{2}$.*

(b) *For every locally extreme point u and every $\varepsilon > 0$, there is $x \in B_X$ such that $\|u - 1_{\mathrm{supp}(u)}T(x)\| \leqslant \varepsilon$.*

(c) *For every y in the unit ball of c_0 and every $\varepsilon > 0$ there is $x \in B_X$ such that $\|y - \frac{1}{2}T(x)\| \leqslant \frac{1}{2} + \varepsilon$.*

Proof. (a) Take $u = \mathrm{sgn}\, y$.

(b) Let u be a locally extreme point of the ball of c_0 and consider the (contractive) projection $1_{\mathrm{supp}(u)} : c_0 \to c_0$ given by multiplication. The hypothesis implies that $\overline{\mathrm{co}}(1_{\mathrm{supp}(u)}TB_X) = B(1_{\mathrm{supp}(u)}c_0)$. Since $1_{\mathrm{supp}(u)}c_0$ is finite dimensional, the closure of $1_{\mathrm{supp}(u)}TB_X$ contains all extreme points of the unit ball of $1_{\mathrm{supp}(u)}c_0$. In particular, it contains u.

(c) We may assume that $y \in c_{00}$. Let $u = \mathrm{sgn}\, y$ and note that $\mathrm{supp}(u) = \mathrm{supp}(y)$. By (b) there is $x \in B_X$ such that $\|u - 1_{\mathrm{supp}(u)}T(x)\| \leqslant \varepsilon$. One thus has

$$\left|y(k) - \tfrac{1}{2}Tx(k)\right| \leqslant \begin{cases} \frac{1}{2} + \varepsilon & k \in \mathrm{supp}(f) \\ \frac{1}{2} & k \notin \mathrm{supp}(f), \end{cases}$$

and the result follows. □

Proof of Theorem 5. Fix $y \in c_0$, with $\|y\| \leqslant 1$ and let $\varepsilon > 0$ be fixed. By Lemma 6 there is $x_1 \in B_X$ such that $\|y - \frac{1}{2}Tx_1\| \leqslant \frac{1}{2} + \varepsilon$. Replacing y by $y - \frac{1}{2}Tx_1$, there is $x_2 \in B_X$ such that

$$\left\| y - \frac{1}{2}Tx_1 - \frac{1}{2}\left(\frac{1}{2}+\varepsilon\right)Tx_2 \right\| \leqslant \left(\frac{1}{2}+\varepsilon\right)^2 .$$

Proceeding inductively one obtains a sequence (x_n) in the unit ball of X such that

$$\left\| y - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{2}+\varepsilon\right)^{i-1} Tx_i \right\| \leqslant \left(\frac{1}{2}+\varepsilon\right)^n \tag{4}$$

holds for all n. Let us estimate the quasi-norm of $\sum_{i=1}^{n}\left(\frac{1}{2}+\varepsilon\right)^{i-1} x_i$.

By the Aoki–Rolewicz theorem [**25**], X has an equivalent p-norm for some $0 < p \leqslant 1$. Hence, for some constant $M = M(p, X)$ and all n one has

$$\begin{aligned}\left\| \sum_{i=1}^{n}\left(\frac{1}{2}+\varepsilon\right)^{i-1} x_i \right\|^p \leqslant M\sum_{i=1}^{n}\left\| \left(\frac{1}{2}+\varepsilon\right)^{i-1} x_i \right\|^p = M\sum_{i=1}^{n}\left(\frac{1}{2}+\varepsilon\right)^{(i-1)p} \\ \leqslant \frac{M}{1-\left(\frac{1}{2}+\varepsilon\right)^p}\end{aligned}$$

for ε small enough. Thus

$$\left\| \sum_{i=1}^{n}\left(\frac{1}{2}+\varepsilon\right)^{i-1} x_i \right\| \leqslant K$$

for some constant K. Taking (4) into account, it is clear that T is almost open. An appeal to the (proof of the) open mapping theorem [**25**] ends the proof. ☐

We can use the above argument to show that B-convex spaces cannot be envelopes of non-locally convex spaces with separating dual. First of all, note that every quasi-norm is equivalent to some p-norm for some $0 < p \leqslant 1$ (this is the Aoki–Rolewicz theorem). This implies that if $\operatorname{co} Y = X$ isometrically and Y has an equivalent p-norm, then $X = \operatorname{co}(\operatorname{co}_p Y)$, where $\operatorname{co}_p Y$ denotes the p-Banach envelope of Y (here the p-convex envelope of a symmetric set A is defined to be the set $\operatorname{co}_p A = \{\sum_{i=1}^{n} t_i a_i : a_i \in A, \sum_{i=1}^{n} |t_i|^p \leqslant 1\}$), and it is so because

$$\overline{\operatorname{co}} B_Y = \overline{\operatorname{co}}(\overline{\operatorname{co}}_p B_Y).$$

Let us say that B has the property (p, θ, κ) if whenever A is a symmetric p-convex subset of B such that $B = \overline{\operatorname{co}} A$ one has

$$B \subset \theta B + \kappa A.$$

For instance, the content of Lemma 6 is that the unit ball of c_0 has all properties $(p, \frac{1}{2}+\varepsilon, 2)$ for $0 < p < 1$ and $\varepsilon > 0$. Another non-trivial example is provided by the unit ball of any B-convex space.

LEMMA 7. *Let X be a B-convex Banach space. Then, for every $0 < p < 1$ there are $\theta < 1$ and $\kappa > 0$ such that B_X has the property (p, θ, κ).*

Proof. This follows from [**2**, Theorem 1.1] which identifies B-convexity with the so-called convex approximation property. Namely, X is B-convex if and only if given a bounded $B \subset X$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\operatorname{co} B \subset \operatorname{co}^{[m]} B + \varepsilon B_X$, where

$$\mathrm{co}^{[m]} B = \left\{ \sum_{i=1}^{m} t_i b_i : \sum_{i=1}^{m} t_i = 1, t_i \geqslant 0, b_i \in B \right\}.$$

Now, if $A \subset B_X$ is a p-convex set such that $B_X = \overline{\mathrm{co}} A$, we have $B_X \subset \mathrm{co}^{[m]} A + \varepsilon B_X$ for all $\varepsilon > 0$ and some $m \in \mathbb{N}$. Since A is p-convex there is $M > 0$ such that $\mathrm{co}^{[m]} A \subset MA$ and so B_X has (p, ε, M), as desired. □

The proof of the following result is contained in that of Theorem 3. We make it explicit for the sake of clarity.

Proposition 7. *If B_X has the property (p, θ, κ) for some $\theta < 1$ then X cannot be the Banach envelope of a non-locally convex p-Banach space with separating dual.*

We obtain an alternative proof of the Kalton result.

Corollary 2. *B-convex spaces cannot be envelopes of non-locally convex spaces with separating dual.*

It is not true that admitting an equivalent norm with the properties (p, θ, κ) guarantees that the space is a K-space; it is well known that H^1 is not a K-space (it contains ℓ_1 as a direct factor) yet its usual norm has all properties $(p, \frac{1}{2}, 1)$ since H^1 has the Radon–Nikodým property and every norm one $f \in H^1$ can be written as the midpoint of two inner functions (see [**10**, **26**]).

7. ... *although it is $(1 + \varepsilon)$-isomorphic!*

In spite of the previous negative result, we show now that there are renormings of c_0, in fact small perturbations of the original norm, that are envelopes of non-locally convex spaces with separating dual.

This shows that the set of Banach envelopes fails to be closed with respect to the Banach–Mazur distance.

Precisely, we show the following. (See Figure 2.)

Proposition 8. *For every $\varepsilon > 0$ there exists a non-locally convex quasi-Banach space with separating dual whose Banach envelope is $(1 + \varepsilon)$-isomorphic to c_0.*

Proof. The construction is a generalization of Kalton's original example (which is 2-isomorphic to c_0) and is based on (a refinement of) an observation used by Talagrand [**34**] to construct a pathological submeasure.

For each rational $\varepsilon > 0$ and each positive integer n for which $n\varepsilon$ is an integer, consider the set

$$S(\varepsilon, n) = \{1, 2, \ldots, (1 + \varepsilon)n\}$$

and let $\Omega = \Omega(\varepsilon, n)$ be the class of all subsets of $S(\varepsilon, n)$ having cardinal n. Finally, set $A_i = \{\omega \in \Omega : i \in \omega\}$ for $1 \leqslant i \leqslant (1 + \varepsilon)n$. □

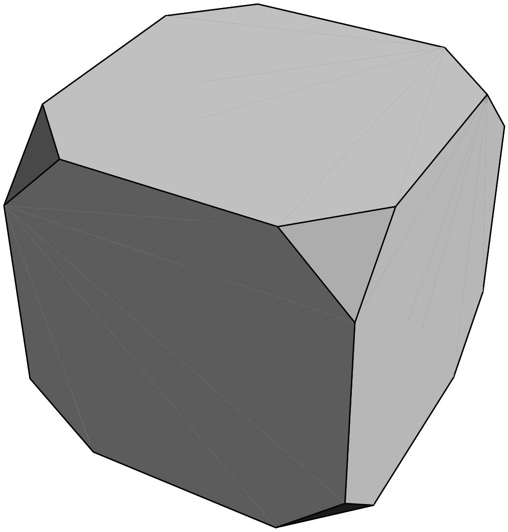

FIGURE 2. *The unit ball of* $\operatorname{co} X_p(\Omega)$ *in three dimensions.*

LEMMA 8. *The collection* $\{A_i\}_{i=1}^{(1+\varepsilon)n}$ *has the following properties.*

(a) *If* $J \subset \{1, 2, \dots, (1+\varepsilon)n\}$ *is such that* $|J| \leqslant \varepsilon n$, *then* $\cup_{i \in J} A_i \neq \Omega$.

(b) $\sum_{i=1}^{(1+\varepsilon)n} 1_{A_i} = n 1_\Omega$.

Proof. (a) Take an $\omega \in \Omega$ that does not intersect J and note that $\omega \in A_i$ if and only if $i \in \omega$. It is clear that $\omega \notin \cup_{i \in J} A_i$.

(b) Let ω be arbitrarily chosen in Ω. Then

$$\sum_{i=1}^{(1+\varepsilon)n} 1_{A_i}(\omega) = \sum_{i=1}^{(1+\varepsilon)n} 1_\omega(i) = |\omega| = n. \qquad \square$$

Now fix p, with $0 < p < 1$, and define a quasi-norm on the space $\ell_\infty(\Omega)$ by putting

$$\|f\|_p = \inf \left\{ \left(\sum_{i=1}^{(1+\varepsilon)n} |c_i|^p \right)^{1/p} : |f| \leqslant \sum_{i=1}^{(1+\varepsilon)n} c_i 1_{A_i} \right\}.$$

Let $X_p(\Omega)$ denote the space $\ell_\infty(\Omega)$ quasi-normed by $\|\cdot\|_p$.

LEMMA 9. (a) $X_p(\Omega)$ *is a* p*-normed lattice.*

(b) $\|f\|_\infty \leqslant \|f\|_p$ *for every* $f \in \ell_\infty(\Omega)$.

(c) $\|1_\Omega\|_p \geqslant (n^{1/p-1})\varepsilon^{1/p}/(1+\varepsilon)$.

(d) $\|f\|_\infty \leqslant \|f\|_{\operatorname{co}(X_p(\Omega))} \leqslant (1+\varepsilon)\|f\|_\infty$ *for all* f.

Proof. (a) and (b) are trivial. To verify (c), let $c_i \geqslant 0$ be so that $1_\Omega \leqslant \sum_{i=1}^{(1+\epsilon)n} c_i 1_{A_i}$. Put

$$J = \left\{ i : c_i \geqslant \frac{1}{(1+\varepsilon)n} \right\}.$$

We claim that $|J| > \varepsilon n$. Suppose on the contrary that $|J| \leqslant \varepsilon n$. By Lemma 8, there is $\omega \in \Omega$ so that $\omega \notin \cup_{i \in J} A_i$. Hence

$$\sum_{i=1}^{(1+\varepsilon)n} 1_{A_i}(\omega) = \sum_{i \notin J} 1_{A_i}(\omega) < \sum_{i \notin J} \frac{1}{(1+\varepsilon)n} < 1,$$

a contradiction. Therefore $|J| \geqslant \varepsilon n$ and

$$\sum_i c_i^p \geqslant \varepsilon n \left(\frac{1}{(1+\varepsilon)n} \right)^p = \frac{\varepsilon}{(1+\varepsilon)^p} n^{1-p},$$

which proves (c).

The first inequality in (d) is clear. As for the other, note that $\operatorname{co} X_p(\Omega)$ is a Banach lattice in its natural order. Let f be such that $\|f\|_\infty \leqslant 1$. Then $|f| \leqslant 1_\Omega$ and since $1_\Omega = (1/n) \sum_{i=1}^{(1+\varepsilon)n} 1_{A_i}$ one has

$$\|f\|_{\operatorname{co} X_p(\Omega)} \leqslant \|1_\Omega\|_{\operatorname{co} X_p(\Omega)} \leqslant \frac{1}{n} \sum_{i=1}^{(1+\varepsilon)n} \|1_{A_i}\|_{X_p(\Omega)} = 1 + \varepsilon,$$

from which the result follows. □

To end with the example, let $\varepsilon > 0$ be any rational number. Choose a sequence $(n_k)_k$ such that $\varepsilon n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Fix $0 < p < 1$ and define X to be the c_0-sum of the spaces $X_p(\Omega(\varepsilon, n_k))$, that is,

$$X = c_0(X_p(\Omega(\varepsilon, n_k))) = \left\{ (f_k)_k : f_k \in X_p(\Omega(\varepsilon, n_k)), \text{ with } \lim_k \|f_k\|_p = 0 \right\}$$

equipped with the p-norm

$$\|(f_k)\|_X = \sup_k \|f_k\|_p.$$

Clearly, X^* separates points in X. Since

$$\operatorname{co} X = \operatorname{co}(c_0(X_p(\Omega(\varepsilon, n_k)))) = c_0(\operatorname{co}(X_p(\Omega(\varepsilon, n_k)))),$$

it follows from Lemma 9 that X is not locally convex. On the other hand Lemma 9(d) shows that the convex envelope is $(1+\varepsilon)$-isomorphic to c_0. □

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