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FUNCTIONAL ANALYSIS

A Lifting Result for Locally Pseudo-Convex Subspaces of L_0

by

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Summary. It is shown that if F is a topological vector space containing a complete, locally pseudo-convex subspace E such that $F/E = L_0$ then E is complemented in F and so $F = E \oplus L_0$. This generalizes results by Kalton and Peck and Faber.

Introduction. Let L_0 denote the space of all (equivalence classes of) measurable functions on $[0,1]$ equipped with the topology of convergence in measure, E a closed subspace of L_0, and $T : L_0 \to L_0/E$ a (linear, continuous) operator. Under what conditions does T lift to an operator $S : L_0 \to L_0$ in the sense that the diagram

$$\begin{tikzcd} & L_0 \arrow[d, "\pi"] \\ L_0 \arrow[ur, "S"] \arrow[r, "T"] & L_0/E \end{tikzcd}$$

commutes? As far as I know this problem was raised by Pełczyński. Kalton and Peck [5, Theorem 3.6] proved that such an S exists if E is locally bounded (that is, a quasi-Banach space); see also [6, Theorem 6.4]. The same is true if E is isomorphic to ω, the space of all sequences, as follows from results of Peck and Starbird [7, Corollary]. The interesting work of Domański about the

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structure of extensions [2] contains alternative proofs of both resuls. Finally, Faber [3, Theorem 2.1] got the corresponding result for locally convex E.

In this short note we generalize the previous results to locally pseudo-convex subspaces of L_0. Actually, we will show that if E is locally pseudo-convex and complete and F is any topological vector space (TVS) containing it, then every operator $L_0 \to F/E$ lifts to F. Thus, the fact that E is a subspace of L_0 plays no rôle here. However we emphasize that there are locally pseudo-convex subspaces of L_0 that are neither locally convex nor locally bounded (nor even locally p-convex for any fixed p): $\prod_{n=1}^{\infty} L_{p(n)}$ is an example if the sequence $0 < p(n) \leq 2$ converges to zero.

In contrast to Faber's proof (which is quite "hard" and depends on specific features of the locally convex subspaces of L_0) our result is obtained straightforwardly from the locally bounded case by means of the universal properties of three basic (and simple) homological constructions: pull-back, push-out and inverse limit.

Before going further we make some conventions. TVSs are assumed to be Hausdorff. Operator means linear and continuous map. If E and F are TVSs, then $L(E, F)$ denotes the space of all operators from E to F. The identity on E is written 1_E.

Let us translate the problem into the language of extensions. An *extension* (of G by E) is a short exact sequence of TVSs and relatively open operators

$$0 \to E \xrightarrow{\imath} F \xrightarrow{\pi} G \to 0. \tag{1}$$

Less technically we can regard F as a TVS containing E as a subspace in such a way that F/E is (isomorphic to) G. We say that (1) *splits* if there is $S \in L(G, F)$ such that $\pi \circ S = 1_G$. And this happens if and only if there is $P \in L(F, E)$ such that $P \circ \imath = 1_E$, that is, if $\imath E$ is a complemented subspace of F.

We now describe the algebraic constructions we shall use in the proof. Some verifications are left to the reader. They are really easy: just try or adapt the corresponding proof for (quasi-) Banach spaces in [4] or [1, Appendix].

1. The pull-back extension. Suppose we are given an extension (1) and an operator $L : H \to G$, where H is a TVS. Then we can construct a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "\imath"] & F \arrow[r, "\pi"] & G \arrow[r] & 0 \\ 0 \arrow[r] & E \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "\pi_H"] \arrow[u, "\pi_F"] & H \arrow[r] \arrow[u, "L"] & 0 \end{tikzcd} \tag{2}$$

as follows: the pull-back space is $\mathrm{PB} = \{(f,h) \in F \times H : \pi f = Lh\}$, with the relative product topology. The maps from PB are the restrictions of the projections. The map $E \to \mathrm{PB}$ is just $e \mapsto (\imath(e), 0)$. It is easily verified that the lower row in (2) is an extension which splits if and only if L lifts to F. And this is so by the following universal property of the pull-back square: if I is a TVS and α and β are operators making the diagram

$$\begin{array}{ccc} F & \xrightarrow{\pi} & G \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle L} \\ I & \xrightarrow{\beta} & H \end{array}$$

commutative, then there is a unique operator $\gamma : I \to \mathrm{PB}$ such that $\alpha = \pi_F \circ \gamma$ and $\beta = \pi_H \circ \gamma$ (the converse is obvious).

Hence the following statements about a pair of TVSs E and H are equivalent:

- Whenever F is a TVS containing E every operator $H \to F/E$ lifts to F.
- Every extension $0 \to E \to I \to H \to 0$ splits.

Thus, the promised generalization of Faber's result is contained in the following:

FACT. *Every extension of L_0 by a complete, locally pseudo-convex space splits.*

Before going into the proof, let us describe

2. The push-out extension. The push-out construction is just the categorical dual of the pull-back. So assume we are given an extension (1) and an operator $T : E \to J$. The push-out of the operators $\imath$ and T is the quotient space $\mathrm{PO} = (F \oplus J)/\Delta$, where $\Delta = \{-\imath(e) \oplus T(e) : e \in E\}$. In our setting Δ is closed because $\imath$ has closed range. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{\imath} & F & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \imath_F} & & \| & & \\ 0 & \longrightarrow & J & \xrightarrow{\imath_J} & \mathrm{PO} & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{3}$$

The arrows ending in PO are induced by the inclusions of F and J into their direct sum $F \oplus J$. The operator $\mathrm{PO} \to G$ sends $(f \oplus j) + \Delta$ to $\pi(f)$. This is clearly a quotient operator and it is easily seen that the lower sequence in (3) is an extension. Moreover this extension splits if and only if T extends to F (in the sense that there is $\tau \in L(F,J)$ such that $\tau \circ \imath = T$). Again, this is immediate from the universal property of the push-out construction: if α

and β are operators making the diagram

$$\begin{array}{ccc} E & \xrightarrow{\imath} & F \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \alpha} \\ J & \xrightarrow{\beta} & K \end{array}$$

commutative, then there is a unique operator $\gamma : \mathrm{PO} \to K$ such that $\alpha = \gamma \circ \imath_F$ and $\beta = \gamma \circ \imath_J$ (the converse is obvious).

3. The inverse limit. The topology of a locally pseudo-convex space E can be obtained through a system of functions

$$\varrho : E \to \mathbb{R}^+ \qquad (\varrho \in \Gamma),$$

where each ϱ is a homogeneous semi-p_ϱ-norm [8, Theorem 3.1.4]. We may assume that given $\alpha, \beta \in \Gamma$ there is $\delta \in \Gamma$ such that $\delta \geq \alpha, \beta$ (in the pointwise sense). For $\varrho \in \Gamma$, let E_ϱ denote the completion of $E/\ker \varrho$. This is clearly a p_ϱ-Banach space and we have an obvious operator $\pi_\varrho : E \to E_\varrho$. Moreover, if $\alpha \geq \beta$ the map π_β factors through E_α and we have a further operator $\pi_\beta^\alpha : E_\alpha \to E_\beta$. It is clear that these form a projective system in the sense that for $\alpha \geq \beta \geq \gamma$ the map $E_\alpha \to E_\gamma$ coincides with the composition $E_\alpha \to E_\beta \to E_\gamma$.

Just as in the locally convex case, it is easily seen that if E is complete, then it is isomorphic to the inverse (projective) limit of the system $\{E_\gamma : \gamma \in \Gamma\}$, that is, the space

$$\operatorname{proj} E_\gamma = \Big\{ (e_\gamma) \in \prod E_\gamma : \pi_\beta^\alpha(e_\alpha) = e_\beta \text{ for all } \alpha \geq \beta \Big\}$$

equipped with the relative product topology. We leave to the reader the verification that the map $e \in E \mapsto (\pi_\gamma(e))_\gamma \in \prod E_\gamma$ defines an isomorphism between E and $\operatorname{proj} E_\gamma$. Every operator $T : F \to E$ gives rise to a system of operators $T_\gamma : F \to E_\gamma$ (namely, $T_\gamma = \pi_\gamma \circ T$), compatible in the sense that for $\alpha \geq \beta$ we have $T_\beta = \pi_\beta^\alpha \circ T_\alpha$.

The universal property of the inverse limit states the converse: if $T_\gamma : F \to E_\gamma$ is a compatible system, then there is a unique operator $T : F \to E$ such that $T_\gamma = \pi_\gamma \circ T$.

Proof of the Fact. Let E be a complete, locally pseudo-convex space. We show that every extension

$$0 \to E \xrightarrow{\imath} F \xrightarrow{\pi} L_0 \to 0$$

splits. If ϱ is a semi-p-norm on E we can apply the push-out procedure to

π_ϱ and obtain the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{\imath} & F & \xrightarrow{\pi} & L_0 & \longrightarrow & 0 \\
 & & {\scriptstyle\pi_\varrho}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E_\varrho & \longrightarrow & \mathrm{PO} & \longrightarrow & L_0 & \longrightarrow & 0
\end{array}$$

We know from [5] that the push-out extension splits and so there is $P_\varrho : F \to E_\varrho$ such that $\pi_\varrho = \imath \circ P_\varrho$. In fact P_ϱ is unique: for if $P : F \to E_\varrho$ is another extension of π_ϱ we have $(P - P_\varrho) \circ \imath = 0$ and so $P - P_\varrho$ factors through L_0. But the only operator from L_0 to a quasi-Banach space is zero, and so $P = P_\varrho$.

We claim that the system $(P_\gamma)_\gamma \in \Gamma$ defines an operator $P : F \to E$ such that $P \circ \imath = 1_E$. Suppose $\alpha \geq \beta$ and let P_α and P_β be as above. We have $\pi_\alpha = P_\alpha \circ \imath$ and $\pi_\beta = P_\beta \circ \imath$. Since $\pi_\beta = \pi_\beta^\alpha \circ \pi_\alpha$ we have $\pi_\beta = \pi_\beta^\alpha \circ P_\alpha \circ \imath$ and by the uniqueness of P_β we see that $P_\beta = \pi_\beta^\alpha \circ P_\alpha$. This implies that there is an operator $P : F \to E$ such that $P_\gamma = \pi_\gamma \circ P$ for all $\gamma \in \Gamma$, which clearly implies that $P \circ \imath = 1_E$ and completes the proof.

Concluding remarks. Of course, the result just proved implies that if E and F are locally pseudo-convex (closed) subspaces of L_0 such that L_0/E and L_0/F are isomorphic, then there is an automorphism of L_0 mapping E onto F.

Let us say that a TVS G has L_0-*structure* if for every neighborhood of the origin U there is a topological decomposition $G = G_1 \oplus \cdots \oplus G_k$ with $G_i \subset U$ for $1 \leq i \leq k$. By [5, Theorem 3.6] (or [2, Proposition 4.3]) every extension of such a G by any quasi-Banach space splits. Moreover, there is no nonzero operator from G into any quasi-Banach space, and so the above proof shows that every extension of G by a complete, locally pseudo-convex space splits. The condition on the operators cannot be removed: indeed, ω has “almost” L_0-structure: if U is a neighborhood of zero, we can write $\omega = F \oplus G$, where F is finite-dimensional and $G \subset U$. It follows that every extension of ω by a quasi-Banach space splits. However, it is shown in [2] (see the counterexamples on p. 166) that there exists an extension $0 \to E \to F \to \omega \to 0$ in which F (and so E) is a Fréchet space that does not split.

The completeness hypothesis is also necessary in the Fact. Indeed, assume E is locally pseudo-convex but not complete and let $\widehat{E}$ be its completion (clearly locally pseudo-convex). Consider the extension $0 \to E \to \widehat{E} \to \widehat{E}/E \to 0$, where the quotient space carries the trivial topology (the only open sets are the empty one and the whole space). Now, let $T : L_0 \to \widehat{E}/E$ be any nonzero linear map; this is clearly an operator that cannot be lifted to $\widehat{E}$ since $L(L_0, \widehat{E}) = 0$. Thus, the lower extension in the pull-back diagram

(which can be defined as in the Hausdorff case and has the same properties)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \widehat{E} & \xrightarrow{\pi} & \widehat{E}/E & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle T} & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathrm{PB} & \longrightarrow & L_0 & \longrightarrow & 0 \end{array}$$

does not split. This is clearly a rewording of [2, "only if" part of Proposition 4.3(c)].

We close with the following

Problem. Does every extension $0 \to L_0 \to F \to L_0 \to 0$ split?

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