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Yet another proof of Sobczyk's theorem

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As everybody knows Sobczyk theorem states that c_0, the space of null sequences with the maximum norm, is 2-complemented in any separable Banach space containing it. Equivalently, if Y is a subspace of a separable Banach space X, then every operator $T : Y \to c_0$ has an extension $\tilde{T} : X \to c_0$ with $\|\tilde{T}\| \leq 2\|T\|$.

Since the publication of [**14**] many proofs of Sobczyk theorem appeared. The survey [**4**] contains every proof whose authors (or some of them) knew when the paper was completed.

After that at least three papers extending Sobczyk theorem to the vector valued setting appeared, namely [**12, 10, 5**]. In my opinion, Rosenthal's proof is very close in spirit to the original arguments of Sobczyk (see [**4**, 2.7]). The proof given by Johnson and Oikhberg reminds me of the proof of Werner [**4**, 2.9]. The proof by Castillo and Moreno has something to do with the argument in [**4**, 2.10].

Before going further, let us say, following Rosenthal, that E has the separable extension property (SEP) if every operator $T : Y \to E$ from a subspace of a separable Banach space X can be extended to an operator $\tilde{T} : X \to E$. If this can be achieved with $\|\tilde{T}\| \leq \lambda\|T\|$ we say that E has the λ-SEP. It is a deep result of Zippin [**15, 16**] that c_0 is the only separable space with the SEP, but there are nonseparable spaces with the SEP: the most obvious ones are injective spaces such as ℓ_∞. It follows from results by Aronszajn and Panitchpakdi [**1**] that $C(K)$ has the 1-SEP if and only if K is an F-space, that is, a compact space where disjoint cozero sets are completely separated (see also [**8, 11, 13**]). Thus, for instance, $\ell_\infty/c_0 = C(\beta\mathbb{N}\backslash\mathbb{N})$ has the 1-SEP. The 'isomorphic part' also follows from recent work by Castillo, Moreno and Suárez [**6**]: the quotient of a space with the SEP by a subspace with the SEP has the SEP. Finally, extensions of two spaces with the SEP have the SEP (see [**3**]). This includes the Johnson-Lindenstrauss space $J\!\!L_\infty$ or the space contructed by Benyamini in [**2**].

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Each of the papers [**12, 10, 5**] just mentioned contains a proof of the fact that if (E_n) is a sequence of Banach spaces having the λ-SEP then $c_0(E_n)$ has the Λ-SEP, where $\Lambda = \Lambda(\lambda)$ (but to be true, each paper considers a different property equivalent to the SEP and proves much more than this).

In this short note I present a new proof of this result based on a kind of open mapping theorem for the strong operator topology. Some preparation is necesary, however, to uncover the core of the argument. First of all, note that it suffices to prove the corresponding result when all the E_n are the same space, say E. For if E_n has the λ-SEP for all n, then $E = \ell_\infty(E_n)$ has it and if $c_0(E)$ has the Λ-SEP then so do all its 1-complemented subspaces, in particular $c_0(E_n)$. Thus our result is as follows:

Theorem 1. *It E has the SEP then so does $c_0(E)$.*

Proof

Notice we have not quantified the SEP in the statement. This is unnecessary because of the following typical result —which answers a question curiously posed in [**12**]. The proof is due to J.M.F. Castillo.

Proposition 1. *If E has the SEP then it has the λ-SEP for some λ.*

Proof. A set of real numbers is bounded if it contains bounded sequences only, hence it suffices to prove that if X_n is a sequence of separable Banach spaces with subspaces Y_n and $T_n : Y_n \to E$ are operators with $\|T_n\| \leq 1$, then there are extensions $\tilde{T}_n : X_n \to E$ such that $\|\tilde{T}_n\| \leq \lambda$, for some finite λ. Fortunately direct sums do exist in the category of Banach spaces: consider the sum operator $T : \ell_1(Y_n) \to E$ given by $T((y_n)) = \sum_n T_n y_n$ and extend it to $\ell_1(X_n)$. □

We pass to the proof of Theorem 1. Let us begin with the observation that an operator $T : Y \to c_0(E)$ is given by a sequence $T_n : Y \to E$ such that $T_n(y) \to 0$ for all $y \in Y$. This just means that the sequence (T_n) converges to zero in the strong operator topology we briefly describe. Let A and B be Banach spaces and $L(A, B)$ the corresponding space of operators. The closed ball of radius r in that space will be denoted $L(A, B)_r$. The strong operator topology (SOT) in $L(A, B)$ is the smallest linear topology for which the sets

$$U = \{T \in L(A, B) : \|Tx\| \leq 1\} \qquad (x \in A)$$

are neighborhoods of the origin. The SOT is never metrizable (unless A is finite dimensional: in this case the SOT agrees with the usual norm topology). However, it is metrizable on bounded sets provided A is separable. Indeed, if (x_n) is bounded and spans a dense subspace of A, then the norm

$$T \longmapsto \sum_{n=1}^{\infty} \frac{\|Tx_n\|}{2^n}$$

induces the (relative) SOT on bounded sets of $L(A, B)$. It is clear that a sequence $T_n : A \to B$ is convergent to zero in the SOT if and only if $T_n x \to 0$

for all $x \in A$ (this already implies that (T_n) is uniformly bounded, by the uniform boundedness principle). Hence we can restate Theorem 1 as follows: if E has the SEP and Y is a subspace of a separable space X, then every SOT-null sequence $T_n : Y \to E$ extends to a SOT-null sequence $\tilde{T}_n : X \to E$. Note that $c_0(E)$ has the λ-SEP iff and only if this can be done with $\|\tilde{T}_n\| \leq \lambda$ provided $\|T_n\| \leq 1$.

The following simple lemma gives a simple criterion for the existence of such extensions.

Lemma 1. *Let $R : S \to T$ be a continuous mapping between topological spaces such that $R(s) = t$. Let S' (resp. T') be a metrizable subset of S (resp. T) containing s (resp. t). The following are equivalent:*

(a) *Every sequence (t_n) converging to t in T' is the image under R or some sequence (s_n) converging to s in S'.*
(b) *R is relatively open at s: if U is a neighborhood of s in S, then $R(U \cap S')$ contains a neighborhood of t in T'.*
(c) *R admits a section (relatively) continuous at t: there is a mapping $\varrho : T' \to S'$ continuous at t and such that $\varrho(t) = s$ and $R \circ \varrho = 1_{T'}$.*

Proof. Let us see that (a) implies (b). If (b) fails there is a neighborhood U of s relative to S' such that t is not interior to $R(U) \cap T'$. Hence there is a sequence $t_n \to t$ with t_n outside $R(U) \cap T'$ for all n. It is clear that (t_n) cannot be the image of any sequence converging to s in S'.

Now we prove that (b) implies (c). This part of the proof borrows from [**7**, Proof of Lemma 2.2]. Let (U_n) be a decreasing base for the topology of S' at s, with $U_1 = S'$. Since R is continuous at s (b) implies that $V_n = R(U_n) \cap T'$ is also a decreasing base of neighborhoods of t in T'. For each n, let $\varrho_n : V_n \to U_n$ any map such that $R \circ \varrho_n = 1_{V_n}$, with $\varrho_n(t) = s$. Now, for $y \neq t$ in T', put $n(y) = \max\{n : y \in V_n\}$. Finally, define $\varrho : T' \to S'$ taking

$$\varrho(y) = \varrho_{n(y)}(y)$$

(and $\varrho(t) = s$). It is clear that $R \circ \varrho = 1_{T'}$ and also that ϱ is continuous at t.

That (c) implies (a) is obvious: if $t_n \to t$ in T', then $s_n = \varrho(t_n)$ is a sequence converging to s whose image under R is the starting sequence. □

Of course this applies to restriction operators (they are always SOT-continuous):

Lemma 2. *Let Y be a subspace of a separable Banach space X and let E be another Banach space. If $R : L(X,E) \to L(Y,E)$ denotes the restriction map, then the following are equivalent:*

(d) *Every sequence of operators $T_n : Y \to E$ converging to zero in the SOT with $\|T_n\| \leq 1$ for all n is the restriction of some SOT null sequence $\tilde{T}_n : X \to E$ with $\|\tilde{T}_n\| \leq \kappa$.*
(e) *Every operator $T : Y \to c_0(E)$ has an extension $\tilde{T} : X \to c_0(E)$ with $\|\tilde{T}\| \leq \kappa \|T\|$.*

(f) *If U is a neighborhood of the origin in the SOT of $L(X,E)$, then $R(U\cap L(X,E)_\kappa)$ contains a neighborhood of the origin in the relative SOT of $L(Y,E)_1$.* □

We are now ready to prove Theorem 1. We shall show that if E has the λ-SEP, then $c_0(E)$ has the $(3\lambda^2+\varepsilon)$-SEP for all $\varepsilon>0$. It suffices to verify that (f) holds true for $\kappa=3\lambda^2+\varepsilon$. The proof is accomplished in two steps. Once U has been fixed we choose an intermediate $\tilde X$ (depending on U) such that $Y\subset\tilde X\subset X$ with $\tilde X/Y$ finite dimensional. This induces a decomposition of R as the composition

$$L(X,E)\xrightarrow{\ R_1\ } L(\tilde X,E)\xrightarrow{\ R_2\ } L(Y,E),$$

where R_i are the corresponding restriction operators. That $R_1(U)$ is large enough will follow from the choice of $\tilde X$. That $R_2(R_1(U))$ is large enough will follow from very easy finite dimensional considerations, thanks to the implication (e) $\Rightarrow$ (f) in Lemma 2.

First step. Let U be a neighborhood of the origin in the SOT of $L(X,E)$. Without loss of generality we may assume that, for some $x_1,\dots,x_k\in X$, one has

$$U=\{T\in L(X,E):\|Tx_i\|\le 1 \text{ for } 1\le i\le k\}$$

Let $\tilde X$ denote the least subspace of X containing $x_1,\dots,x_k$ and Y. By the very definition of λ-SEP we see that for each $r>0$ the set $R_1(U\cap L(X,E)_{\lambda r})$ contains $\tilde U\cap L(\tilde X,E)_r$, where

$$\tilde U=\{T\in L(\tilde X,E):\|Tx_i\|\le 1 \text{ for } 1\le i\le k\}.$$

This was the first step. I emphasize that we have not proved that operators $\tilde X\to c_0(E)$ extend to X since $\tilde X$ depends on U. □

Second step. Finally, we prove that every operator $T:Y\to c_0(E)$ extends to an operator $\tilde T:\tilde X\to c_0(E)$ with $\|\tilde T\|\le(3+\varepsilon)\lambda\|T\|$. Let Z denote $\tilde X/Y$ and $\pi:\tilde X\to Z$ the natural quotient map. Since Z is finite dimensional there is a finite dimensional $F\subset\tilde X$ such that for every $z\in Z$ there is $f\in F$, with $\|f\|\le(1+\varepsilon)\|z\|$ such that $\pi(f)=z$. Writing $G=Y\cap F$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & F & \xrightarrow{\ \pi\ } & Z & \longrightarrow & 0\\ & & {\scriptstyle\imath_G}\big\downarrow & & {\scriptstyle\imath_F}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Y & \longrightarrow & \tilde X & \xrightarrow{\ \pi\ } & Z & \longrightarrow & 0 \end{array}$$

with exact rows. Notice that (the restriction) $T\circ\imath_G$ can be easily extended to an operator $\tilde T:F\to c_0(E)$ with $\|\tilde T\|\le\lambda\|T\|$: indeed, as G is finite dimensional, the corresponding sequence $T_n\circ\imath_G:G\to E$ converges to zero in norm! So, just extend in each coordinate, with $\|\tilde T_n\|\le\lambda\|T_n\circ\imath_G\|$. The rest is straightforward once one realizes that we have a push-out diagram (we refer the reader to [**9**] or [**3**] for explanations). Whatever this means the relevant information is that, due to the 'form' of the above diagram, $\tilde X$ is well isomorphic to a certain quotient of $Y \oplus F$ from where T extends easily. In any case some control of the constants is needed and so we verify these facts by hand.

So consider the direct sum space $Y \oplus F$ (with the sum norm) and the map $S : Y \oplus F \to \tilde{X}$ sending (y, f) into $y + f$. Clearly $\|S\| \leq 1$. On the other hand, given $x \in \tilde{X}$ we can write $x = (x - f) + f$, where f is any element of F such that $\pi(f) = \pi(x)$, with $\|f\| \leq (1 + \varepsilon)\|x\|$. It follows that $x - f$ belongs to Y and $S(x - f, f) = x$, in particular S is onto, and we see that $\|(x - f, f)\| \leq (3 + \varepsilon)\|x\|$. Let $\Delta = \ker S$, that is:

$$\Delta = \{(y, f) \in Y \times F : y + f = 0\} = \{(g, -g) : g \in G\}.$$

The push-out space associated to the pair of operators (actually inclusions) $G \to Y$ and $G \to F$ is just $\mathrm{PO} = (Y \oplus F)/\Delta$. We have seen that it is $(3 + \varepsilon)$-isomorphic to $\tilde{X}$ via the sum map. Consider the map $L : Y \oplus F \to c_0(E)$ given by $L(y, f) = T(y) + \tilde{T}(f)$. It is clear that $\|L\| \leq \|\tilde{T}\| \leq \lambda\|T\|$ and also that L vanishes on Δ. Hence it defines a operator $\tilde{L} : \mathrm{PO} \to c_0(E)$ and composition with $S^{-1} : \tilde{X} \to \mathrm{PO}$ gives an extension of T to $\tilde{X}$ of norm at most $(3 + \varepsilon)\lambda\|T\|$. □

Concluding remarks

Theorem 1 suggests some questions.

- Must $D \otimes_\varepsilon E$ have the SEP if both D and E have it? By Theorem 1 this is so when $D = c_0$ since $c_0 \otimes_\varepsilon E = c_0(E)$, but I do not know even if $\ell_\infty \otimes_\varepsilon \ell_\infty$ has the SEP!
- Let Y be a subspace of a separable space X such that every operator $Y \to E$ extends to X. Does every operator $Y \to c_0(E)$ extend to X? By the main result in [**5**] the answer is 'yes' if X/Y has the BAP.
- Let Y be a subspace of a separable space X such that every operator $Y \to E_i$ extends to X $(i = 1, 2)$. Does every operator $Y \to E_1 \otimes_\varepsilon E_2$ extend to X?

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