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RESEARCH ARTICLE

Memorandum on multiplicative bijections and order

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Abstract Let $C(X,\mathbb{I})$ denote the semigroup of continuous functions from the topological space X to $\mathbb{I}=[0,1]$, equipped with the pointwise multiplication. The paper studies semigroup homomorphisms $C(Y,\mathbb{I})\to C(X,\mathbb{I})$, with emphasis on isomorphisms. The crucial observation is that, in this setting, homomorphisms preserve order, so isomorphisms preserve order in both directions and they are automatically lattice isomorphisms. Applications to uniformly continuous and Lipschitz functions on metric spaces are given. Sample result: if Y and X are complete metric spaces of finite diameter without isolated points, every multiplicative bijection $T:\mathrm{Lip}(Y,\mathbb{I})\to\mathrm{Lip}(X,\mathbb{I})$ has the form $Tf=f\circ\tau$, where $\tau:X\to Y$ is a Lipschitz homeomorphism.

Keywords Semigroups of continuous functions · Homomorphism · Representation

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Introduction

This paper contemplates $C(X, \mathbb{I})$ as a semigroup under pointwise multiplication. As usual, $C(X, \mathbb{I})$ denotes the set of all continuous functions from the topological space X to $\mathbb{I} = [0, 1]$. Besides its semigroup structure, the set $C(X, \mathbb{I})$ is a lattice with the pointwise order, that is, $f \leq g$ meaning $f(x) \leq g(x)$ for all $x \in X$. Also, as a subset of the Banach algebra of bounded continuous functions $f : X \to \mathbb{R}$, it is a topological space with the metric

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|.$$

We emphasize that such notions as 'homomorphism', 'isomorphism', and the like refer to the 'default' semigroup setting, unless otherwise stated.

Plan of the paper The main purpose of the paper is to describe semigroup isomorphisms (multiplicative bijections) $T : C(Y, \mathbb{I}) \longrightarrow C(X, \mathbb{I})$.

The crucial observation (Lemma 3) is that, in this setting, a homomorphism must preserve order, so isomorphisms preserve order in both directions and therefore they are automatically lattice isomorphisms. This allows one to use the basic material on lattices contained in Sect. 1.

From this we obtain that if X and Y are compact spaces, then there is a homeomorphism $\tau : X \to Y$ and a continuous function $\mathfrak{t} : X \to [0, \infty]$ such that

$$(f(\tau(x))^-)^{\mathfrak{t}(x)} \leq Tf(x) \leq (f(\tau(x))^+)^{\mathfrak{t}(x)}$$

for all $f \in C(Y, \mathbb{I})$ and all $x \in X$. Moreover, $0 < \mathfrak{t}(x) < \infty$ on a dense set where, a fortiori, one has

$$Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}. \tag{1}$$

In general we will not have this nice representation on the whole domain X, even if $Y = X$ is compact (Example 1). In fact, this holds if and only if T is continuous (Proposition 2).

It should be noted that some 'balance' condition on the couple Y, X is necessary to get a homeomorphism between X and Y out from an isomorphism T. Indeed, if X is completely regular and βX is its Stone-Čech compactification, then $C(X, \mathbb{I})$ and $C(\beta X, \mathbb{I})$ are obviously isomorphic while X is homeomorphic to βX only if X is compact.

We present some applications in Sect. 3. We give a complete description of the isomorphisms between the semigroups of continuous functions on metric spaces: if Y and X are metric spaces (compact or not), every isomorphism has a representation as in (1) for all $x \in X$. A similar result is obtained for uniformly continuous functions when Y and X are complete.

Although we have no complete description of the isomorphisms of semigroups of Lipschitz functions, let us mention the following sample result (Corollary 2): if Y and X are complete metric spaces of finite diameter without isolated points, then every semigroup isomorphism $T : \mathrm{Lip}(Y, \mathbb{I}) \to \mathrm{Lip}(X, \mathbb{I})$ has the form $Tf = f \circ \tau$, where $\tau : X \to Y$ is a Lipschitz homeomorphism—there is no $\mathfrak{t}$ here!

Our closing application concerns measurable functions. Under rather mild assumptions on the measure spaces $(Y, \mathfrak{Y}, \nu)$ and $(X, \mathfrak{X}, \mu)$ we show that every multiplicative mapping $T : L^\infty(\mu, \mathbb{I}) \to L^\infty(\nu, \mathbb{I})$ has the form given in (1), where $\tau : X \to Y$ is a measurable isomorphism and $\mathfrak{t} : X \to (0, \infty)$ a measurable function. We decided to present a simple, self-contained, (lattice-free), proof because this result has some interest in operator theory and rounds-off earlier results by Molnár on effect algebras. See Sect. 3 for unexplained terms.

Predecessors As the reader can imagine this is not the first paper on semigroups of continuous functions. To the best of our knowledge the first one is Milgram's [14], a paper of classical elegance where it is shown that if Y and X are compact spaces and $T : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ is a semigroup isomorphism, then there is a homeomorphism $\tau : X \to Y$, a finite (possibly empty) set $S \subset X$ of isolated points, a continuous function $\mathfrak{t} : X \backslash S \to (0, \infty)$, and for each $x \in S$ an automorphism σ_x of $\mathbb{R}$, such that

$$Tf(x) = \begin{cases} f(\tau(x))^{\mathfrak{t}(x)} & \text{for } x \in X \backslash S, \\ \sigma_x(f(\tau(x))) & \text{for } x \in S. \end{cases}$$

Milgram uses what he called O-ideals. Although this approach could be used in our setting, dealing with functions taking values in $\mathbb{I}$ is more difficult: for instance, the statement 'f vanishes at some point of X' cannot be expressed within the semigroup structure of $C(X, \mathbb{I})$. Of course that statement is equivalent to 'f has no inverse' in $C(X, \mathbb{R})$. See Remark 1 for more on this.

Later on, Shirota [18] stated that two realcompact spaces Y and X are homeomorphic if and only if the semigroups $C(X, \mathbb{R})$ and $C(Y, \mathbb{R})$ are isomorphic. The argument given in [18] depends on lattice theory, but the proof seems to contain some gaps. A complete proof appeared much later in [7]. Curiously enough the proof by Császár is very close in spirit to Milgram's. To be true, Császár considers more general semigroups $C(X, \mathbb{S})$, where $\mathbb{S}$ is an adequate semigroup of real numbers and so Theorem 1 should be credited to him. Neither [18] nor [7] give any representation of the corresponding isomorphisms.

Very recently Marovt has published a paper [13] establishing the form on the automorphisms of $C(X, \mathbb{I})$, with X compact metric (roughly the statement of Theorem 3 but assuming $X = Y$ to be compact). Marovt's main motivation was the study of sequential isomorphisms of effect algebras. See Remark 3 for explanations.

Notations Topological spaces are assumed to be completely regular and Hausdorff. Operations and relations in $C(X, \mathbb{I})$ are defined 'pointwise': for instance $f < g$ means $f(x) < g(x)$ for all $x \in X$.

To avoid any possible confusion with exponents, given a mapping $f : X \to Y$ and $A \subset Y$, we write $f^{\leftarrow}(A)$ for the set $\{x \in X : f(x) \in A\}$ and we use the same notation for the inverse of f, provided it exists. The characteristic function of the set A is denoted 1_A, while the identity map on X is $\mathbf{1}_X$. The set of constant functions from X to $\mathbb{I}$ is denoted $\mathbb{I}_X$. Apart from these conventions our notation is standard: we follow Willard [21] for topology matters, Birkhoff [3] for lattice theory, Weaver [20] for Lipschitz functions, and Cohn [6] for measure theory.

Finally, we will use a recent result by Albiac and Kalton which allows us to represent certain real Banach algebras as the algebra of all real-valued continuous functions on a compact space: if $\mathfrak{A}$ is a (real, unital) Banach algebra whose norm satisfies the inequality

$$2\|fg\| \leq \|f^2 + g^2\| \quad (f, g \in \mathfrak{A}),$$

then, as a Banach algebra, $\mathfrak{A}$ is isometrically isomorphic to $C(X, \mathbb{R})$, for some compact space X. See [1, 2] for the remarkably simple proof.

1 Background on lattice homomorphisms

In this section we gather some basic material on the *lattices* $C(X, \mathbb{I})$ and their homomorphisms. We will not give proofs and we refer the reader to [4], where the corresponding results for $C(X, \mathbb{R})$ are stated and proved. Replacing $\mathbb{R}$ by $\mathbb{I}$ requires only minor changes in the proofs and we leave it to the reader.

Let A and B be lattices. A lattice homomorphism is a mapping $T : A \to B$ such that $T(f \vee g) = Tf \vee Tg$ and $T(f \wedge g) = Tf \wedge Tg$ for all $f, g \in A$.

Proposition 1 (See Proposition 1 in [4]) *Let Y and X be compact spaces and $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ a lattice homomorphism preserving* 0 *and* 1. *Then there is a unique continuous mapping $\tau : X \to Y$ such that*

$$t(x, f(\tau(x))^-) \leq Tf(x) \leq t(x, f(\tau(x))^+) \quad (f \in C(Y), x \in X),$$

where $t(x, c) = Tc(x)$. Here, 0^- *should be treated as* 0 *and* 1^+ *as* 1.

The above τ shall be referred as the map *associated* to T. It is easily seen that sending T to τ we get a (contravariant) functor, which clearly implies that τ is a homeomorphism if T is a lattice isomorphism—an old result by Kaplansky [12].

One may wonder under what conditions one can get a representation like $Tf(x) = t(x, f(\tau(x)))$ for *all* $x \in X$. We have the following result, where L is said to be *increasing* if $f < g$ implies $Lf < Lg$.

Lemma 1 (See Lemma 3 in [4]) *Let Y and X be compact spaces*, $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ *a lattice isomorphism, and let t and τ be as in Proposition* 1. *The following statements are equivalent*:

(a) *T is continuous on $\mathbb{I}_Y$ for the topology of pointwise convergence in $C(X, \mathbb{I})$.*
(b) *T is continuous in the topology of pointwise convergence.*
(c) *$Tf(x) = t(x, f(\tau(x)))$, for all f and x.*
(d) *$T^{\leftarrow}$ (the inverse of T) is increasing.*
(e) *T is continuous.*

It is proved in [4, Theorem 1] that it T is a lattice isomorphism, then $Tf(x) = t(x, f(\tau(x)))$ holds true for all x in some dense G_δ subset of X, possibly depending on T. Also, one has:

Lemma 2 (See Lemma 5 in [4]) *Let Y and X be compact spaces, $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ a lattice isomorphism and let $x \in X$ satisfy one of the following conditions*:

- *There is a sequence (x_n) converging to x, with $x_n \neq x$ for all n.*
- *X is locally connected at x.*

Then $Tf(x) = t(x, f(\tau(x)))$ for all $f \in C(Y, \mathbb{I})$.

2 Multiplicative bijections

In this section we move to our main subject, that is, multiplicative mappings. The following lemma relates multiplication and order, answering a question raised in [4].

Lemma 3 *Every multiplicative mapping $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ is order preserving. Consequently, every multiplicative bijection $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ is a lattice isomorphism.*

Proof First, note that for each $m, n \in \mathbb{N}$ and f one has $T(f^{m/n}) = (Tf)^{m/n}$. Indeed, from $(f^{1/n})^n = f$ we have $(T(f^{1/n}))^n = Tf$, that is, $T(f^{1/n}) = (Tf)^{1/n}$. As T is multiplicative we have $T(f^{m/n}) = (Tf)^{m/n}$.

Now, let $f \leq g$. Fix $n \in \mathbb{N}$, and define $h : Y \to \mathbb{I}$ as

$$h(y) = 0 \quad \text{if } g(y) = 0 \quad \text{and} \quad h(y) = \frac{f(y) f^{\frac{1}{n}}(y)}{g(y)} \quad \text{otherwise.}$$

Clearly, $h \leq f^{1/n} \leq 1$ and it is obvious that $f \cdot f^{1/n} = h \cdot g$.

Notice that h is continuous on Y. The continuity at every point where $h(y) \neq 0$ is clear; while, if $h(y) = 0$, then so $f(y) = 0$, and it follows from the estimate $h \leq f^{1/n}$ and the continuity of $f^{1/n}$. Finally, from

$$T(f)T(f)^{\frac{1}{n}} = T(f)T(f^{\frac{1}{n}}) = T(ff^{\frac{1}{n}}) = T(hg) = T(h)T(g) \leq T(g)$$

we get $Tf \leq Tg$, since n is arbitrary. This completes the proof. □

Now, we have:

Theorem 1 (Császár) *Two compact spaces Y and X are homeomorphic if and only if the multiplicative semigroups $C(X, \mathbb{I})$ and $C(Y, \mathbb{I})$ are isomorphic.*

Proof This follows from the corresponding result for lattices, in view of Lemma 3. □

So, let T be a semigroup isomorphism and define $t : X \times \mathbb{I} \to \mathbb{I}$ as before, that is, $t(x, c) = Tc(x)$. For fixed x, the map $t(x, \cdot)$ is a multiplicative endomorphism of $\mathbb{I}$. It is proved in [13] that each multiplicative selfmap on $\mathbb{I}$ preserving 0 and 1 has the form $c \mapsto c^{\mathfrak{t}}$ for some $\mathfrak{t} \in [0, \infty]$. Here, for $\mathfrak{t} \in \{0, \infty\}$, we understand

$$c^{\mathfrak{t}} = \lim_{s \to \mathfrak{t}} c^s \quad (0 < s < \infty).$$

Hence an endomorphism of $\mathbb{I}$ preserving 0 and 1 is continuous if and only if it assumes some value in (0, 1), equivalently, if it is continuous both at 0 and 1. Thus with these conventions we have

$$Tc(x) = t(x, c) = c^{\mathfrak{t}(x)},$$

for a unique mapping $\mathfrak{t} : X \to [0, \infty]$. Quite clearly, $\mathfrak{t}$ is continuous, and in fact, we can recover it from the image under T of any constant $c \in (0, 1)$. Indeed, for $0 < c < 1$, one has

$$\mathfrak{t}(x) = \log_c Tc(x),$$

if we agree that $\log_c 0 = \infty$ for $0 < c < 1$. Now, Proposition 1 yields:

Theorem 2 *Let $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ be a multiplicative bijection, where Y and X are compact spaces. Then there is a homeomorphism $\tau : X \to Y$ and a continuous function $\mathfrak{t} : X \to [0, \infty]$ such that*

$$(f(\tau(x))^-)^{\mathfrak{t}(x)} \leq Tf(x) \leq (f(\tau(x))^+)^{\mathfrak{t}(x)} \tag{2}$$

for all $f \in C(Y, \mathbb{I})$ and all $x \in X$, where 0^- is treated as 0 and 1^+ as 1. Moreover, the open set $D = \mathfrak{t}^{\leftarrow}(0, \infty)$ is dense in X and one has $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$ for all f and all $x \in D$.

Proof It only remains to see that D is dense. Since $X = D \oplus \mathfrak{t}^{\leftarrow}(0) \oplus \mathfrak{t}^{\leftarrow}(\infty)$ it suffices to see that $\mathfrak{t}^{\leftarrow}(0)$ and $\mathfrak{t}^{\leftarrow}(\infty)$ have empty interior. Let us verify it for the later case, the former being similar. Assume x interior to $\mathfrak{t}^{\leftarrow}(\infty)$. Let $f \in C(Y, [0, \frac{1}{2}])$ be a function vanishing outside $\tau[\mathfrak{t}^{\leftarrow}(\infty)]$ and such that $f(\tau(x)) = \frac{1}{2}$. Then (2) implies that $Tf = 0$, an absurd. □

Now, bearing Lemma 1 in mind, we can add some more criteria for a multiplicative bijection to be continuous. Notice that the equivalence between (e) and (f) fails for mere lattice isomorphisms.

Proposition 2 *Let Y and X be compact spaces. For a multiplicative bijection $T : C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ the following statements are equivalent:*

(a) *For each $x \in X$ one has $Tc(x) \to 0$ as $c \to 0^+$ and $Tc(x) \to 1$ as $c \to 1^-$.*
(b) *$0 < \mathfrak{t}(x) < \infty$ for all $x \in X$.*
(c) *$Tf(x) = (f(\tau(x)))^{\mathfrak{t}(x)}$ for all f and x.*
(d) *There is $0 < f < 1$ such that $0 < Tf < 1$.*
(e) *T is continuous.*
(f) *$T^{\leftarrow}$ is continuous.*

Proof Everything follows from Lemma 1, but that (f) is equivalent to the other statements. But (e) and (f) are equivalent, since (d) holds for T if and only if it holds for its inverse. □

Example 1 (Compare to [8]) Let X be a completely regular space which is not pseudo-compact, and let $\mathfrak{t}: X \to (0, \infty)$ be a (possibly unbounded) continuous function. For $f \in C(X, \mathbb{I})$, define Tf by

$$Tf(x) = f(x)^{\mathfrak{t}(x)}.$$

This is a semigroup automorphism of $C(X, \mathbb{I}) = C(\beta X, \mathbb{I})$. Clearly, T is continuous at 1 if and only if $\mathfrak{t}$ is bounded from above; it is continuous at 0 if and only if $\mathfrak{t}$ is bounded away from 0.

Remark 1 Let us consider the following particular case of the above example. Take $X = \mathbb{N}$, with the discrete topology and let $\mathfrak{t}: \mathbb{N} \to (0, \infty)$. Put

$$Tf(n) = f(n)^{\mathfrak{t}(n)} \quad (n \in \mathbb{N}).$$

As $C(\mathbb{N}, \mathbb{I}) = C(\beta\mathbb{N}, \mathbb{I})$, we can consider T as an automorphism of the semigroup $C(\beta\mathbb{N}, \mathbb{I})$. Also, we can regard $\mathfrak{t}$ as a function from $\beta\mathbb{N}$ to $[0, \infty]$. Consider the constant $\frac{1}{2}$ as a function in $C(\beta\mathbb{N}, \mathbb{I})$. Then $T\frac{1}{2}$ vanishes on $t^{\leftarrow}(\infty)$. This subset of $\beta\mathbb{N}$ will be empty only if $\mathfrak{t}$ is bounded on $\mathbb{N}$. As T is an automorphism of $C(\beta\mathbb{N}, \mathbb{I})$ we see that the condition $f > 0$ cannot be expressed within the semigroup structure of $C(\beta\mathbb{N}, \mathbb{I})$. Of course, that condition just means 'f is an inversible square' in the larger semigroup $C(\beta\mathbb{N}, \mathbb{R})$.

3 Applications

In this section we present some applications to certain distinguished semigroups of functions defined on metric spaces. We use d to denote distance in any metric space: this notation is clear unless we need to consider two metrics on the same set.

3.1 Continuous functions on metric spaces

Theorem 3 *Let X and Y be metrizable spaces (or even completely regular spaces where every point is G_δ), $\tau: X \to Y$ a homeomorphism and $\mathfrak{t}: X \to (0, \infty)$ a continuous function. Then the map $T: C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ given by $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$ is a multiplicative bijection. All multiplicative bijections arise in this way.*

Proof The first part is contained in Example 1. As for the converse, let $T: C(Y, \mathbb{I}) \to C(X, \mathbb{I})$ be a multiplicative bijection. By the universal property of the Stone-Čech compactification we can regard T as a multiplicative bijection between $C(\beta Y, \mathbb{I})$ and $C(\beta X, \mathbb{I})$. By Theorem 2 there is a homeomorphism $\tau: \beta X \to \beta Y$ and a continuous function $\mathfrak{t}: \beta X \to [0, \infty]$ such that

$$(f(\tau(x))^-)^{\mathfrak{t}(x)} \le Tf(x) \le (f(\tau(x))^+)^{\mathfrak{t}(x)} \quad (x \in \beta X).$$

It is well-known that the only G_δ points in βX are those of X, and similarly for Y (this can be seen in the classical treatise [11]). It follows that τ acts as a homeomorphism between X and Y. Moreover every G_δ point satisfies (at least) one of the conditions of Lemma 2 and therefore $0 < \mathfrak{t}(x) < \infty$ for all $x \in X$ and $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$, as required. □

3.2 Uniformly continuous functions

Theorem 4 *Let X and Y be complete metric spaces, $\tau : X \to Y$ a uniform homeomorphism and $\mathfrak{t} : X \to (0,\infty)$ a uniformly continuous function satisfying the following condition*:

($\heartsuit$) *If $\mathfrak{t}(x_n)$ converges to 0 or ∞, then there is $\varepsilon > 0$ and m such that $d(x_n, x) \geq \varepsilon$ for all $n \geq m$ and $x \neq x_n$.*

Then the map $T : U(Y,\mathbb{I}) \to U(X,\mathbb{I})$ given by $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$ is a multiplicative bijection. All multiplicative bijections arise in this way.

Proof Let us recall that f is uniformly continuous if and only if $d(x_n, y_n) \to 0$ implies $f(x_n) - f(y_n) \to 0$. Also, we will use the fact that, in a complete metric space, every sequence contains either a convergent subsequence or a uniformly 'separated' subsequence—the distance between two different terms is bounded from below by a fixed positive number.

We now prove the first statement. We may assume $Y = X$ and $\tau = \mathbf{1}_X$. Let $\mathfrak{t} : X \to (0,\infty)$ be a uniformly continuous function satisfying ($\heartsuit$). We must verify that $f \longmapsto f^{\mathfrak{t}}$ is an automorphism of $U(X,\mathbb{I})$. But $1/\mathfrak{t}$ has the same properties as $\mathfrak{t}$ and so one only has to show that $f^{\mathfrak{t}}$ is uniformly continuous when f is.

Take (x_n) and (y_n) so that $d(x_n, y_n) \to 0$. Let us see that

$$f(x_n)^{\mathfrak{t}(x_n)} - f(y_n)^{\mathfrak{t}(y_n)} \to 0. \tag{3}$$

Passing to a subsequence if necessary we may and do assume $\mathfrak{t}(x_n)$ converges to some point in $[0,\infty]$. If that limit is 0 or ∞, then $x_n = y_n$ for n large enough and so there is nothing to prove. Otherwise (3) is obvious.

As for the converse, let $T : U(Y,\mathbb{I}) \to U(X,\mathbb{I})$ be a multiplicative bijection. By general representation results, given a metric (or uniform) space Z there is a compact space κZ (in fact a compactification of Z) such that $U(Z,\mathbb{I}) = C(\kappa Z,\mathbb{I})$. Applying Theorem 2 we get a homeomorphism $\tau : \kappa X \to \kappa Y$ and a continuous function $\mathfrak{t} : \kappa X \to [0,\infty]$ such that

$$(f(\tau(x))^-)^{\mathfrak{t}(x)} \leq Tf(x) \leq (f(\tau(x))^+)^{\mathfrak{t}(x)} \quad (x \in \kappa X).$$

As X is complete, the only G_δ points in κX are those of X, and similarly for Y (see [9, Lemma 1]). It follows that τ maps X to Y, as a uniform homeomorphism. Moreover every G_δ point satisfies one of the conditions of Lemma 2 and therefore $0 < \mathfrak{t}(x) < \infty$ for all $x \in X$ and

$$Tf(x) = f(\tau(x))^{\mathfrak{t}(x)} \quad (x \in X).$$

It remains to prove that $\mathfrak{t}$ is uniformly continuous and satisfies ($\heartsuit$). At this stage of the proof we may and do assume $Y = X$ and $\tau = \mathbf{1}_X$.

Let us verify ($\heartsuit$) first. Suppose $\mathfrak{t}(x_n) \to \infty$. As $\mathfrak{t}$ is continuous (x_n) cannot contain convergent subsequences and so there is $\delta > 0$ and k such that $d(x_n, x_m) \geq \delta$ for $n \neq m$ and $n, m \geq k$. We claim there is $\varepsilon > 0$ and m such that $d(x_n, x) \geq \varepsilon$ if $n \geq m$

and $x \neq x_n$. For if not, passing to a sequence if necessary (we don't relabel it), there is (y_n) with $0 < d(x_n, y_n) \to 0$. We note in passing that we may assume $y_n \neq y_m$ for $n \neq m$ and $x_n \neq y_m$ for all n and m. It is not hard to see that there is $f \in U(X, \mathbb{I})$ such that

$$f(x_n) = 1 - \frac{1}{\mathfrak{t}(x_n)} \quad \text{and} \quad f(y_n) = 1$$

for every $n \in \mathbb{N}$. We leave the details to the reader. We then have

$$Tf(x_n) = \left(1 - \frac{1}{\mathfrak{t}(x_n)}\right)^{\mathfrak{t}(x_n)} \longrightarrow \frac{1}{e}$$

as $n \to \infty$, while $Tf(y_n) = 1$ and so Tf is not uniformly continuous. The same conclusion obtains when $\mathfrak{t}(x_n) \to 0$.

Finally, let us verify that $\mathfrak{t}$ is uniformly continuous. Take sequences (x_n) and (y_n) such that $d(x_n, y_n) \to 0$. We prove that $\mathfrak{t}(x_n) - \mathfrak{t}(y_n) \to 0$. Clearly, we can assume $(\mathfrak{t}(x_n))$ and $(\mathfrak{t}(y_n))$ convergent in $[0, \infty]$. If one of the two limits is 0 or ∞, then $x_n = y_n$ for n large enough and we are done. Otherwise the two sequences must have the same limit, for if not $T\frac{1}{2}$ cannot be uniformly continuous. This completes the proof. $\square$

Corollary 1 *For a given metric space X exactly one of the following alternatives holds*:

- *X contains an infinite uniformly isolated subset.*
- *Every automorphism of $U(X, \mathbb{I})$ is continuous in the topology of uniform convergence.*

Proof Of course, we can assume X complete since each alternative holds for X if and only if it holds for its completion. Now the result follows from Theorem 4 and Example 1. $\square$

3.3 Lipschitz functions

Given a metric space X we write $\mathrm{Lip}(X, \mathbb{I})$ for the set of Lipschitz functions from X to $\mathbb{I}$. Although in general the product of real-valued Lipschitz functions need not to be Lipschitz, $\mathrm{Lip}(X, \mathbb{I})$ is always closed under pointwise multiplication. In what follows we describe the isomorphisms between these semigroups. In this case, our approach is different, based on the ideas of [5] and, ultimately, on Shirota's [18]. Let us start with the following.

Definition 1 Given $f, g \in \mathrm{Lip}(X, \mathbb{I})$, we write $f \subset g$ if, whenever $h \in \mathrm{Lip}(X, \mathbb{I})$, $hg = 0$ implies $hf = 0$.

If f is a function on X, we write U_f for the interior of its support. Here, the support of f is the closure of the set $\{x \in X : f(x) \neq 0\}$. The following result has an obvious proof we leave to the reader.

Lemma 4 (Mainly Shirota) *Given* $f, g \in \mathrm{Lip}(X, \mathbb{I})$, *one has* $f \subset g$ *if and only if* $U_f \subset U_g$ *in the usual set theoretic sense.*

Recall that a *regular* open set is one that equals the interior of its closure. Let $R(X)$ denote the set of all regular open sets of X: it is a lattice, when ordered by inclusion.

By the very definition, U_f is regular for each $f \in \mathrm{Lip}(X, \mathbb{I})$—actually, for each continuous f. Conversely, if U is a regular open set in X, then there is $f \in \mathrm{Lip}(X, \mathbb{I})$ so that $U = U_f$—take $f(x) = \min\{d(x, U^c), 1\}$. Thus the real meaning of Lemma 4 is that the lattice structure of $R(X)$ can be obtained from the semigroup $\mathrm{Lip}(X, \mathbb{I})$ so that homomorphisms of semigroups of Lipschitz functions will induce homomorphisms between the corresponding lattices of regular open sets. The following result is a particular case of [5, Lemma 6].

Lemma 5 *Let* X *and* Y *be complete metric spaces. If* $\mathfrak{L} : R(X) \to R(Y)$ *is an isomorphism of lattices, then there is a dense subspace* $X' \subset X$ *and a mapping* $\tau : X' \to Y$ *such that, for each* $x \in X'$,

$$\{\tau(x)\} = \bigcap_{x \in U} \mathfrak{L}(U).$$

Proof With a slight abuse of notation, we define a map from X to the subsets of Y thus:

$$\tau(x) = \bigcap_{x \in U} \mathfrak{L}(U).$$

Let X' be the set of those points $x \in X$ for which $\tau(x)$ contains exactly one point of Y. Let W be a nonempty open subset of X. We want to see that W meets X'. Take a nonempty $U_1 \in R(X)$ such that $\overline{U}_1 \subset W$ and $\operatorname{diam} U_1 \leq 1$. Choose a nonempty $V_1 \subset \mathfrak{L}(U_1)$, with $\operatorname{diam} V_1 \leq 1$. Then choose a nonempty $U_2 \subset \mathfrak{L}^{-1}(V_1)$ with $\overline{U}_2 \subset U_1$ and $\operatorname{diam} U_2 \leq 1/2$. Next, take a nonempty $V_2 \subset \mathfrak{L}(U_2)$ such that $\overline{V}_2 \subset V_1$ and $\operatorname{diam} V_2 \leq 1/2$. In this way we get sequences (U_n) and (V_n) in $R(X)$ and $R(Y)$, respectively, such that, for each n:

- $\overline{U}_{n+1} \subset U_n$ and $\overline{V}_{n+1} \subset V_n$.
- U_n and V_n have diameter at most $1/n$.
- $\mathfrak{L}(U_{n+1}) \subset V_n \subset \mathfrak{L}(U_n)$.

Now, it is clear that there are $x \in X$ and $y \in Y$ such that

$$\{x\} = \bigcap_n U_n = \bigcap_n \overline{U}_n \quad \text{and} \quad \{y\} = \bigcap_n V_n = \bigcap_n \overline{V}_n.$$

From where it follows that $\tau(x) = \{y\}$. Moreover $x \in W$ and so X' is dense in X. □

Before going into the main result for Lipschitz functions let us make explicit the following remark.

Lemma 6 *Let* $T : \mathrm{Lip}(Y, \mathbb{I}) \to \mathrm{Lip}(X, \mathbb{I})$ *be an isomorphism and let* $f, g, h \in \mathrm{Lip}(Y, \mathbb{I})$. *Then* f *and* g *agree on* U_h *if and only if* Tf *and* Tg *agree on* U_{Th}.

Proof Just observe that f and g agree on U_h if and only if $hf = hg$. □

Theorem 5 *Let* $T : \mathrm{Lip}(Y, \mathbb{I}) \to \mathrm{Lip}(X, \mathbb{I})$ *be an isomorphism, where* Y *and* X *are complete metric spaces. Then there is a uniform homeomorphism* $\tau : X \to Y$ *and a continuous function* $\mathfrak{t} : X \to (0, \infty)$ *such that* $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$, *and* $\mathfrak{t}(x) = 1$ *unless* x *is isolated.*

If, in addition, X *has finite diameter, then* τ *is Lipschitz from* X *to* Y.

Proof We consider the mapping $\mathfrak{L} : R(X) \to R(Y)$ sending U_{Tf} to U_f. The definition makes sense by Lemma 4. Clearly, $\mathfrak{L}$ preserves the order in both directions, hence it is a lattice isomorphism and so Lemma 5 applies: there is a dense $X' \subset X$ and a mapping $\tau : X' \to Y$ such that, for each $x \in X'$,

$$\bigcap_{x \in U} \mathfrak{L}(U)$$

reduces to the point $\tau(x)$. Our immediate aim is to show that τ is uniformly continuous. We must prove that if (x_n) and (y_n) are sequences in X' such that $d(x_n, y_n) \to 0$, then so $d(\tau x_n, \tau y_n) \to 0$. If we assume the contrary, passing to a subsequence if necessary, one finds $\varepsilon > 0$ and two sets $U, V \in R(Y)$ such that $d(U, V) \geq \varepsilon$, U contains $\tau(x_n)$ and V contains $\tau(y_n)$ for every n. This is true, but not entirely trivial—we refer the reader to [5, Sect. 1.1] for a detailed proof. Now, take $f \in \mathrm{Lip}(Y, \mathbb{I})$ so that $f|_U = 0$ and $f|_V = 1$. For instance one may take

$$f(y) = \min\{d(y, U)/\varepsilon, 1\}.$$

Now, taking U', V' and f' so that $\mathfrak{L}(U') = U, \mathfrak{L}(V') = V$ and $Tf' = f$ one has $x_n \in U'$, $y_n \in V'$ (by the very definition of τ) and $f'|_{U'} = 0$, $f'|_{V'} = 1$ (by Lemma 6). Hence $f'(x_n) = 0$ and $f'(y_n) = 1$, so f' is not uniformly continuous, let alone Lipschitz and we have reached a contradiction.

Since τ is uniformly continuous on X' and X' is dense in X we can extend it to a uniformly continuous mapping from X to Y still denoted τ. That τ is in fact a uniform homeomorphism will be proved later.

Next, we claim that the value of Tf at $x \in X'$ depends only on the value of f at $\tau(x)$. This obviously follows from Lemma 6 if x is isolated so we assume x is not isolated in the ensuing argument.

Assume on the contrary there are $f, g \in \mathrm{Lip}(Y, \mathbb{I})$ with $f(\tau(x)) \neq g(\tau(x))$ and $Tf(x) = Tg(x)$. Choose sequences (x_n) and (y_n) in X' in such a way that

$$d(y_n, x) \leq \frac{d(x_n, x)}{2} \quad \text{and} \quad d(x_{n+1}, x) \leq \frac{d(y_n, x)}{2},$$

for all $n \in \mathbb{N}$. Set $\alpha_n = d(x_n, x)$, $\beta_n = d(y_n, x)$ and let U_n (respectively, V_n) be the ball of radius $\alpha_n/4$ centered at x_n (respectively, the ball of radius $\beta_n/4$ centered at y_n). Notice that $U_n \cap V_m = \emptyset$ for all n and m, while both $U_n \cap U_m$ and $V_n \cap V_m$ are empty unless $n = m$. Write $U = \bigcup_n U_n$, $V = \bigcup_n V_n$ and $Z = U \cup V$. Let $h : Z \to \mathbb{I}$ be the function agreeing with Tf on U and with Tg in V. We see h is Lipschitz on Z.

Indeed, let L be a common Lipschitz constant for Tf and Tg and pick $u, v \in Z$. If u and v are in U, then

$$|h(u) - h(v)| = |Tf(u) - Tf(v)| \leq Ld(u, v),$$

and similarly if both u and v are in V. Now, if $u \in U$ and $v \in V$, we may assume if fact that $u \in U_n$ and $v \in V_m$, with $n \leq m$ (the case $n \geq m$ is similar). We have

$$d(u, x) \geq \frac{3\alpha_n}{4} \quad \text{and} \quad d(v, x) \leq \frac{5\beta_m}{4} \leq \frac{5\alpha_n}{8},$$

so $d(u, v) \geq \alpha_n/8$ and

$$\begin{aligned}|h(u) - h(v)| &= |Tf(u) - Tg(v)| \leq |Tf(u) - Tf(x)| + |Tg(x) - Tg(v)| \\ &\leq L\left(\frac{5\alpha_n}{4} + \frac{5\beta_m}{4}\right) \leq 2L\alpha_n,\end{aligned}$$

hence the Lipschitz constant of h is not greater than $16L$. It is a classical result in function theory that Lipschitz functions can be extended preserving both the Lipschitz constant and the sup norm (see [20, Theorem 1.5.6(a)]). So let us extend h to some member in $\mathrm{Lip}(X, \mathbb{I})$ we still call h. Now, let $h' \in \mathrm{Lip}(Y, \mathbb{I})$ be such that $Th' = h$. Then, by Lemma 6, h' agrees with f in a neighbourhood of each $\tau(x_n)$ and agrees with g in a neighbourhood of each $\tau(y_n)$. Since

$$\lim_{n\to\infty} f(\tau(x_n)) = f(\tau(x)) \neq g(\tau(x)) = \lim_{n\to\infty} g(\tau(y_n))$$

h' is discontinuous at $\tau(x)$, a contradiction which proves what we claimed.

As a consequence we have $Tf(x) = t(x, f(\tau(x)))$ for every $f \in \mathrm{Lip}(Y, \mathbb{I})$ and every $x \in X$, where $t(x, c) = Tc(x)$. By the form of the endomorphisms of $\mathbb{I}$ one has in fact $Tf(x) = f(\tau(x))^{\mathfrak{t}(x)}$, where $\mathfrak{t} : X \to [0, \infty]$ is continuous. The surjectivity of T now implies that $\mathfrak{t}(x) \in (0, \infty)$ for all $x \in X$. Elementary considerations on the inverse of T give that τ is a uniform homeomorphism: let $S : \mathrm{Lip}(X, \mathbb{I}) \to \mathrm{Lip}(Y, \mathbb{I})$ be the inverse of T. Then

$$Sf(y) = f(\sigma(y))^{\mathfrak{s}(y)},$$

with $\sigma : Y \to X$ uniformly continuous. It is really easy to see that σ and τ are inverse for each other—and also that $\mathfrak{s}(y)\mathfrak{t}(x) = 1$ provided $y = \tau(x)$.

Let us verify that τ is Lipschitz when Y has finite diameter. We may use τ to transfer the distance of Y to X thus:

$$d'(x, x') = d(\tau(x), \tau(x')).$$

Then (X, d') is isometric to Y (through τ) and the mapping $L : \mathrm{Lip}((X, d'), \mathbb{I}) \to \mathrm{Lip}((X, d), \mathbb{I})$ defined by

$$Lf(x) = f(x)^{\mathfrak{t}(x)}$$

is an isomorphism. After a moment's reflection we realize that it suffices to see that if d' is a finite metric, uniformly equivalent to d on X, and there is a continuous function $\mathfrak{t}$ such that the mapping

$$Tf(x) = f(x)^{\mathfrak{t}(x)}$$

defines an isomorphism between $\mathrm{Lip}((X, d'), \mathbb{I})$ and $\mathrm{Lip}((X, d), \mathbb{I})$, then the identity is Lipschitz from (X, d) to (X, d').

Assuming the contrary, for each $n \in \mathbb{N}$, we get $x_n, y_n \in X$ such that

$$n \cdot d(x_n, y_n) < d'(x_n, y_n) \leq \mathrm{diam}(X, d').$$

This already implies that $d(x_n, y_n)$ converges to zero and so $d'(x_n, y_n)$ does, because d and d' are uniformly equivalent.

By the extension result mentioned above, we may complete the proof that $\mathbf{1}_X : (X, d) \to (X, d')$ is Lipschitz assuming $X = \{x_n, y_n : n \in \mathbb{N}\}$.

As X does not contain uniformly isolated infinite subsets, we may proceed as in the proof of Theorem 4 to verify that $\mathfrak{t}$ (whence $1/\mathfrak{t}$) takes values in a compact subset of $(0, \infty)$—as we did to prove property ($\heartsuit$). It follows that $\mathfrak{t}$ is d-Lipschitz since $(\frac{1}{2})^{\mathfrak{t}} = T\frac{1}{2}$ is. Now, fix $0 < c < 1$ and let $f : X \to [c, 1]$ be d'-Lipschitz. Then $Tf(x) = f(x)^{\mathfrak{t}(x)}$ defines a d-Lipschitz function. Therefore so $\log f^{\mathfrak{t}} = \mathfrak{t} \cdot \log f$ is d-Lipschitz. Since f is bounded away from zero and $1/\mathfrak{t}$ is d-Lipschitz we see that f is d-Lipschitz. Hence

$$\mathrm{Lip}((X, d'), [c, 1]) \subset \mathrm{Lip}((X, d), [c, 1]).$$

But d'-Lipschitz functions are bounded and since Lipschitz functions form a linear space we have $\mathrm{Lip}((X, d'), \mathbb{R}) \subset \mathrm{Lip}((X, d), \mathbb{R})$. From where it follows that $\mathbf{1}_X : (X, d) \to (X, d')$ is Lipschitz; see [10, Theorem 3.9].

We conclude the proof by showing that $\mathfrak{t}(x) = 1$ unless x is isolated. Moving to a compact set if necessary we may assume that $Tf(y) = f(y)^{\mathfrak{t}(y)}$ defines an automorphism of $\mathrm{Lip}(X, \mathbb{I})$. If $\mathfrak{t}(x) < 1$, then $\mathfrak{t}(y) < 1 - \varepsilon$ for y in a neighbourhood of x and Tf is not Lipschitz if $f(y) = \min\{1, d(x, y)\}$. □

Corollary 2 *Let Y and X be complete metric spaces, both of finite diameter and without isolated points. Every isomorphism $T : \mathrm{Lip}(Y, \mathbb{I}) \to \mathrm{Lip}(X, \mathbb{I})$ has the form $Tf = f \circ \tau$, where $\tau : X \to Y$ is a Lipschitz homeomorphism (in both directions).*

Remark 2 The hypothesis on the diameters cannot be removed in the preceding results. Indeed, if X is a metric space, with (generally unbounded) distance function d, then $d' = d/(1 + d)$ is a bounded metric on X and it is not hard to see that $\mathrm{Lip}((X, d'), \mathbb{I}) = \mathrm{Lip}((X, d), \mathbb{I})$. Needless to say d' and d are Lipschitz equivalent if and only if d is bounded.

3.4 Measurable functions

Given a measure space $(X, \mathfrak{X}, \mu)$ we write $L^\infty(\mu)$ for the Banach algebra of all essentially bounded measurable functions $f : X \to \mathbb{R}$, with 'pointwise' operations, the essential supremum norm, and the traditional conventions about identifying functions equal almost everywhere. The meaning of $L^\infty(\mu, \mathbb{I})$ should be obvious.

To avoid any measure theoretic pathology, in the following result we consider only *standard* measures: that is, σ-finite measures defined on the Borel sets of a Polish (separable and complete metric) space.

Theorem 6 *Let $(X, \mathfrak{X}, \mu)$ and $(Y, \mathfrak{Y}, \nu)$ be standard measure spaces, $\tau : X \to Y$ a measurable isomorphism and $\mathfrak{t} : X \to (0, \infty)$ a measurable function. Then the map $T : L^\infty(\nu, \mathbb{I}) \to L^\infty(\mu, \mathbb{I})$ given by*

$$Tf(x) = (f(\tau(x)))^{\mathfrak{t}(x)} \tag{4}$$

is an isomorphism. All isomorphisms arise in this way.

Before going into the proof, let us remark that $L^\infty(\mu, \mathbb{I})$ is a complete lattice: every subset of $L^\infty(\mu, \mathbb{I})$ has a supremum in $L^\infty(\mu, \mathbb{I})$. Note that if $S \subset L^\infty(\mu, \mathbb{I})$ is countable, then $\bigvee S$ can be computed pointwise.

We will exploit this fact thanks to the following.

Lemma 7 *Order isomorphisms $L^\infty(\nu, \mathbb{I}) \to L^\infty(\mu, \mathbb{I})$ preserve almost everywhere convergence. Hence so multiplicative bijections do.*

Proof Note that $f_n \to f$ almost everywhere if and only if

$$f = \bigwedge_n \bigvee_{k \geq n} f_k = \bigvee_n \bigwedge_{k \geq n} f_k$$

and that order isomorphisms preserve arbitrary joins and meets.

As for the second statement, notice that there is a compact space $\mathfrak{M}$ such that $L^\infty(\mu, \mathbb{R})$ is isomorphic to $C(\mathfrak{M}, \mathbb{R})$, as a Banach algebra. Hence $L^\infty(\mu, \mathbb{I}) = C(\mathfrak{M}, \mathbb{I})$ as ordered semigroups and so Lemma 3 applies. However there is no need of representation since the *proof* of Lemma 3 works directly in this case. □

Proof of Theorem 6 The first part is nearly obvious. As for the second one, let T be a multiplicative bijection. Clearly, T must send idempotents into idempotents. So, given a measurable $A \subset Y$ there is $A' = \Psi(A)$ (well defined up to null sets) such that $T1_A = 1_{A'}$. Let us analyze the action of Ψ. First, it is clear that Ψ defines a bijection between the measure algebras. Moreover, Ψ preserves Boolean operations since

$$1_{A \cap B} = 1_A \cdot 1_B = 1_A \wedge 1_B \quad \text{and} \quad 1_{A \cup B} = 1_A \vee 1_B.$$

The hypothesis on the measures guarantees that $\Psi(A) = \tau^{\leftarrow}(A)$ for some measurable isomorphism $\tau : X \to Y$, by a classical result of von Neumann [17] (see also [19])

This can be used to 'localize' the action of T as follows: given $A \in \mathfrak{Y}$ and $f \in L^\infty(\nu, \mathbb{I})$ one has

$$T(1_A f) = T1_A Tf = 1_{\Psi(A)} Tf.$$

Thus f and g agree almost everywhere on A (equivalently, $1_A f = 1_A g$ in $L^\infty(\nu, \mathbb{I})$) if and only if Tf and Tg agree almost everywhere on $\Psi(A)$. In particular, $f < g$ almost everywhere in Y if and only if $Tf < Tg$ almost everywhere in X.

Next, we identify the function $\mathfrak{t}: X \to (0, \infty)$. For each rational r in $\mathbb{I}$, let us fix a version of Tr, so that we consider $Tr(x)$ defined for every $x \in X$. Note that for fixed $r, s \in \mathbb{I}$, one has $Trs(x) = Tr(x)Ts(x)$ almost everywhere. Thus, the set

$$\{x \in X : Trs(x) \neq Tr(x)Ts(x) \text{ for some } r, s \in \mathbb{I} \cap \mathbb{Q}\}$$

has measure zero. Hence we can choose new versions (that we don't relabel) of the functions Tr in such a way that $Trs(x) = Tr(x)Ts(x)$ for all $r \in \mathbb{I} \cap \mathbb{Q}$, with $T1 = 1$ and $T0 = 0$. We then have that for fixed $x \in X$, the mapping $r \in \mathbb{I} \cap \mathbb{Q} \mapsto Tr(x) \in \mathbb{I}$ is multiplicative and unital. After a moment's reflection we realize that there is $\mathfrak{t} = \mathfrak{t}(x) \in [0, \infty]$ such that

$$Tr(x) = r^{\mathfrak{t}(x)} \quad (x \in X, r \in \mathbb{I} \cap \mathbb{Q})$$

and that the function $\mathfrak{t} : X \to [0, \infty]$ so defined is measurable. Now, since $0 < T\frac{1}{2}(x) < 1$ almost everywhere on X we can choose a new version of $\mathfrak{t}$ taking values in $(0, \infty)$. Again, we don't relabel it.

Finally, we will prove that (4) holds almost everywhere on X. If f is a simple function taking rational values, then $f = \sum_{i=1}^n r_i 1_{A_i}$, with $X = A_1 \oplus \cdots \oplus A_n$ and so

$$Tf = T\left(\bigvee_{i=1}^n r_i 1_{A_i}\right) = \bigvee_{i=1}^n T(r_i 1_{A_i}) = \bigvee_{i=1}^n Tr_i T1_{A_i} = \bigvee_{i=1}^n r_i^{\mathfrak{t}} 1_{\tau^{\leftarrow}(A_i)} = (f \circ \tau)^{\mathfrak{t}}.$$

Now, fix $f \in L^\infty(\nu, \mathbb{I})$. Let (f_n) be an increasing sequence of $\mathbb{Q}$-valued simple functions converging pointwise to f. We already know that $Tf_n(x) = (f_n(\tau(x)))^{\mathfrak{t}(x)}$ almost everywhere. On the other hand,

$$Tf_n(x) = (f_n(\tau(x)))^{\mathfrak{t}(x)} \to (f(\tau(x)))^{\mathfrak{t}(x)} \quad (n \to \infty)$$

for every $x \in X$. A quick look at Lemma 7 ends the proof. □

Remark 3 Let $\mathfrak{A}$ be a unital C*-algebra. The so called *effects* are the self-adjoint elements f of $\mathfrak{A}$ such that $0 \le f \le 1$. Denote the set of effects of $\mathfrak{A}$ by $E(\mathfrak{A})$. The *sequential product* in $E(\mathfrak{A})$ is defined by $f \circ g = f^{1/2} \cdot g \cdot f^{1/2}$. This makes $E(\mathfrak{A})$ into a semigroup. When $\mathfrak{A}$ is commutative it is *-isomorphic to $C(\mathfrak{M}, \mathbb{C})$ for some compact space $\mathfrak{M}$ and $E(\mathfrak{A})$ equipped with the sequential product equals our old friend $C(\mathfrak{M}, \mathbb{I})$. These notions originate in the quantum theory of measurement. See [15] and the references therein.

In the recent paper [15] (see algo the monograph [16]) Molnár proved that if $\mathfrak{B}$ and $\mathfrak{A}$ are von Neumann algebras and $T : E(\mathfrak{B}) \to E(\mathfrak{A})$ is a *sequential isomorphism* then, for $\mathfrak{C} = \mathfrak{A}, \mathfrak{B}$, there are direct sum decompositions $\mathfrak{C} = \mathfrak{C}_0 \oplus \mathfrak{C}_1 \oplus \mathfrak{C}_2$, within the category of von Neumann algebras, with $\mathfrak{C}_0$ commutative, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ having no commutative factors, and three bijections: a multiplicative $T_0 : E(\mathfrak{B}_0) \to E(\mathfrak{A}_0)$ one, a *-isomorphism, $T_1 : \mathfrak{B}_1 \to \mathfrak{A}_1$ and an anti *-isomorphism $T_2 : \mathfrak{B}_2 \to \mathfrak{A}_2$ such that $T = T_0 \oplus T_1 \oplus T_2$ holds on $E(\mathfrak{B})$.

It is well-known that the commutative summands can be represented as $L^\infty(\mu_{\mathfrak{C}}, \mathbb{C})$ algebras, and so $\mathfrak{B}_0$ and $\mathfrak{A}_0$ are *-isomorphic (use either Theorem 1 or the proof of Theorem 6). When $\mathfrak{A}$ (hence $\mathfrak{B}$) acts on a separable Hilbert space the underlying measures are standard and Theorem 6 provides a sharp description of the 'irregular part' T_0.

Before leaving the C* setting, let us remark that, although Lemma 7 implies that isomorphisms between $L^\infty(\mu, \mathbb{I})$ semigroups are always continuous in the topology of convergence in measure this is not the case for the weak* topology—that is, the relative $\sigma(L^\infty, L^1)$ topology. In fact we do have the following:

Corollary 3 *Let λ denote Lebesgue measure on the Borel sets on the unit interval. An automorphism of $L^\infty(\lambda, \mathbb{I})$ is weak* continuous if and only if it extends to a *-automorphism.*

Proof The 'if part' is clear since *-isomorphisms are weak* continuous. As for the converse, let T be a weak* continuous automorphism of $L^\infty(\lambda, \mathbb{I})$. After composing with $\tau^{\leftarrow}$ we can and do assume $Tf = f^{\mathfrak{t}}$ and we must show that $\mathfrak{t} = 1$ almost everywhere. It suffices to see that $T\frac{1}{2} = \frac{1}{2}$. Let (f_n) be a sequence of idempotents converging to $\frac{1}{2}$ in the weak* topology of $L^\infty(\lambda, \mathbb{I})$. A more concrete choice could be

$$f_n = \frac{1 + r_n}{2},$$

where r_n is the n-th Rademacher's function—that is, $r_n(t)$ is the signum of $\cos(2^n \pi t)$ for $0 \le t \le 1$. It is well-known that (r_n) is weak* null in $L^\infty(\lambda)$. Now, since T leaves every f_n fixed, we have

$$\frac{1}{2} = \lim_n f_n = \lim_n T f_n = T\left(\frac{1}{2}\right),$$

and the proof is complete. □

4 Concluding remarks and questions

We take our leave of the reader with some questions stemming from the results of the paper.

Problem 1 The second part of Theorem 2 implies that each multiplicative bijection T is completely determinated by the associated maps τ and $\mathfrak{t}$. It would be interesting to characterize those functions $\mathfrak{t} : X \to [0, \infty]$ associated to automorphisms of $C(X, \mathbb{I})$. In particular, must be $X = \beta D$? This would complete Theorem 2.

Problem 2 Let X be a compact metric space. For which functions $\mathfrak{t} : X \to (0, \infty)$ is $f \mapsto f^{\mathfrak{t}}$ an automorphism of $\mathrm{Lip}(X, \mathbb{I})$?

Problem 3 Describe the isomorphisms $C^\infty(Y,\mathbb{I}) \to C^\infty(X,\mathbb{I})$ when X and Y are smooth manifolds.

Problem 4 Describe the automorphisms of the group $L^\infty(\mu,\mathbb{T})$. Here $\mathbb{T}$ is the multiplicative group of complex numbers of modulus one.

Note added in prrof Problem 1 has been soved by J. Araujo, Multiplicative bijections of semigroups of interval-valued continuous functions. Proc. Am. Math. Soc. 137 (2009) 171–178. The solution of Problem 3 is contained in “Some preserved problems for algebras of smooth functions”, a paper by the two first named authors to appear in Arkiv för Matematik.

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