A Lifting Result for Locally Pseudo-Convex Subspaces of $L_0$

by

Félix CABELLO SÁNCHEZ

Presented by Aleksander PELCZYŃSKI

Summary. It is shown that if $F$ is a topological vector space containing a complete, locally pseudo-convex subspace $E$ such that $F/E = L_0$ then $E$ is complemented in $F$ and so $F = E \oplus L_0$. This generalizes results by Kalton and Peck and Faber.

Introduction. Let $L_0$ denote the space of all (equivalence classes of) measurable functions on $[0,1]$ equipped with the topology of convergence in measure, $E$ a closed subspace of $L_0$, and $T : L_0 \to L_0/E$ a (linear, continuous) operator. Under what conditions does $T$ lift to an operator $S : L_0 \to L_0$ in the sense that the diagram

\[
\begin{array}{ccc}
L_0 & \xrightarrow{S} & L_0 \\
\downarrow \pi & & \downarrow \\
L_0/E & \xrightarrow{T} & L_0/E
\end{array}
\]

commutes? As far as I know this problem was raised by Pełczyński. Kalton and Peck [5, Theorem 3.6] proved that such an $S$ exists if $E$ is locally bounded (that is, a quasi-Banach space); see also [6, Theorem 6.4]. The same is true if $E$ is isomorphic to $\omega$, the space of all sequences, as follows from results of Peck and Starbird [7, Corollary]. The interesting work of Domański about the

2000 Mathematics Subject Classification:

Key words and phrases: space of measurable functions, lifting, extension, pull-back, push-out.

Supported in part by DGICYT project MTM2004-02635.
structure of extensions [2] contains alternative proofs of both resuts. Finally, Faber [3, Theorem 2.1] got the corresponding result for locally convex $E$.

In this short note we generalize the previous results to locally pseudo-convex subspaces of $L_0$. Actually, we will show that if $E$ is locally pseudo-convex and complete and $F$ is any topological vector space (TVS) containing it, then every operator $L_0 \to F/E$ lifts to $F$. Thus, the fact that $E$ is a subspace of $L_0$ plays no role here. However we emphasize that there are locally pseudo-convex subspaces of $L_0$ that are neither locally convex nor locally bounded (nor even locally $p$-convex for any fixed $p$): $\prod_{n=1}^{\infty} L_p(n)$ is an example if the sequence $0 < p(n) \leq 2$ converges to zero.

In contrast to Faber’s proof (which is quite “hard” and depends on specific features of the locally convex subspaces of $L_0$) our result is obtained straightforwardly from the locally bounded case by means of the universal properties of three basic (and simple) homological constructions: pull-back, push-out and inverse limit.

Before going further we make some conventions. TVSs are assumed to be Hausdorff. Operator means linear and continuous map. If $E$ and $F$ are TVSs, then $L(E, F)$ denotes the space of all operators from $E$ to $F$. The identity on $E$ is written $1_E$.

Let us translate the problem into the language of extensions. An extension (of $G$ by $E$) is a short exact sequence of TVSs and relatively open operators

\[ 0 \to E \xrightarrow{i} F \xrightarrow{\pi} G \to 0. \tag{1} \]

Less technically we can regard $F$ as a TVS containing $E$ as a subspace in such a way that $F/E$ is (isomorphic to) $G$. We say that (1) splits if there is $S \in L(G, F)$ such that $\pi \circ S = 1_G$. And this happens if and only if there is $P \in L(F, E)$ such that $P \circ i = 1_E$; that is, if $iE$ is a complemented subspace of $F$.

We now describe the algebraic constructions we shall use in the proof. Some verifications are left to the reader. They are really easy; just try or adapt the corresponding proof for (quasi-) Banach spaces in [4] or [1, Appendix].

1. The pull-back extension. Suppose we are given an extension (1) and an operator $L : H \to G$, where $H$ is a TVS. Then we can construct a commutative diagram

\[ 0 \longrightarrow E \xrightarrow{i} F \xrightarrow{\pi} G \longrightarrow 0 \]

\[ 0 \longrightarrow E \xrightarrow{\text{PB}} \xrightarrow{\pi_H} H \longrightarrow 0 \quad \tag{2} \]
as follows: the pull-back space is \( PB = \{(f, h) \in F \times H : \pi f = L h\} \), with the relative product topology. The maps from \( PB \) are the restrictions of the projections. The map \( E \to PB \) is just \( e \mapsto (\iota(e), 0) \). It is easily verified that the lower row in (2) is an extension which splits if and only if \( L \) lifts to \( F \). And this is so by the following universal property of the pull-back square: if \( I \) is a TVS and \( \alpha \) and \( \beta \) are operators making the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\pi} & G \\
\uparrow & & \uparrow L \\
I & \xrightarrow{\beta} & H
\end{array}
\]

commutative, then there is a unique operator \( \gamma : I \to PB \) such that \( \alpha = \pi_F \circ \gamma \) and \( \beta = \pi_H \circ \gamma \) (the converse is obvious).

Hence the following statements about a pair of TVSs \( E \) and \( H \) are equivalent:

- Whenever \( F \) is a TVS containing \( E \) every operator \( H \to F/E \) lifts to \( F \).
- Every extension \( 0 \to E \to I \to H \to 0 \) splits.

Thus, the promised generalization of Faber’s result is contained in the following:

**Fact.** Every extension of \( L_0 \) by a complete, locally pseudo-convex space splits.

Before going into the proof, let us describe

2. The push-out extension. The push-out construction is just the categorical dual of the pull-back. So assume we are given an extension (1) and an operator \( T : E \to J \). The push-out of the operators \( \iota \) and \( T \) is the quotient space \( PO = (F \oplus J)/\Delta \), where \( \Delta = \{-\iota(e) + T(e) : e \in E\} \). In our setting \( \Delta \) is closed because \( \iota \) has closed range. We have a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{} & E & \xrightarrow{\iota} & F & \xrightarrow{\pi} & G & \xrightarrow{} & 0 \\
\downarrow T & & \downarrow \iota_F & & \Downarrow \pi_F & & \Downarrow & & \\
0 & \xrightarrow{} & J & \xrightarrow{\iota_J} & PO & \xrightarrow{} & G & \xrightarrow{} & 0
\end{array}
\]

The arrows ending in \( PO \) are induced by the inclusions of \( F \) and \( J \) into their direct sum \( F \oplus J \). The operator \( PO \to G \) sends \( (f \oplus j) + \Delta \) to \( \pi(f) \). This is clearly a quotient operator and it is easily seen that the lower sequence in (3) is an extension. Moreover this extension splits if and only if \( T \) extends to \( F \) (in the sense that there is \( \tau \in L(F, J) \) such that \( \tau \circ \iota = T \)). Again, this is immediate from the universal property of the push-out construction: if \( \alpha \)
and $\beta$ are operators making the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{r} & F \\
\downarrow T & & \downarrow \alpha \\
J & \xrightarrow{\beta} & K
\end{array}
\]

commutative, then there is a unique operator $\gamma : PO \to K$ such that $\alpha = \gamma \circ r_F$ and $\beta = \gamma \circ r_J$ (the converse is obvious).

3. The inverse limit. The topology of a locally pseudo-convex space $E$ can be obtained through a system of functions

$$
\varrho : E \to \mathbb{R}^+ \quad (\varrho \in \Gamma),
$$

where each $\varrho$ is a homogeneous semi-$p_\varrho$-norm [8, Theorem 3.14]. We may assume that given $\alpha, \beta \in \Gamma$ there is $\delta \in \Gamma$ such that $\delta \geq \alpha, \beta$ (in the pointwise sense). For $\varrho \in \Gamma$, let $E_\varrho$ denote the completion of $E/\ker \varrho$. This is clearly a $p_\varrho$-Banach space and we have an obvious operator $\pi_\varrho : E \to E_\varrho$. Moreover, if $\alpha \geq \beta$ the map $\pi_\beta$ factors through $E_\alpha$ and we have a further operator $\pi_\alpha^\beta : E_\alpha \to E_\beta$. It is clear that these form a projective system in the sense that for $\alpha \geq \beta \geq \gamma$ the map $E_\alpha \to E_\gamma$ coincides with the composition $E_\alpha \to E_\beta \to E_\gamma$.

Just as in the locally convex case, it is easily seen that if $E$ is complete, then it is isomorphic to the inverse (projective) limit of the system $\{E_\gamma : \gamma \in \Gamma\}$, that is, the space

$$
\text{proj } E_\gamma = \left\{(e_\gamma) \in \prod E_\gamma : \pi_\beta^\alpha(e_\alpha) = e_\beta \text{ for all } \alpha \geq \beta\right\}
$$

equipped with the relative product topology. We leave to the reader the verification that the map $e \in E \mapsto (\pi_\gamma(e))_\gamma \in \prod E_\gamma$ defines an isomorphism between $E$ and $\text{proj } E_\gamma$. Every operator $T : F \to E$ gives rise to a system of operators $T_\gamma : F \to E_\gamma$ (namely, $T_\gamma = \pi_\gamma \circ T$), compatible in the sense that for $\alpha \geq \beta$ we have $T_\beta = \pi_\beta^\alpha \circ T_\alpha$.

The universal property of the inverse limit states the converse: if $T_\gamma : F \to E_\gamma$ is a compatible system, then there is a unique operator $T : F \to E$ such that $T_\gamma = \pi_\gamma \circ T$.

Proof of the Fact. Let $E$ be a complete, locally pseudo-convex space. We show that every extension

$$
0 \to E \xrightarrow{r} F \xrightarrow{p} L_0 \to 0
$$

splits. If $\varrho$ is a semi-$p$-norm on $E$ we can apply the push-out procedure to
\( \pi_\varnothing \) and obtain the diagram

\[
\begin{array}{c}
0 \longrightarrow E \xrightarrow{\pi_\varnothing} F \xrightarrow{\pi} L_0 \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow E_\varnothing \longrightarrow PO \longrightarrow L_0 \longrightarrow 0
\end{array}
\]

We know from [5] that the push-out extension splits and so there is \( P_\varnothing : F \to E_\varnothing \) such that \( \pi_\varnothing = \iota \circ P_\varnothing \). In fact \( P_\varnothing \) is unique: for if \( P : F \to E_\varnothing \) is another extension of \( \pi_\varnothing \) we have \((P - P_\varnothing) \circ \iota = 0 \) and so \( P - P_\varnothing \) factors through \( L_0 \). But the only operator from \( L_0 \) to a quasi-Banach space is zero, and so \( P = P_\varnothing \).

We claim that the system \((P_\gamma)_{\gamma \in \Gamma} \) defines an operator \( P : F \to E \) such that \( P \circ \iota = 1_E \). Suppose \( \alpha \geq \beta \) and let \( P_\alpha \) and \( P_\beta \) be as above. We have \( \pi_\alpha = P_\alpha \circ \iota \) and \( \pi_\beta = P_\beta \circ \iota \). Since \( \pi_\beta = \pi_\alpha \circ \pi_\alpha \) we have \( \pi_\beta = \pi_\alpha \circ P_\alpha \circ \iota \) and by the uniqueness of \( P_\beta \) we see that \( P_\beta = \pi_\beta \circ P_\alpha \). This implies that there is an operator \( P : F \to E \) such that \( P_\gamma = \pi_\gamma \circ P \) for all \( \gamma \in \Gamma \), which clearly implies that \( P \circ \iota = 1_E \) and completes the proof.

**Concluding remarks.** Of course, the result just proved implies that if \( E \) and \( F \) are locally pseudo-convex (closed) subspaces of \( L_0 \) such that \( L_0/E \) and \( L_0/F \) are isomorphic, then there is an automorphism of \( L_0 \) mapping \( E \) onto \( F \).

Let us say that a TVS \( G \) has \( L_0 \)-structure if for every neighborhood of the origin \( U \) there is a topological decomposition \( G = G_1 \oplus \cdots \oplus G_k \) with \( G_i \subset U \) for \( 1 \leq i \leq k \). By [5, Theorem 3.6] (or [2, Proposition 4.3]) every extension of such a \( G \) by any quasi-Banach space splits. Moreover, there is no nonzero operator from \( G \) into any quasi-Banach space, and so the above proof shows that every extension of \( G \) by a complete, locally pseudo-convex space splits. The condition on the operators cannot be removed: indeed, \( \omega \) has “almost” \( L_0 \)-structure; if \( U \) is a neighborhood of zero, we can write \( \omega = F \oplus G \), where \( F \) is finite-dimensional and \( G \subset U \). It follows that every extension of \( \omega \) by a quasi-Banach space splits. However, it is shown in [2] (see the counterexamples on p. 166) that there exists an extension \( 0 \to E \to F \to \omega \to 0 \) in which \( F \) (and so \( E \)) is a Fréchet space that does not split.

The completeness hypothesis is also necessary in the Fact. Indeed, assume \( E \) is locally pseudo-convex but not complete and let \( \hat{E} \) be its completion (clearly locally pseudo-convex). Consider the extension \( 0 \to E \to \hat{E} \to \hat{E}/E \to 0 \), where the quotient space carries the trivial topology (the only open sets are the empty one and the whole space). Now, let \( T : L_0 \to \hat{E}/E \) be any nonzero linear map; this is clearly an operator that cannot be lifted to \( \hat{E} \) since \( L(L_0, \hat{E}) = 0 \). Thus, the lower extension in the pull-back diagram
(which can be defined as in the Hausdorff case and has the same properties)

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & \tilde{E} & \longrightarrow & \pi \ E & \longrightarrow & 0 \\
\| & & \uparrow & & \uparrow & & \uparrow T & & \\
0 & \longrightarrow & E & \longrightarrow & \text{PB} & \longrightarrow & L_0 & \longrightarrow & 0
\end{array}
\]

does not split. This is clearly a rewording of [2, “only if” part of Proposition 4.3(c)].

We close with the following

**Problem.** Does every extension \( 0 \rightarrow L_0 \rightarrow F \rightarrow L_0 \rightarrow 0 \) split?

**Acknowledgements.** I thank Jesús M. F. Castillo for explaining to me—again—how a projective limit works (and many other things), and Javier Cabello Sánchez for reading a preliminary \LaTeX{}-script of this note.

**References**


Félix Cabello Sánchez
Departamento de Matemáticas
Universidad de Extremadura
Avenida de Elvas
06071 Badajoz, Spain
E-mail: fcabello@unex.es
Web: http://kohmogorov.unex.es/~fcabello

Received March 15, 2006;
received in final form October 2006