Version April 17, 2009

Nonlinear centralizers in homology, and vice versa

Félix Cabello Sánchez

Introduction

Background. Quasi-linear maps entered into Banach space theory in [**13**], where Enflo, Lindenstrauss and Pisier constructed a nontrivial extension $\ell^2 \rightarrowtail X \twoheadrightarrow \ell^2$ thus solving the "three-space" problem.

Then Kalton developed a rather satisfactory theory in [**17**] showing that extensions of quasi-Banach spaces $Y \rightarrowtail X \twoheadrightarrow Z$ are in correspondence with quasi-linear maps $\Omega : Z \to Y$. The clean formula $\|(y,z)\|_\Omega = \|y - \Omega(z)\|_Y + \|z\|_Z$ connecting X and Ω is due to Ribe [**29**]. Both Kalton and Ribe gave examples of (non-trivial) quasi-linear maps $\Omega : \ell^1 \to \mathbb{K}$ thus producing nontrivial extensions $\mathbb{K} \rightarrowtail X \twoheadrightarrow \ell^1$ in which the middle space X cannot be locally convex because the Hahn-Banach theorem cannot apply to the identity on $\mathbb{K}$. Another counterexample was obtained independently and more or less simultaneously by Roberts in [**30**].

Soon afterwards appeared the admirable [**24**]. There, Kalton and Peck use a kind of vector-valued version of Ribe's map to construct quasi-linear maps on any quasi-Banach space with unconditional basis. Let us consider this point in more detail as it is one of the main motivations of the present paper. What was proved in [**24**] is that given any Lipschitz function $\theta : \mathbb{R} \to \mathbb{R}$ the mapping Ω_θ defined by $\Omega_\theta(f) = f\theta(\log(|f|/\|f\|_X))$ is quasi-linear on every quasi-Banach space with unconditional basis X —we invariably regard the elements of such a X as functions $f : \mathbb{N} \to \mathbb{K}$— and non-trivial as long as the basis of X contains no subsequence equivalent to the usual basis of c_0 and θ is unbounded on $\mathbb{R}_-$. Taking $X = \ell^2$ and θ as the identity on $\mathbb{R}$ one obtains another solution to Palais problem —the nowadays called Kalton-Peck space Z_2. All this can be seen in [**9**, Chapter 1] or [**1**, Chapter 16].

The crucial observation here is that the maps appearing in [**24**] are more than quasi-linear: they are ℓ^∞-centralizers, that is, they satisfy an estimate of the form

$$\|\Omega(af) - a\Omega(f)\| \le C\|a\|_\infty \|f\| \qquad (a \in \ell^\infty, f \in X).$$

Equivalently, the induced extension $Y \rightarrowtail X \twoheadrightarrow Z$ lives in the category of ℓ^∞ modules.

These examples suggested a correspondence between centralizers defined on different sequence spaces.

This theme was pursued by Kalton himself in the more general setting of function spaces [**20**]. Fix a measure μ and consider the corresponding L^p spaces for $0 < p \le \infty$. It is proved in [**20**, Theorem 5.1] that, when $p > 1$, every L^∞-centralizer Ω on L^p can be pushed to a centralizer $\Omega^{[1]}$ on L^1 defined by

$$\Omega^{[1]}(f) = u|f|^{1/q}\Omega(|f|^{1/p}),$$

where $u|f|$ is the polar decomposition of $f \in L^1$. Moreover, all centralizers on L^1 arise in this form [**20**, Theorem 8.1]. As self extensions of L^p spaces in the category of quasi-Banach L^∞-modules are all induced by centralizers we have

$$\mathrm{Ext}(L^p) = \mathrm{Ext}(L^1) \qquad (1 < p < \infty).$$

Research supported in part DGICYT projects MTM2004-02635 and MTM2007-6994-C02-02.

Actually [**20**] contain much more general results for (Banach) function spaces, but we will concentrate on Lebesgue spaces in this introduction.

The basic problem left open in [**20**, p. 83] was to decide whether this correspondence extends to $0 < p < 1$.

Plan of the paper. Let us explain the plan of the paper and present the main results. Section 1 is preliminary. Section 2 deals with sequence spaces. We address Kalton problem by considering first extensions $\ell^q \rightarrowtail X \twoheadrightarrow \ell^p$ for $0 < p, q < \infty$. We use an ultraproduct technique to transfer centralizers to homomorphisms on ultrapowers which allows one to apply Raynaud's representation of ultrapowers of L^p spaces to conclude that $\operatorname{Ext}(\ell^p, \ell^q)$ vanishes unless $p = q$. Then we use the functor Hom and the Hom-Ext sequences to show that $\operatorname{Ext}(\ell^p)$ is independent on $0 < p < \infty$. This answers Kalton's question in the affirmative, but only for sequence spaces.

In Section 3 we revisit Kalton 'pushing-down' method, we generalize it for centralizers acting between any two function spaces and we explain the homological meaning behind it. To sum up we have that if the extension of L^∞-modules $Y \rightarrowtail X \twoheadrightarrow Z$ comes from a centralizer (which is true for all extensions under rather mild assumptions on the quasi-Banach function spaces Z and Y) and V is another function space, then the 'tensorized' sequence

$$0 \longrightarrow Y \otimes V \longrightarrow X \otimes V \longrightarrow Z \otimes V \longrightarrow 0$$

makes sense and is exact —the symbol $\otimes$ indicates tensor product in the category of quasi-Banach modules over L^∞, which is the 'default' setting in this paper. We have taken time out to explain this point in some detail because the fact that tensor products preserve exactness is an extremely rare phenomenon that cannot be observed within locally convex categories. As for applications, let us mention that $\operatorname{Ext}(L^p, L^q)$ vanishes if p and q are different numbers in $[1, \infty]$.

Although we do not known if local convexity can be removed in the preceding result we have a complete solution for symmetric centralizers. Indeed, we shall see in Section 4 that if p and q are different numbers in $(0, \infty]$, then every symmetric centralizer from L^p to L^q is trivial (Theorem 11). After that we prove (Theorem 12) that the space of symmetric centralizers on L^p does not depend essentially on $0 < p < \infty$. Some applications to Hardy classes and derivations are given at the end of the Section —these are extensions to any p of some results already known for $p > 1$.

Section 5 contains counterexamples against the vague intuition that $\operatorname{Ext}(Z, Y)$ 'should' vanish if there are 'few' homomorphisms from Z to Y. Indeed we show that there are Orlicz function spaces with $\operatorname{Hom}(Z, Y) = 0$ and $\operatorname{Ext}(Z, Y) \neq 0$. A technical point of the proof is deferred to the ensuing Appendix.

A number of remarks and open questions are sprinkled throughout the paper.

1. Centralizers and extensions

Let A be a Banach algebra that for all purposes in this paper will be L^∞. A quasi-normed module over A is a quasi-normed space X together with a jointly continuous outer multiplication $A \times X \to X$ satisfying the traditional algebraic requirements. If the underlying space is complete (that is, a quasi-Banach space) we speak of a quasi-Banach module. Given A-modules X and Y, a homomorphism $T : X \to Y$ is an operator such that $T(ax) = aT(x)$ for all $a \in A$ and $x \in X$. Operators and homomorphisms are assumed to be continuous unless otherwise stated. If no continuity is assumed, we speak of linear maps and morphisms. Let us fix some notations: $L(X, Y)$ denotes the space of operators from X to Y. We use $\operatorname{Hom}_A(X, Y)$ for the homomorphisms and $\mathcal{M}_A(X, Y)$ for morphisms. If there is no possible confussion about the underlying algebra A, we omit the subscript.

1.1. Extensions. An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0 \tag{1}$$

The open mapping theorem [**25**] guarantees that $\imath$ embeds Y as a closed submodule of X in such a way that the corresponding quotient X/Y is isomorphic to Z. Two extensions $0 \to Y \to X_i \to Z \to 0$ $(i = 1, 2)$ are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle u} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

By the three-lemma [**16**], and the open mapping theorem, u must be an isomorphism. We say that (1) splits if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This just means that Y is a complemented submodule of X (that is, there is a homomorphism $X \to Y$ which is a left inverse for the inclusion $Y \to X$; equivalently, there is a homomorphism $Z \to X$ which is a right inverse for the quotient $X \to Z$) and implies that X is isomorphic to the direct sum $Y \oplus Z$ (the converse is not true in general). Given quasi-Banach modules Y and Z, we denote by $\operatorname{Ext}_A(Z, Y)$ (or just $\operatorname{Ext}_A(Z)$ when $Y = Z$) the set of all possible A-module extensions (1) modulo equivalence. By using pull-back and push-out constructions, it can be proved (see [**6**] for the details in the F-space setting) that $\operatorname{Ext}_A(Z, Y)$ carries a "natural" structure of commutative A-bimodule (without topology) in such a way that trivial extensions correspond to 0. (The usual approach using injective or projective representations completely fails dealing with quasi-Banach modules since there are neither injective nor projective objects.) Thus, $\operatorname{Ext}_A(Z, Y) = 0$ means "every A-module extension $0 \to Y \to X \to Z \to 0$ is trivial".

Taking A as the underlying field $\mathbb{K}$, one recovers extensions in the quasi-Banach space setting. We write $\operatorname{Ext}_{\mathbb{K}}(Z, Y)$ for the space of all extensions of Z by Y (modulo equivalence).

1.2. Function spaces as modules. From now on, μ will denote a fixed countably additive measure on measure on the measure space $\mathcal{S}$. We will assume μ sigma-finite or, at least, decomposable. Our ambient space will be L^0, the space of all measurable functions on $\mathcal{S}$ with the topology of convergence in measure on sets of finite measure and we apply the usual convention about identifying functions equal almost everywhere. L^∞ denotes the Banach algebra of all essentially bounded measurable functions on $\mathcal{S}$ equipped with the essential supremum norm and 'pointwise' operations.

Our 'default' category will be that of quasi-Banach modules over L^∞. Accordingly, we use minimal notation in this setting and so $\operatorname{Hom}(X, Y), \mathcal{M}(X, Y), \operatorname{Ext}(X, Y)$, and the like always refer to the algebra L^∞ for a measure that should be clear for the context —and will never be the ground field $\mathbb{K}$.

A function space is defined as a linear subspace X of L^0 equipped with a quasi-norm $\|\cdot\|_X$ such that if $f \in L^0, g \in X$ and $|f| \le g$, then $f \in X$ and $\|f\|_X \le \|g\|_X$. The reader will notice that this nothing different from being a (quasi-normed) L^∞-module of measurable functions and so we consider function spaces as L^∞-modules under 'pointwise' multiplication. On the other hand every function space is automatically a lattice under the pointwise order —the order of L^0. We invariably assume X saturated, that is, $1_A \in X$ for $\mu(A) < \infty$ and the inclusion of X into L^0 continuous. Let us introduce some relevant properties that function spaces may or not have.

Definition 1. A (saturated, quasi-normed) function space X is said to be:

- Minimal if simple integrable functions are dense.

- Maximal if whenever (f_n) is an increasing sequence of non-negative functions in X converging almost everywhere to f and $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$.
- Order-continuous if whenever $(f_i)_{i\in I}$ is a decreasing net in X such that $\bigwedge_i f_i = 0$ one has $\|f_i\| \to 0$ as i increases in I.
- r-convex if there is M such that
$$\left\| \left(\sum_{i=1}^{k} |f_i|^r \right)^{1/r} \right\| \leq M \left(\sum_{i=1}^{k} \|f_i\|^r \right)^{1/r} \qquad (f_i \in X).$$
- r-concave, if the reversed inequality holds for some $M > 0$.
- Lattice-convex if it is r-convex for some $r > 0$.

Every order continuous function space is minimal. The spaces L^p are order continuous as well as maximal for every $0 < p < \infty$. It is clear that r-convexity implies r-normability. Moreover, X is r-convex if and only if $X^r = \{f \in L^0 : |f|^{1/r} \in X\}$ is (isomorphic to) a Banach space with the quasi-norm $\|f\|_{X^r} = \||f|^{1/r}\|_X^r$. In this case, X is also s-convex for every $0 < s < r$ [**18**, lemma 2.1].

Lattice convexity is introduced and treated in detail in [**18**]. We are not giving the original definition, but an equivalent condition [**18**, Theorem 2.2]. Every function space which is q-concave for some finite q turns out to be lattice-convex [**18**, Theorem 4.1].

1.3. Centralizers and the extensions they induce.

Definition 2. Let Z and Y be (quasi-normed) function spaces.

- A centralizer from Z to Y is a homogeneous mapping $\Omega : Z \to L^0$ (sic) satisfying an estimate
$$\|\Omega(af) - a\Omega(f)\|_Y \leq C\|a\|_\infty \|f\|_Z.$$
Homogeneous means $\Omega(\lambda f) = \lambda \Omega f$ for all $\lambda \in \mathbb{K}, f \in Z$. We write $\mathcal{C}(Z,Y)$ (or just $\mathcal{C}(Z)$ if $Y = Z$) for the set of all centralizers from Z to Y and we denote by $C[\Omega]$ the least constant for which the preceding inequality holds.
- We say that $\Omega : Z \to L^0$ is quasi-linear from Z to Y and we write $\Omega \in \mathcal{Q}(Z,Y)$ if
$$\|\Omega(f+g) - \Omega(f) - \Omega(g)\|_Y \leq Q(\|f\|_Z + \|g\|_Z)$$
for some constant Q independent on $f, g \in Z$. The least constant for which the preceding inequality holds is denoted $Q[\Omega]$.

Before going further let us state the following simple remark. The proof is the same as [**20**, Lemma 4.2].

Lemma 1. *If Z and Y are function spaces, then $\mathcal{C}(Z,Y) \subset \mathcal{Q}(Z,Y)$. Moreover, there is a constant K depending only on Z and Y such that $Q[\Omega] \leq KC[\Omega]$ whenever Ω is a centralizer.*$\square$

We now indicate the connection between centralizers and extensions. Suppose Ω is a centralizer from Z to Y. Define $Y \oplus_\Omega Z = \{(g,f) \in L^0 \times Z : g - \Omega f \in Y\}$ quasi-normed by $\|(g,f)\|_\Omega = \|g - \Omega f\|_Y + \|f\|_Z$. Clearly, the map $\imath : Y \to Y \oplus_\Omega Z$ sending g to $(g,0)$ preserves the quasi-norm, while the map $\pi : Y \oplus_\Omega Z \to Z$ given as $\pi(g,f) = f$ is open, so that we have an extension of quasi-Banach spaces

$$0 \longrightarrow Y \xrightarrow{\imath} Y \oplus_\Omega Z \xrightarrow{\pi} Z \longrightarrow 0 \tag{2}$$

Actually only quasi-linearity is necessary here. The condition that Ω is a centralizer implies that the multiplication $a(g,f) = (ag, af)$ makes $Y \oplus_\Omega Z$ into an L^∞-module in such a way that (2) becomes an extension of modules.

It is proved in [**20**, Theorem 4.5] that if Z is minimal and Y maximal, then every extension of modules $Y \rightarrowtail X \twoheadrightarrow Z$ comes from a centralizer, up to equivalence. It is easily seen that two centralizers Ω and Φ (from Z to Y) induce equivalent extensions if and only if there is an L^∞-morphism $h : Z \to L^0$ such that $\|\Omega(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_Z$. We write $\Omega \sim \Phi$ in this case and $\Omega \approx \Phi$ if the preceding inequality holds for $h = 0$. In particular Ω induces a trivial extension if and only if $\|\Omega(f) - h(f)\|_Y \leq K\|f\|_Z$ for some morphism $h : Z \to L^0$. We then say that Ω is a trivial centralizer and we write $\operatorname{dist}(\Omega, h)$ for the least possible constant K in the preceding inequality. Actually we need these estimates only for f in a dense submodule $Z_0 \subset Z$. Let us state this formarly as follows and leave the proof for the reader.

Lemma 2. *Let Z and Y be quasi-Banach function modules. There is a constant K depending only on Z and Y such that, if $\Omega \in \mathcal{C}(Z, Y)$ and there is a dense submodule $Z_0 \subset Z$ and $\phi \in \mathcal{M}(Z_0, L^0)$ satisfying $\|\Omega f - \phi(f)\| \leq D\|f\|$ for some D and all $f \in Z_0$, then there is $\phi^* \in \mathcal{M}(Z, L^0)$ extending ϕ with $\|\Omega f - \phi^*(f)\| \leq KD\|f\|$ for all $f \in Z$.* □

We shall write $\mathcal{C}_\sim(Z, Y)$ and $\mathcal{C}_\approx(Z, Y)$ to denote the set $\mathcal{C}(Z, Y)$ factored by $\sim$ and $\approx$, respectively.

1.4. Morphisms, homomorphisms and multiplication operators. We are about to see that morphisms between function modules are given by multiplication. Recall that we are considering only decomposable measures.

Lemma 3. *Let Z be a function space and $h : Z \to L^0$ any mapping satisfying $h(af) = ah(f)$ for $a \in L^\infty$ and $f \in Z$. Then there is $\phi \in L^0$ such that $h(f) = \phi f$ for all $f \in Z$.*

Proof. As $C[h] = 0$, h must be linear, by Lemma 1. Next, we prove the Lemma assuming finite the underlying measure, so that Z contains L^∞. Set $\phi = h(1)$. We want to see that $h(f) = \phi f$ for every $f \in X$. Pick $f \in X$ and write $f' = 1 + |f|$. We have $1 = (1/f')f'$ and since $1/f'$ is in L^∞ we have $\phi = (1/f')h(f')$ and so $h(f') = f'\phi$, from where it follows that $h(f) = \phi f$.

Finally, assuming μ decomposable we fix a decomposition $\mathcal{S} = \oplus_i S_i$ into sets of finite measure and we set $\phi = \sum_i h(1_{S_i})$, where the sum is taken 'pointwise'. As $1_{S_i}h(f) = h(1_{S_i}f)$ we have $h(f) = \sum_i 1_{S_i}h(f) = \sum_i h(1_{S_i}f) = \sum_i h(1_{S_i})f = \phi f$. □

Given (quasi-normed) function modules X and Y we consider the set of multiplication operators $\mathrm{M}(X, Y) = \{\phi \in L^0 : \phi f \in Y \text{ for every } f \in X\}$ with the operator (quasi-) norm

$$\|\phi\| = \sup\{\|\phi f\|_Y : \|f\|_X \leq 1\}.$$

We close the Section with the following remarks we record here for future references.

Lemma 4. *Let X and Y be quasi-Banach function spaces.*

(a) $\mathrm{Hom}(X, Y) = \mathrm{M}(X, Y) = \mathcal{M}(X, Y)$.
(b) $\mathrm{Hom}(X, L^1) = X'$, *the Köthe dual of X.*
(c) *If X is order continuous and $T : X \to Y$ is an operator such that* $\operatorname{supp} Tf \subset \operatorname{supp} f$ *for every $f \in X$, then T is a homomorphism of L^∞-modules.*
(d) *If $q \leq p$ one has* $\mathrm{Hom}(L^p, L^q) = L^r$, *where $p^{-1} + r^{-1} = q^{-1}$. Otherwise* $\mathrm{Hom}(L^p, L^q)$ *is isomorphic to $L^\infty(\alpha)$, where α is the atomic part of μ.*

Proof. (a) In view of Lemma 7 we only need to check that if $\phi \in L^0$ is such that $\phi f \in Y$ for all $f \in X$, then $\|\phi : X \to Y\| < \infty$, which is immediate from the Closed Graph Theorem as given in [**25**, Theorem 1.6 and Corollary 1.7]. Part (b) is straightforward from (a).

(c) Let us first assume μ finite, so that $1_A \in X$ for every measurable $A \subset \mathcal{S}$. Put $\phi = T1_{\mathcal{S}}$. We want to see that $Tf = \phi f$ for all f. Take a measurable A and let B the complement of A. We have $\phi = T1_A + T1_B$, with $\operatorname{supp} T1_A \subset A$ and $\operatorname{supp} T1_B \subset B$. It follows that $T1_A = \phi 1_A$ and so $Tf = \phi f$ if f is simple. As T has closed graph we sees that T is given by multiplication by ϕ and, in particular, it is a homomorphism of L^∞-modules.

If μ is merely decomposable, fix a decomposition $\mathcal{S} = \oplus_i S_i$, with $\mu(S_i) < \infty$ for every i. Then T maps $X(S_i)$ (those functions to $Y(S_i)$ and so, for each i, there is $\phi_i \in L^0(S_i)$ (actually in $Y(S_i)$) such that $Tf = \phi_i f$ for all $f \in X(S_i)$. Considering the pointwise sum $\phi = \sum_i \phi_i$, as every $f \in X$ can be written as $f = \sum_i 1_{S_i} f$ (summation in the quasi-norm of X) we have

$$Tf = T\left(\sum_i 1_{S_i} f\right) = \sum_i T(1_{S_i} f) = \sum_i \phi_i 1_{S_i} f = \phi f,$$

from where it follows that T is an L^∞-homomorphism.

Part (d) clearly follows from Lemma 7. Notice however that if we consider α as a function on the set of atoms of μ so that $\alpha(A) = \mu(A)$, then $\phi \in L^0(\alpha)$ is bounded in $\mathrm{M}(L^p, L^q)$ if and only if $\alpha^{\frac{1}{p}-\frac{1}{q}}\phi$ is in $L^\infty(\alpha)$. $\square$

Notation and conventions. Throughout the paper Δ_X will denote the least constant Δ for which the weak triangle inequality $\|x+y\| \leq \Delta(\|x\| + \|y\|)$ holds for the quasi-norm of X. We use M for a constant independent on functions and centralizers, but perhaps depending on the involved spaces and which may vary from line to line.

2. Sequence spaces

In this Section we consider mainly sequence spaces, so the ground algebra is ℓ^∞ and we work in the category of quasi-Banach ℓ^∞-modules. In particular, (homo)morphisms and extensions refer to ℓ^∞ unless otherwise stated.

2.1. Splitting.

The main result of the Section is the following.

THEOREM 1. *If $p \neq q$, then* $\operatorname{Ext}(\ell^p, \ell^q) = 0$.

The proof uses ultrapowers and one simple idea of [**5**]. We need only very basic ultraproduct theory and we refer the reader to Sims' booklet [**33**] for definitions and information on the topic.

Let μ be a fixed σ-finite measure and consider the corresponding spaces L^p for $0 < p \leq \infty$. If U is a countably incomplete ultrafilter on a given index set I we can construct the ultrapowers $(L^p(\mu))_U = L^p_U$ for $0 < p \leq \infty$. It was known from the very beginning that for each fixed $0 < p < \infty$ one can represent L^p_U as $L^p(\nu)$ for some measure ν. This was proved in [**3**] for $1 \leq p < \infty$ and then for $0 < p < 1$ in [**32**]. The fourth section of [**32**] contains a serious gap and so the case $p = 0$ remains obscure. I thank Fernando Albiac for this remark.

The discovering of Raynaud that ν can be chosen so that $L^p(\nu)$ represents L^p_U for every finite p is crucial for us. The results we shall use are stated in the highly abstract setting of non-commutative spaces and perhaps some explanations are in order. First one considers L^1_U. This is an abstract L-space and so by a result of Kakutani there is a measure ν such that the lattice L^1_U is isometrically isomorphic to $L^1(\nu)$ through a Riesz isomorphism we denote Λ. The measure ν cannot be σ-finite unless L^1 is finite-dimensional but by a result by Maharam can be taken decomposable. This implies that $L^\infty(\nu)$ equals the conjugate of $L^1(\nu)$ and since every element of L^∞_U defines a functional on L^1_U we have an embedding $\Lambda_\infty : L^\infty_U \to L^\infty(\nu)$ which is in fact a homomorphism of algebras, so that every $L^\infty(\nu)$-module becomes an L^∞_U-module by restriction.

Now, for $0 < p < \infty$, let $S^p : L^p \to L^1$ be the Mazur map sending $f = u|f|$ to $u|f|^p$ and define $\Lambda_p : L^p_U \to L^p(\nu)$ as $\Lambda_p = (S^p_\nu)^{-1} \circ \Lambda \circ (S^p)_U$. These maps have the following properties [**28**, Theorems 3.6 and 5.1]:

- Λ_p is a surjective linear isometry between L^p_U and $L^p(\nu)$ which preserves the lattice and the L^∞_U-module structures.

- If p, q and r satisfy $p^{-1} + q^{-1} = r^{-1}$ and (f_i) and (g_i) are bounded families in L^p and L^q respectively, then
$$\Lambda_r[(f_i g_i)] = (\Lambda_p[(f_i)])(\Lambda_q[(g_i)]),$$
where the product appearing in the left-hand side is the obvious one $L^p \times L^q \to L^r$, while that of the right-hand side is the pointwise product $L^p(\nu) \times L^q(\nu) \to L^r(\nu)$. The square brackets denote the class of the given family in the corresponding ultrapower.

Lemma 5. *With the above notations we have* $\mathrm{Hom}_{L_U^\infty}(L_U^p, L_U^q) = \mathrm{Hom}_{L^\infty(\nu)}(L^p(\nu), L^q(\nu))$ *in the sense that if* $T : L_U^p \to L_U^q$ *is a homomorphism of* L_U^∞*-modules, then the corresponding operator*
$$\tilde{T} = \Lambda_q \circ T \circ \Lambda_p^{-1} : L^p(\nu) \to L^q(\nu)$$
is an homomorphism of $L^\infty(\nu)$*-modules.*

Proof. Clearly, $\tilde{T}$ preserves the product by elements of L_U^∞. Thus according to Lemma 4 (c) it suffices to see that for every non-negative $f \in L^p(\nu)$ we have $\operatorname{supp} f = B$ (up to equivalence in the measure algebra of ν), where B is given by
$$1_B = \bigwedge\{e \in L_U^\infty : f = ef, e = e^2\},$$
and the meet is computed in the lattice $L^\infty(\nu)$. (The support of $f \in L^0(\nu)$, defined as the set where f is not zero, depends only on the $L^\infty(\nu)$ structure: the set of those idempotents in $L^\infty(\nu)$ leaving f invariant must have a greatest lower bound in $L^\infty(\nu)$, necessarily of the form 1_A for some measurable A. Clearly, $A = \operatorname{supp} f$ in the measure algebra of ν.)

Indeed, take $g \in L^p(\nu)$ having support contained in $B \setminus \operatorname{supp} f$. Then f and g are disjoint in $L^p(\nu)$ and so are $\Lambda_p^{-1}(f)$ and $\Lambda_p^{-1}(g) = 0$ in L_U^p. Therefore they have disjoint representatives and we can find projections e_f, e_g in L_U^∞ such that $e_i h = \delta_{ih} h$ for $i, h \in \{f, g\}$. We have $1_B \le e_f$ and so $1_B e_g \le e_f e_g = 0$. As $g = 1_B e_g g$ we see that $g = 0$. It follows that each σ-finite subset of $B \setminus \operatorname{supp} f$ has measure zero and using a decomposition we get $\nu(B \setminus \operatorname{supp} f) = 0$. □

We are now ready for the proof of Theorem 1. As the reader can imagine we will show that every centralizer from ℓ^p to ℓ^q is trivial. Although the basic idea is the same in the two cases, the information we get from the proof and the technical details vary depending on whether $p > q$ or $p < q$ and so we treat these cases separately. From now on, ℓ^0 denotes the space of all sequences —it is L^0 for counting measure on the integers.

Proposition 1. *If* $0 < q < p < \infty$*, then for every* $\Omega \in \mathcal{C}(\ell^p, \ell^q)$ *there is* $\phi \in \ell^0$ *such that* $\|\Omega f - \phi f\|_q \le M\|f\|_p$ *for some* M *and all* $f \in \ell^p$*.*

Proof. Suppose on the contrary $\mathcal{C}(\ell^p, \ell^q)$ contains a non-trivial centralizer Ω. Set $\Omega_n = \Omega \circ \pi_n$, where π_n is the obvious projection of ℓ^p onto the subspace spanned by $e_1, e_2, \dots, e_n$. It is easily seen (though not entirely trivial) that one also has
$$\operatorname{dist}(\Omega_n, \mathcal{M}(\ell^p, \ell^0)) \to \infty \qquad (n \to \infty).$$
These distances are all finite because Ω_n factors through a finite dimensional space. For each n we take a morphism ϕ_n such that
$$\delta_n = \operatorname{dist}(\Omega_n, \phi_n) \le \operatorname{dist}(\Omega_n, \mathcal{M}(\ell^p, \ell^0)) + 1/n.$$
Of course, $\delta_n \to \infty$ as $n \to \infty$. Put
$$v_n = \frac{\Omega_n - \phi_n}{\delta_n},$$
so that v_n is a homogeneous mapping from ℓ^p to ℓ^q (not ℓ^0!) with $\|v_n : \ell^p \to \ell^q\| \le 1$ and $C[v_n][\le \delta_n^{-1} C[\Omega] \to 0$ as $n \to \infty$.

Let U be a free ultrafilter on the integers and consider the corresponding ultrapowers ℓ^p_U and ℓ^q_U. We can use the (probably nonlinear) maps v_n to define $v : \ell^p_U \to \ell^q_U$ by

$$v[(f_n)] = [(v_n(f_n))].$$

Let us check that v is well defined. First, suppose $[(f_n)] = 0$, that is, $\|f_n\| \to 0$ along U. Then $\|v_n(f_n)\| \leq \|f_n\| \to 0$ along U. Suppose now $[(f_n)] = [(g_n)]$. We must prove that $[(v_n(f_n))] = [(v_n(g_n))]$. But

$$\lim_U \|v_n(f_n) - v_n(g_n)\| = \lim_U \|v_n(f_n) - v_n(g_n) - v_n(f_n - g_n)\| \leq \lim_U Q[v_n](\|g_n\| + \|f_n - g_n\|) = 0$$

and the definition of v makes sense. Now it is nearly obvious that v is a continuous homomorphism of ℓ^∞_U-modules. By the discussion of Raynaud's results above, v is given by multiplication by a member of ℓ^r_U, where $1/r + 1/p = 1/q$ and we can take a bounded sequence (u_n) in ℓ^r representing v in the sense that

$$v[(f_n)] = [(u_n f_n)]$$

whenever (f_n) is a bounded sequence in ℓ^p. This implies that $\operatorname{dist}(v_n, u_n) \to 0$ along U. In particular, for every $\varepsilon > 0$, the set

$$S = \{n \in \mathbb{N} : 0 < \operatorname{dist}(\delta_n^{-1}(\Omega_n - \phi_n), u_n) < \varepsilon\}$$

is in U and it contains infinitely many indices n. For these n we get

$$\operatorname{dist}(\Omega_n, \phi_n + \delta_n u_n) < \varepsilon \delta_n < 2\varepsilon \operatorname{dist}(\Omega_n, \mathcal{M}(\ell^p, \ell^0)),$$

in striking contradiction with our choice of ϕ_n. □

Proposition 2. *Given $\Omega \in \mathcal{C}(\ell^p, \ell^q)$ we define $\phi : \mathbb{N} \to \mathbb{K}$ by $\phi(k) = \Omega(e_k)(k)$, where (e_k) is the unit basis of ℓ^p. If $0 < p < q < \infty$, then* $\operatorname{dist}(\Omega, \phi) < \infty$.

We give now a proof based on the same idea, thus rendering this Section self-contained. A more efficient proof will be given later in Proposition 6.

Proof. With the notations of Proposition 1 we put $\delta_n = \operatorname{dist}(\Omega \circ \pi_n, \phi \circ \pi_n)$. Suppose $\delta_n \to \infty$. As before we define $v : \ell^p \to \ell^q$ taking the ultraproduct of the maps

$$v_n = \frac{\Omega \circ \pi_n - \phi \circ \pi_n}{\delta_n}$$

We will prove that $v = 0$.

To see this consider the isometric Riesz isomorphism $\Lambda : \ell^1_U \to L^1(\nu)$ and we recall that each atom of ν is associated to a unique positive extreme point of the unit ball of $L^1(\nu)$. As Λ preserves extreme points, we see that atoms of ν come from extreme points of the ball of ℓ^1_U. On the other hand each extreme point in the ball of an ultrapower has a representative consisting of extreme points in the ball of the model space and therefore the (positive) extreme points of the unit ball of ℓ^1_U have all the form $[(e_{i(n)})]$ for some function $i : \mathbb{N} \to \mathbb{N}$. Actually $[(e_{i(n)})] = [(e_{j(n)})]$ in ℓ^1_U if and only if the set $\{n \in \mathbb{N} : i(n) = j(n)\}$ belongs to U so that the atoms of ν are in correspondence with the elements of the set-theoretic ultrapower $\mathbb{N}^U$. Thus we have a decomposition $L^1(\nu) = \ell^1(\mathbb{N}^U) \oplus_1 L^1(\gamma)$, where γ is the continuous part of ν, and so $L^r(\nu) = \ell^r(\mathbb{N}^U) \oplus_r L^r(\gamma)$ for $0 < r < \infty$. Set $\tilde{v} = \Lambda_q \circ v \circ \Lambda_p^{-1}$. This is a homomorphism of $L^\infty(\nu)$-modules from $L^p(\nu)$ to $L^q(\nu)$ and so $\tilde{v}$ vanishes on $L^p(\gamma)$ (see Lemma 4(d)). On the other hand, v vanishes on every $[(e_{i(n)})]$ by the very definition and therefore $v = 0$. This is a contradiction since taking for each $n \in \mathbb{N}$ some normalized $f_n \in \ell^p_n$ such that

$$\|\Omega(f_n) - \phi f_n\|_q \geq \frac{\delta_n}{2}$$

we have $\|v[(f_n)]\| \geq \frac{1}{2}$ in ℓ^q_U. □

A simple inspection to the proofs of Propositions 1 and 2 reveals the following.

Corollary 1. *Let $0 < p, q < \infty$ be different real numbers. There is a constant $K = K(p,q)$ such that for each $\Omega \in \mathcal{C}(\ell^p, \ell^q)$ there is $\phi \in \ell^0$ satisfying* $\operatorname{dist}(\Omega, \phi) \leq KC[\Omega]$. □

Remark 1. (Uniform approximation) It seems very likely that if Z and Y are quasi-Banach function spaces such that every $\Omega \in \mathcal{C}(Z, Y)$ is trivial, then we can find $\phi \in L^0$ such that $\operatorname{dist}(\Omega, \phi) \leq KC[\Omega]$ for some K independent on Ω. This is true, for instance, if Z has a dense submodule algebraically isomorphic to L^∞ (which is always the case if Z is a sequence space with unconditional basis) or if both Z and Y are symmetric spaces on $[0, \infty)$. We refer the reader to [**17**, Proposition 3.3] for the affirmative answer in the pure linear setting (all linear spaces are free modules over the underlying field) and to [**4**] for some related ideas.

2.2. Uses of the Hom-Ext sequence. We now introduce a bit of homology to compare different spaces of centralizers. Let $Y \rightarrowtail X \twoheadrightarrow Z$ be an extension of quasi-Banach L^∞-modules. If E is another L^∞-module we can apply $\operatorname{Hom}(-, E)$ to get an exact sequence

$$0 \to \operatorname{Hom}(Z, E) \to \operatorname{Hom}(X, E) \to \operatorname{Hom}(Y, E) \to \operatorname{Ext}(Z, E) \to \cdots \tag{3}$$

The arrows involving extensions are just morphisms: no topology is defined on the spaces of extensions. In any case, if $\operatorname{Ext}(Z, E)$ vanishes, then (3) represents an extension of $\operatorname{Hom}(Y, E)$ by $\operatorname{Hom}(Z, E)$.

Suppose now Z, Y and E are function spaces and the starting extension is given by a centralizer Ω so that $X = Y \oplus_\Omega Z$. Then $\operatorname{Hom}(Z, E) = \mathrm{M}(Z, E)$ and $\operatorname{Hom}(Y, E) = \mathrm{M}(Y, E)$ are function spaces —perhaps they are not saturated, but this is irrelevant here. If, besides, for every centralizer $\Psi \in \mathcal{C}(Z, E)$ there is $\psi \in L^0$ such that $\operatorname{dist}(\Psi, \psi) \leq KC[\Psi]$, for some $K > 0$, we can define a centralizer Φ from $\mathrm{M}(Y, E)$ to $\mathrm{M}(Z, E)$ as follows. Take $f \in \mathrm{M}(Y, E)$ and consider the composition

$$f \circ \Omega : Z \to L^0 \to L^0,$$

where $f : L^0 \to L^0$ is the corresponding multiplication operator. Then $f \circ \Omega$ is in $\mathcal{C}(Z, E)$, and $C[f \circ \Omega] \leq \|f : Y \to E\| C[\Omega]$. Therefore there is $\phi \in L^0$ satisfying $\operatorname{dist}(f \circ \Omega, \phi) \leq KC[f \circ \Omega]$ and we can define a homogeneous map $\Phi : \mathrm{M}(Y, E) \to L^0$ taking $\phi = \Phi(f)$, so that

$$\|f\Omega(g) - (\Phi f)g\|_E \leq KC[\Omega]\|f\|\|g\|_E \qquad (f \in \mathrm{M}(Y, E), g \in Z). \tag{4}$$

This is in fact a centralizer to $\mathrm{M}(Z, E)$. Indeed, take $f \in \mathrm{M}(Y, E)$ and $a \in L^\infty$, with $\|a\|_\infty \leq 1$. We have

$$\begin{aligned}
\|\Phi(af) - a\Phi f\|_{\mathrm{M}(Z,E)} &= \sup_{\|g\|_Z \leq 1} \|\Phi(af)g - a\Phi fg\|_E \\
&= \sup_{\|g\|_Z \leq 1} \|\Phi(af)g - af\Omega g + af\Omega g - f(\Omega ag) + f\Omega(ag) - (\Phi f)ag\|_E \\
&\leq (2M + 1)\Delta_E^2 C[\Omega]\|f\|_{\mathrm{M}(Y,E)}.
\end{aligned}$$

All relevant properties of Φ follow from (4). Let us show that

$$\operatorname{Hom}(Y \oplus_\Omega Z, E) = \mathrm{M}(Z, E) \oplus_\Phi \mathrm{M}(Y, E)$$

under the pairing $(f, g)(y, z) = fz - gy$. One has

$$\begin{aligned}
\|(f, g)(y, z)\|_E = \|fz - gy\|_E &= \|fz - \Phi(g)z + \Phi(g)z - gy\|_E \\
&\leq M(\|f - \Phi g\|_{\mathrm{M}(Y,E)}\|z\|_Z + \|(\Phi g)z - gy\|_E) \\
&\leq M(\|f - \Phi g\|_{\mathrm{M}(Y,E)}\|z\|_Z + \|g\Omega z - gy\|_E + \|g\|_{\mathrm{M}(Y,E)}\|z\|_Z) \\
&\leq M(\|f - \Phi g\|_{\mathrm{M}(Y,E)}\|z\|_Z + \|g\|_{\mathrm{M}(Y,E)}\|\Omega z - y\|_Y + \|g\|_{\mathrm{M}(Y,E)}\|z\|_Z) \\
&\leq M\|(f, g)\|_\Phi \|(y, z)\|_\Omega
\end{aligned}$$

This defines a homomorphism u making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{M}(Z,E) \arrow[r] \arrow[d, equal] & \mathrm{M}(Z,E)\oplus_\Phi \mathrm{M}(Y,E) \arrow[r] \arrow[d, "u"] & \mathrm{M}(Y,E) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(Z,E) \arrow[r] & \mathrm{Hom}(Y\oplus_\Omega Z,E) \arrow[r] & \mathrm{Hom}(Y,E) \arrow[r] & 0 \end{tikzcd}$$

By the five-lemma u must be bijective. By the open mapping theorem (or better, by Roelcke's lemma [**12**, Lemma A]) it is isomorphic. A direct proof is also available.

Let us consider the transformation $\Omega\in\mathcal{C}(Z,Y)\longmapsto\Phi\in\mathcal{C}(\mathrm{M}(Y,E),\mathrm{M}(Z,E))$ in more detail. Although there is some arbitrariness in the definition of Φ, if Φ' is another centralizer such that

$$\|f\Omega(g)-(\Phi' f)g\|_E\le M\|f\|\|g\|_E$$

for some M and all $f\in\mathrm{M}(Y,E)$ and $g\in E$, then $\Phi'\approx\Phi$. So, after a moment's reflection we see that (4) defines a mapping $\mathcal{C}_\approx(Z,Y)\to\mathcal{C}_\approx(\mathrm{M}(Y,E),\mathrm{M}(Z,E))$ we may denote $\mathrm{Hom}(-,E)$. It should be obvious that this map is in fact a morphism over L^∞. Besides this correspondence sends $\mathrm{M}(Z,L^0)$ to $\mathcal{M}(\mathrm{M}(Z,E),L^0)$ and so it defines a morphism $\mathcal{C}_\sim(Z,Y)\to\mathcal{C}_\sim(\mathrm{M}(Y,E),\mathrm{M}(Z,E))$ we still denote $\mathrm{Hom}(-,E)$. As we have seen, this is coherent with the usual meaning of $\mathrm{Hom}(-,E)$ acting between (classes of) extensions.

We can use $\mathrm{Hom}(-,\ell^r)$ to 'dualize' extensions in a very similar way as one uses $L(-,\mathbb{K})$ to take adjoints in Banach spaces. After all notice that, for any $0<p,q<\infty$ and $1/r=1/p+1/q$, we have

$$\mathrm{Hom}(\ell^p,\ell^r)=\ell^q \qquad\text{and}\qquad \mathrm{Hom}(\ell^q,\ell^r)=\ell^p$$

so $\mathrm{Hom}(\mathrm{Hom}(\ell^p,\ell^r),\ell^r)=\ell^p$ whenever $0<r<p<\infty$. Thus we see that ℓ^p is 'reflexive' (with ℓ^r as target space) even for $0<p<1$! For an arbitrary ℓ^∞ module E one only has a homomorphism $\imath: E\to\mathrm{Hom}(\mathrm{Hom}(E,\ell^r),\ell^r)$ given by $(\imath e)(u)=u(e)$.

The following couple of results provides the affirmative answer to Kalton's question we mentioned in the Introduction for sequence spaces.

Theorem 2. *Let $0<p,q<\infty$ be fixed. Then $\mathrm{Hom}(-,\ell^r)$ defines an isomorphism between $\mathrm{Ext}(\ell^p)$ and $\mathrm{Ext}(\ell^q)$, where r is given by $1/p+1/q=1/r$.*

Proof. (Linear.) Suppose we are given an extension of ℓ^∞-modules $\ell^p\rightarrowtail X\twoheadrightarrow\ell^p$. Apply $\mathrm{Hom}(-,\ell^r)$ to get

$$0\to\mathrm{Hom}(\ell^p,\ell^r)\to\mathrm{Hom}(X,\ell^r)\to\mathrm{Hom}(\ell^p,\ell^r)\to\mathrm{Ext}(\ell^p,\ell^r)\to\cdots \tag{5}$$

But $\mathrm{Ext}(\ell^p,\ell^r)=0$, so the above diagram is in fact a self-extension of ℓ^q. It is pretty obvious that this procedure preserves equivalence and so it defines a morphism $\mathrm{Hom}(-,\ell^r):\mathrm{Ext}(\ell^p)\longrightarrow\mathrm{Ext}(\ell^q)$.

To see that it is indeed an isomorphism, just consider $\mathrm{Hom}(-,\ell^r)$ as a map from $\mathrm{Ext}(\ell^q)$ to $\mathrm{Ext}(\ell^p)$ —taking into account that $r<q$— and let us check that the two maps are inverse to each other. Indeed, if we apply $\mathrm{Hom}(-,\ell^r)$ to (5), we obtain another extension

$$0\to\mathrm{Hom}(\mathrm{Hom}(\ell^p,\ell^r),\ell^r)\to\mathrm{Hom}(\mathrm{Hom}(X,\ell^r),\ell^r)\to\mathrm{Hom}(\mathrm{Hom}(\ell^p,\ell^r),\ell^r)\to 0. \tag{6}$$

But after the identification $\ell^p=\mathrm{Hom}(\mathrm{Hom}(\ell^p,\ell^r),\ell^r)$ this extension is equivalent to the starting one since the diagram

$$\begin{tikzcd} \ell^q \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\imath"] & \ell^p \arrow[d, equal] \\ \mathrm{Hom}(\mathrm{Hom}(\ell^q,\ell^r),\ell^r) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}(X,\ell^r),\ell^r) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}(\ell^q,\ell^r),\ell^r) \end{tikzcd}$$

is commutative.

(Nonlinear) We know every self-extension of ℓ^p comes from a centralizer. With the same notation as before, taking $Z = Y = \ell^p$ and $E = \ell^r$, so that $\mathrm{M}(Z, E) = \mathrm{M}(Y, E) = \ell^q$, we have seen that for every $\Omega \in \mathcal{C}(\ell^p)$ there is $\Phi \in \mathcal{C}(\ell^q)$ such that

$$\|f\Omega(g) - \Phi(f)g\|_r \leq M\|f\|_q\|g\|_p. \tag{7}$$

And this correspondence defines a morphism $\mathrm{Hom}(-, \ell^r) : \mathcal{C}_\sim(\ell^p) \longrightarrow \mathcal{C}_\sim(\ell^q)$. Taking now $Z = Y = \ell^q$ and $E = \ell^r$ again we can regard $\mathrm{Hom}(-, \ell^r)$ as a morphism from $\mathcal{C}_\sim(\ell^q)$ to $\mathcal{C}_\sim(\ell^p)$. That these morphisms are inverse to each other is obvious from (7). □

Please notice the 'contravariant' nature of the isomorphism appearing in Theorem 2. Actually, if $p = q$ so that $r = p/2$, the involutive automorphism $\mathrm{Hom}(-, \ell^r)$ is not the identity on $\mathrm{Ext}(\ell^p, \ell^p)$. In particular, $\mathrm{Hom}(-, \ell^1)$ is just Banach space duality from $\mathrm{Ext}(\ell^p)$ to $\mathrm{Ext}(\ell^q)$ when $1 < p, q < \infty$ are conjugate exponents in the sense that $1/p + 1/q = 1$.

A 'covariant' isomorphism can be obtained through $\mathrm{Hom}(\ell^s, -)$ as follows.

Suppose $\ell^p \rightarrowtail X \twoheadrightarrow \ell^p$ is an extension of ℓ^∞-modules. Fix $s > p$ and apply $\mathrm{Hom}(\ell^s, -)$ to get

$$0 \to \mathrm{Hom}(\ell^s, \ell^p) \to \mathrm{Hom}(\ell^s, E) \to \mathrm{Hom}(\ell^s, \ell^p) \to \mathrm{Ext}(\ell^s, \ell^p) \to \cdots \tag{8}$$

As $\mathrm{Ext}(\ell^s, \ell^p) = 0$, the above diagram is in fact a self-extension of ℓ^q, where q is given by $1/s + 1/q = 1/p$. We have the following.

Theorem 3. *Let $0 < p < q < \infty$ be fixed. Then $\mathrm{Hom}(\ell^s, -)$ defines an isomorphism between $\mathrm{Ext}(\ell^p)$ and $\mathrm{Ext}(\ell^q)$, where s is given by $1/s + 1/q = 1/p$.*

Proof. Let us describe the action of $\mathrm{Hom}(\ell^s, -)$ on centralizers. Let Ω be a centralizer giving E. Regarding $f \in \ell^q$ as a homomorphism from ℓ^s to ℓ^p we can consider the composition $\Omega \circ f : \ell^s \to \ell^p \to \ell^0$. This is a centralizer from ℓ^s to ℓ^p, with $C[\Omega \circ f] \leq C[\Omega]\|f\|_q$. Now, Corollary 1 gives $\phi \in \ell^0$ such that $\mathrm{dist}(\phi, \Omega \circ f) \leq MC[\Omega]\|f\|_q$. Writing $\phi = \Theta(f)$ we obtain a centralizer Θ on ℓ^q satisfying

$$\|\Theta(f)g - \Omega(fg)\|_p \leq MC[\Omega]\|f\|_q\|g\|_s \qquad (f \in \ell^q, g \in \ell^s). \tag{9}$$

It is moreover clear that if Θ' is another centralizer satisfying a similar inequality, then $\Theta \approx \Theta'$. That sending Ω to Θ defines a morphism from $\mathcal{C}_\sim(\ell^p)$ to $\mathcal{C}_\sim(\ell^q)$ should be obvious.

Next we find an inverse for $\mathrm{Hom}(\ell^s, -)$. Although the most natural way of proceeding is to apply Kalton's downloading which is described in detail in next Section, we can invert the action of $\mathrm{Hom}(\ell^s, -)$ by composing two functors of type $\mathrm{Hom}(-, \ell^r)$. Notice that the composition of two contravariant functors is covariant.

We must adjust r and t so that $\mathrm{Hom}(\mathrm{Hom}(\ell^q, \ell^r), \ell^t) = \ell^p$. We fix any $r < q$, so that $\mathrm{Hom}(\ell^q, \ell^r) = \ell^{\tilde{q}}$, where $q^{-1} + \tilde{q}^{-1} = r^{-1}$. Then t is given by $q^{-1} + \tilde{q}^{-1} = t^{-1}$ and we have

$$t^{-1} = r^{-1} - q^{-1} + p^{-1} = r^{-1} + s^{-1}.$$

We prove now that the composition

$$\mathrm{Hom}(-, \ell^t) \circ \mathrm{Hom}(-, \ell^r) : \mathrm{Ext}(\ell^q) \to \mathrm{Ext}(\ell^{\tilde{q}}) \to \mathrm{Ext}(\ell^p)$$

is left inverse to $\mathrm{Hom}(\ell^s, -) : \mathrm{Ext}(\ell^p) \to \mathrm{Ext}(\ell^q)$. This clearly implies that $\mathrm{Hom}(\ell^s, -)$ is an isomorphism since both $\mathrm{Hom}(-, \ell^r)$ and $\mathrm{Hom}(-, \ell^t)$ are.

So, let Ω be in $\mathcal{C}(\ell^p)$ and let $\Theta \in \mathcal{C}(\ell^q)$ be the centralizer given by (9). Next, we apply $\mathrm{Hom}(-, \ell^r)$ to Θ thus obtaining a centralizer $\Phi \in \mathcal{C}(\ell^{\tilde{q}})$ satisfying

$$\|\Phi(h)f - h\Theta(f)\|_r \leq M\|h\|_{\tilde{q}}\|f\|_q. \tag{10}$$

Finally, applying $\mathrm{Hom}(-, \ell^t)$ to Φ we get a centralizer $\Psi \in \mathcal{C}(\ell^p)$ such that

$$\|\Psi(k)h - k\Phi(f)\|_t \leq M\|k\|_p\|h\|_{\tilde{q}}. \tag{11}$$

We complete the proof by showing that $\Psi \approx \Omega$. To see this, take any $k \in \ell^p$ and $h \in \ell^{\tilde{q}}$ and write $k = fg$, with $f \in \ell^q, g \in \ell^s$ and $\|k\|_p = \|f\|_q \|g\|_s$. According to (9), (10) and (11) we have

$$\|\Theta(f)gh - \Omega(fg)h\|_t \leq M\|f\|_q\|g\|_s\|h\|_{\tilde{q}}$$
$$\|fg\Phi(h) - \Theta(f)gh\|_t \leq M\|f\|_q\|g\|_s\|h\|_{\tilde{q}}$$
$$\|\Psi(fg)h - fg\Phi(h)\|_t \leq M\|f\|_q\|g\|_s\|h\|_{\tilde{q}}.$$

And, therefore,

$$\|\Psi(k) - \Theta(k)\|_p = \|(\Psi(k) - \Theta(k)) : \ell^{\tilde{q}} \to \ell^t\| \leq M\|k\|_p,$$

as required. □

Remark 2 (Linear approximation vs. non-linear approximation). There is an important difference between Propositions 1 and 2. Indeed, while the approximating morphism ϕ depends linearly (actually 'modularly') on Ω when $p < q$ this cannot be achieved when $p > q$ for any values of p and q. Precisely, if $p > q$ there is no linear map $\Lambda : \mathcal{C}(\ell^p, \ell^q) \to \ell^0$ such that $\operatorname{dist}(\Omega, \Lambda(\Omega)) \leq MC[\Omega]$. Otherwise $\operatorname{Hom}(-, \ell^q)$ would be zero at $\operatorname{Ext}(\ell^p)$. In fact there is no linear Λ such that $\operatorname{dist}(\Omega, \Lambda(\Omega)) < \infty$. We will refrain from entering into further details here.

3. Kalton's downloading revisited

In this Section we present an explicit method that allows one to 'shift' centralizers applying a suitable transformation that replaces both the domain and the target space by new ones. We will prove in due course (Subsection 3.8) that this process is nothing different from taking a tensor product.

3.1. Product spaces. Let X and Y be quasi-normed function spaces on the same measure space. The space of products is

$$XY = \{u \in L^0 : u = fg \text{ for some } f \in X, g \in Y\}.$$

For $u \in XY$, put

$$\|u\|_{XY} = \inf\{\|f\|\|g\| : u = fg, f \in X, g \in Y\}.$$

Lemma 6. *XY is a quasi-normed function space. Moreover, if X is a p-Banach space and Y is a q-Banach space, then XY is an r-Banach space, where r is given by $1/r = 1/p + 1/q$.*

Proof. Let $u_1, u_2 \in XY$. We show that $u_1 + u_2$ belongs to XY and that $\|u_1 + u_2\| \leq 4\Delta_X\Delta_Y(\|u_1\| + \|u_2\|)$. Suppose $u_i = f_i g_i, i = 1, 2$ and let $h = |g_1| + |g_2|$. Then $g_i = a_i h$, with $\|a_i\|_\infty \leq 1$ and clearly $\|h\| \leq \Delta_Y(\|g_1\| + \|g_2\|)$. Hence,

$$u_1 + u_2 = f_1g_1 + f_2g_2 = f_1a_1h + f_2a_2h = (a_1f_1 + a_2f_2)h \in XY.$$

Moreover,

$$\|u_1 + u_2\| \leq \|a_1f_1 + a_2f_2\|\|h\| \leq \Delta_X\Delta_Y(\|f_1\| + \|f_2\|)(\|g_1\| + \|g_2\|).$$

Taking into account that, for $\alpha, \beta > 0$, $u_1 = \alpha f_1\alpha^{-1}g_1$ and $u_2 = \beta f_2\beta^{-1}g_2$, we obtain

$$\|u_1 + u_2\| \leq \Delta_X\Delta_Y(\alpha\|f_1\| + \beta\|f_2\|)(\alpha^{-1}\|g_1\| + \beta^{-1}\|g_2\|).$$

Minimizing the right-hand side on the preceding inequality, we get

$$\|u_1 + u_2\| \leq \Delta_X\Delta_Y\left(\|f_1\|^{1/2}\|g_1\|^{1/2} + \|f_2\|^{1/2}\|g_2\|^{1/2}\right)^2 \leq \Delta_X\Delta_Y\Delta_{1/2}(\|f_1\|\|g_1\| + \|f_2\|\|g_2\|).$$

From where it follows that XY is a quasi-normed module of measurable functions. Saturation and the continuity of the embedding $XY \subset L^0$ follows easily from the corresponding properties of X and Y.

We now prove completeness and the 'moreover' part at once. First, observe that a quasi-normed space Z is (isomorphic to) a s-Banach space if and only if every series $\sum_n z_n$ such that $(\|z_n\|) \in \ell^s$ converges in Z.

Let $\sum_n u_n$ be a series in XY, with $(\|u_n\|) \in \ell^r$, where $1/r = 1/p + 1/q$. We show that $\sum_n u_n$ converges in XY. Without loss of generality, we may assume $u_n \geq 0$ for all n. Take sequences $(s_n) \in \ell^p$ and $(t_n) \in \ell^q$ such that $s_n t_n > \|u_n\|$ and, for each n choose positive $f_n \in X$ and $g_n \in Y$ such that $u_n = f_n g_n$, with $\|f_n\|_X \leq s_n$ and $\|g_n\|_Y \leq t_n$. Put $f = \sum_{n=1}^\infty f_n$ and $g = \sum_{n=1}^\infty g_n$ and let $v = fg$.

Clearly, $\sum_{n=1}^m u_n \leq v$ for all m, so that we can define a measurable function taking

$$u(t) = \sum_{n=1}^\infty u_n(t) \qquad (t \in \mathcal{S}).$$

Obviously $u \in XY$, since XY is a function module and u is dominated by v, which belongs to XY. The proof will be complete if we show that $u = \sum_{n=1}^\infty u_n$ in XY. But, for each m, one has

$$u - \sum_{n=1}^m u_n \leq \left(\sum_{n=m+1}^\infty f_n \right) \left(\sum_{n=m+1}^\infty g_n \right)$$

and since

$$\lim_{m\to\infty} \left\| \sum_{n=m+1}^\infty f_n \right\|_X = \lim_{m\to\infty} \left\| \sum_{n=m+1}^\infty g_n \right\|_Y = 0$$

we are done. □

We now compute some product spaces which are important for applications.

Proposition 3. (a) *For $0 < p, q \leq \infty$ we have $L^p L^q = L^r$, where $1/r = 1/p + 1/q$.*
(b) *If X is a (Banach-) Köthe space and X' is its Köthe dual, then $X'X = L^1$.*

Proof. (a) First, notice that if $f \in L^p$ and $g \in L^q$, then $fg \in L^r$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$, by Hölder inequality. On the other hand, if $h \in L^r$, we can write $h = u|h|^{r/p}|h|^{r/q}$, where u is unimodular, $|h|^{r/p}$ belongs to L^p and $|h|^{r/q}$ belongs to L^q. Clearly, $\|h\|_r = \||h|^{r/p}\|_p \||h|^{r/q}\|_q$.

(b) This is just a rewording of the factorization of Lozanovskiĭ [**27**]: for every $h \in L^1$ and $\varepsilon > 0$ there exist $f \in X$ and $g \in X'$ such that $h = fg$ and $\|f\|_X \|g\|_{X'} \leq (1+\varepsilon)\|h\|_1$. □

3.2. Downloading. The following result is a generalization of [**20**, Theorem 5.1], where Kalton considered only the case in which $Z = Y$ is a Köthe space and $V = Z'$ so that $YV = ZV = L^1$.

Theorem 4. *Let Y, Z and V be function modules on the same measure and $\Omega \in \mathcal{C}(Z, Y)$ a centralizer. Then there is $\Omega' \in \mathcal{C}(ZV, YV)$ such that*

$$\|\Omega'(fg) - \Omega(f)g\|_{YV} \leq KC[\Omega]\|f\|_Z \|g\|_V \qquad (f \in Z, g \in V), \tag{12}$$

where K is a constant depending only on Z, Y and V. Moreover, if Θ is another centralizer satisfying the corresponding estimate, then $\Theta \approx \Omega'$.

Proof. As a preparation, we prove that if $f_1, f_2 \in Z$ and $g_1, g_2 \in V$ are such that $f_1 g_1 = f_2 g_2$, then

$$\|\Omega(f_1)g_1 - \Omega(f_2)g_2\|_{YV} \leq C\left(\|f_1\|_Z \|g_1\|_V + \|f_2\|_Z \|g_2\|_V\right),$$

for a suitable constant C. Indeed, let $h = f_1 g_1 = f_2 g_2$ and take $f = |f_1| + |f_2|$. Then from

$$\|\Omega(f_i) - f_i f^{-1}\Omega(f)\|_Y \leq C[\Omega]\|f\|_Z \qquad (i = 1, 2),$$

we get

$$\|\Omega(f_i)g_i - hf^{-1}\Omega(f)\|_{YV} \leq C[\Omega]\|f\|_Z \|g_i\|_V \qquad (i = 1, 2).$$

And so,

$$\begin{aligned}\|\Omega(f_1)g_1-\Omega(f_2)g_2\|_{YV} &\leq \Delta_{YV}C[\Omega]\|f\|_Z(\|g_1\|_V+\|g_2\|_V)\\ &\leq 2\Delta_V\Delta_Y\Delta_Z C[\Omega](\|f_1\|_Z+\|f_2\|_Z)(\|g_1\|_V+\|g_2\|_V).\end{aligned}$$

By homogeneity of Ω we also have

$$\|\Omega(f_1)g_1-\Omega(f_2)g_2\|_{YV}\leq 2\Delta_V\Delta_Y\Delta_Z C[\Omega](\alpha\|f_1\|_Z+\beta^{-1}\|f_2\|_Z)(\alpha^{-1}\|g_1\|_V+\beta\|g_2\|_V)$$

for all $\alpha,\beta>0$. And arguing as in the proof of Lemma 6, we obtain

$$\|\Omega(f_1)g_1-\Omega(f_2)g_2\|_{YV}\leq 4\Delta_V\Delta_Y\Delta_Z C[\Omega]\left(\|f_1\|_Z\|g_1\|_V+\|f_2\|_Z\|g_2\|_V\right).$$

Now, fix $\varepsilon>0$. For $h\in ZV$, write $h=f_hg_h$, with $f_h\in Z, g_h\in V$ and $\|f_h\|_Z\|g_h\|_V\leq(1+\varepsilon)\|h\|_{ZV}$. We may and do assume that f_h depends homogeneously on h (that is, $f_{\lambda h}=\lambda f_h$). Finally, define $\Omega':ZV\to L^0$ by $\Omega'(h)=\Omega(f_h)g_h$.

Let $f\in Z, g\in V$ and $h=fg$. Then,

$$\begin{aligned}\|\Omega'(fg)-\Omega(f)g\|_{YV} &= \|\Omega'(h)-\Omega(f)g\|_{YV}\\ &= \|\Omega(f_h)g_h-\Omega(f)g\|_{YV}\\ &\leq 4\Delta_V\Delta_Y\Delta_Z C[\Omega]\left(\|f_h\|_Z\|g_h\|_V+\|f\|_Z\|g\|_V\right)\\ &\leq 4(2+\varepsilon)\Delta_V\Delta_Y\Delta_Z C[\Omega]\|f\|_Z\|g\|_V.\end{aligned}$$

It remains to see that Ω' is a centralizer from ZV to YV. Take $h\in ZV$ and $a\in L^\infty$. Then,

$$\begin{aligned}\|\Omega'(ah)-a\Omega'(h)\|_{YV} &= \|\Omega'(ah)-a\Omega(f_h)g_h\|_{YV}\\ &= \|\Omega'(ah)-\Omega(f_h)ag_h\|_{YV}\\ &\leq 4(2+\varepsilon)\Delta_V\Delta_Y\Delta_Z C[\Omega]\|f_h\|_Z\|ag_h\|_V\\ &\leq 4(2+\varepsilon)\Delta_V\Delta_Y\Delta_Z C[\Omega]\|a\|_\infty\|h\|_{ZV}.\end{aligned}$$

The 'moreover' part is obvious. □

It is important to realize that the 'correspondence' between Ω and Ω' given by (12) induces a morphism from $\mathcal{C}_\approx(Z,Y)$ to $\mathcal{C}_\approx(ZV,YV)$ we will denote by $-\cdot V$. Clearly, $-\cdot V$ sends trivial centralizers to trivial centralizers and therefore it defines also a morphism from $\mathcal{C}_\sim(Z,Y)$ to $\mathcal{C}_\sim(ZV,YV)$ we still denote $-\cdot V$.

Unfortunately we cannot invert the action of $-\cdot V$ even when $Z=Y=L^p$ and $V=L^q$ unless the resulting product space is locally convex, so we need a direct criterion to prove the nontriviality of Ω'. Fortunately the following simple idea works for most function spaces appearing in nature.

Lemma 7. *Notations of Theorem 4.*

(a) *If* $\operatorname{Hom}(V,YV)=Y$*, then* Ω' *is trivial if and only if so is* Ω.
(b) *If* Y *and* V *are maximal, lattice-convex function spaces, then* $\operatorname{Hom}(V,YV)=Y$.

Proof. Let us prove (a). The hypothesis means that every homomorphism $V\to YV$ has the form $v\mapsto yv$ for some $y\in Y$ and that $\|y:V\to YV\|$ is equivalent to $\|y\|_Y$.

Suppose Ω' is trivial. Then there is $\phi\in L^0$ such that

$$\|\Omega'(fg)-\phi(fg)\|_{VY}\leq M\|fg\|_{ZV}\qquad(f\in Z, g\in V).$$

Taking (12) into account we have

$$\|(\Omega f)g-(\phi f)g\|_{VY}\leq M\|f\|_Z\|g\|_V\qquad(f\in Z, g\in V).$$

Therefore we have $\Omega\approx\phi$ and we are done.

As for (b), first of all notice that $X^r=\{ug^r: |u|=1, g\in X_+\}$ and $\|g^r\|_{X^r}=\|g\|_X^r$. Also, the arithmetic rule $X^rY^r=(XY)^r$ holds isometrically. Next we prove that if X and Y

are function modules, then $\mathrm{M}(X^r, Y^r) = \mathrm{M}(X,Y)^r$, with equality of quasi-norms. Indeed, let $\varphi \in \mathrm{M}(X^r, Y^r)_+$, so that for each $x \in X_+$ one has $\varphi x^r \in Y^r$. Then

$$\|\varphi^{1/r}x\|_Y^r = \|\varphi x^k\|_{Y^r} \leq \|\varphi\|\|x^r\|_{X^r} = \|\varphi\|\|x\|_X^r.$$

Therefore $\varphi^{1/r}$ belongs to $\mathrm{M}(X,Y)$ and $\|\varphi^{1/r} : X \to Y\| \leq \|\varphi : X^r \to Y^r\|^{1/r}$, that is, $\mathrm{M}(X^r, Y^r) \subset \mathrm{M}(X,Y)^r$ and $\|\varphi\|_{\mathrm{M}(X,Y)^r} \leq \|\varphi\|_{\mathrm{M}(X^r,Y^r)}$. The reversed containment is as follows. Let $\psi \in \mathrm{M}(X,Y)_+$, so that $\|\psi x\|_Y \leq \|\psi\|_{\mathrm{M}(X,Y)}\|x\|_X$. Then,

$$\|\psi^r x^r\|_{Y^r} = \|\psi x\|_Y^r \leq \|\psi\|_{\mathrm{M}(X,Y)}^r \|x\|_X^r = \|\psi\|_{\mathrm{M}(X,Y)}^r \|x^r\|_{X^r},$$

that is, $\|\psi^r\|_{\mathrm{M}(X^r,Y^r)} \leq \|\psi\|_{\mathrm{M}(X,Y)}^r = \|\psi^r\|_{\mathrm{M}(X,Y)^r}$, as desired.

Now we recall a result by Calderón [**8**] stating that if X and Y are Banach function modules, then the intermediate spaces $X^\theta Y^{1-\theta}$ Banach spaces for every $0 < \theta < 1$, and in particular so is $X^{1/2}Y^{1/2}$. Also we need the following recent result of Schep [**31**, Theorem 2.8]: if E and F are maximal Banach function spaces and EF is (isomorphic to) a Banach space, then $\mathrm{M}(E, EF) = F$.

After this preparation, let us complete the proof. We already know that $\mathrm{Hom}(V, VY) = \mathrm{M}(V, VY)$. The hypothesis implies that V^s and Y^t are Banach spaces for suitable s and t and so are V^r and Y^r, where $r = \min\{s,t\}$. As $V^{\frac{r}{2}}, Y^{\frac{r}{2}}$ and $V^{\frac{r}{2}}Y^{\frac{r}{2}} = (VY)^{\frac{r}{2}}$ are all Banach spaces we have

$$\mathrm{M}(V^{\frac{r}{2}}, (VY)^{\frac{r}{2}}) = \mathrm{M}(V^{\frac{r}{2}}, V^{\frac{r}{2}}Y^{\frac{r}{2}}) = Y^{\frac{r}{2}},$$

so

$$Y = (Y^{\frac{r}{2}})^{\frac{2}{r}} = \mathrm{M}(V^{\frac{r}{2}}, (VY)^{\frac{r}{2}})^{\frac{2}{r}} = \mathrm{M}(V^{\frac{r}{2}\frac{2}{r}}, (VY)^{\frac{r}{2}\frac{2}{r}}) = \mathrm{M}(V, VY),$$

which completes the proof. □

3.3. Intermediate spaces. In some applications of Theorem 4 one needs to know which spaces X have L^p as a factor (that is, $X = L^pV$ for suitably chosen V) and which spaces Y are factors of L^q (this means $L^q = YW$ for some W). The following result contains some partial answers.

PROPOSITION 4. *Let X be a quasi-Banach function module.*

(a) *If X is p-convex, then $L^p = X \cdot \mathrm{M}(X, L^p)$.*
(b) *If X is q-concave (hence maximal), then $X = L^q \cdot \mathrm{M}(L^q, X)$.*

PROOF. As before we will use lattice-convexity to pass to the locally convex setting, where the results appear in Schep [**31**]. Notice that $(L^p)^r = L^{p/r}$ for $0 < p, r < \infty$. Let us prove (a) and leave the proof of (b) to the reader. It suffices to see that $(L^p)^p$ equals $(X \cdot \mathrm{M}(X, L^p))^p$. The former space is just L^1, while the latter can be computed as

$$(X \cdot \mathrm{M}(X, L^p))^p = X^p(\mathrm{M}(X, L^p))^p = X^p\,\mathrm{M}(X^p, (L^p)^p) = X^p\,\mathrm{M}(X^p, L^1).$$

But X^p is a Banach function space and $\mathrm{M}(X^p, L^1)$ is quite clearly its Köthe dual. Now, take a look at Proposition 3. □

3.4. Intermission: minimal extensions and K-spaces. A minimal extension (of Z) is a short exact sequence of quasi-Banach spaces $\mathbb{K} \rightarrowtail X \twoheadrightarrow Z$, where $\mathbb{K}$ is the ground field. When every minimal extension of Z splits we say that Z is a K-space —K is not, but should be for Kalton [**14**]. It is important for us that, for $p \in (0, \infty]$, (the infinite-dimensional) L^p is a K-space if and only if $p \neq 1$ ([**17, 26, 29, 30**]). The deepest of these results is, by far, that L^∞ is a K-space.

Let A be a Banach algebra with unit 1 and X a quasi-Banach A-module. Then the right modules Z^* and $\mathrm{Hom}_A(Z, A^*)$ are naturally isomorphic: consider the map $1_* : \mathrm{Hom}_A(Z, A^*) \to Z^*$ given by composition (on the left): $1_*(h) = 1 \circ h$. This is clearly a homomorphism of (right) modules. Let $u : Z^* \to \operatorname{Hom}_A(Z, A^*)$ be given by $((u(z^*))(z))(a) = z^*(az)$. It is easily verified that 1_* and u are mutually inverse homomorphisms.

We now show that, when the underlying algebra is a K-space the 'first derived' version of the identity $Z^* = \operatorname{Hom}_A(Z, A^*)$ holds true.

THEOREM 5. *Let A be a commutative and unital Banach algebra whose underlying Banach space is a K-space. Then $\operatorname{Ext}_{\mathbb{K}}(Z, \mathbb{K}) = \operatorname{Ext}_A(Z, A^*)$ for all A-modules Z.*

PROOF. We give only the main steps. The way from $\operatorname{Ext}_A(Z, A^*)$ to $\operatorname{Ext}_{\mathbb{K}}(Z, \mathbb{K})$ is taking push-out with $1 : A^* \to \mathbb{K}$ as in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^* & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle 1}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathrm{PO} & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

As for other direction, let $\mathbb{K} \rightarrowtail E \twoheadrightarrow Z$ be a minimal extension of Z. Applying $L(A, -)$ we have

$$0 \to L(A, \mathbb{K}) \to L(A, E) \to L(A, Z) \to \operatorname{Ext}_{\mathbb{K}}(A, \mathbb{K}) \to \cdots \tag{13}$$

But A is a K-space and this is in fact an extension of $L(A, Z)$ by $A^* = L(A, \mathbb{K})$ in the category of left-modules on A. On the other hand there is a natural embedding $\jmath : Z \to L(A, Z)$ given by $\jmath z(a) = az$ and we can form the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(A,\mathbb{K}) & \longrightarrow & L(A,E) & \longrightarrow & L(A,Z) & \longrightarrow & 0 \\ & & \big\| & & \big\uparrow & & \big\uparrow{\scriptstyle \jmath} & & \\ 0 & \longrightarrow & A^* & \longrightarrow & \mathrm{PB} & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

whose lower row is the extension we were looking for. □

COROLLARY 2. *Let Z be an L^∞-module whose underlying quasi-Banach is a K-space. Then $\operatorname{Ext}(Z, L^1) = 0$. In particular $\operatorname{Ext}(L^p, L^1) = 0$ for every $p \neq 1$.*

PROOF. We have $\operatorname{Ext}_{L^\infty}(Z, (L^\infty)^*) = 0$. But L^1 is complemented in $(L^\infty)^*$ by a homomorphism (namely, the Radon-Nikodým projection) and the result follows. □

3.5. Esoteric remark on minimal extensions. Theorem 5 implies that if Z is any quasi-Banach L^∞-module, then $\operatorname{Ext}_{\mathbb{K}}(Z, \mathbb{K}) = \operatorname{Ext}_{L^\infty}(Z, (L^\infty)^*)$. Of course, the module $(L^\infty)^*$ is very large and one may ask under which conditions can be replaced by L^1. As L^1 is complemented by a homomorphism $(L^\infty)^*$ we see that $\operatorname{Ext}_{L^\infty}(Z, L^1)$ is a complemented submodule of $\operatorname{Ext}_{\mathbb{K}}(Z, \mathbb{K}) = \operatorname{Ext}_{L^\infty}(Z, (L^\infty)^*)$ for arbitrary Z.

We want to stress that the usual procedure to get a module extension out from a minimal extension of would lead to a kind of 'centralizer' from Z to $(L^\infty)^*$.

Kalton isolated in [**21**, p. 490 (4.4-4.6)] the subclass of quasi-linear maps $\Phi : L^1 \to \mathbb{K}$ which are associated to centralizers $\Psi \in \mathcal{C}(L^1)$ in the sense that

$$|\Phi(f) - \int_{\mathcal{S}} \Psi(f) d\mu| \leq M\|f\|_1 \qquad (f \in L^1_0) \tag{14}$$

under the name of 'semilinear functionals'. To be true Kalton deals with mappings Φ defined on (dense semi-ideals of) L^1_+, but a genuine quasi-linear map can be obtained as $f \mapsto \Phi(f^+) - \Phi(f^-)$, where $f = f^+ - f^-$ is the decomposition of f into its positive and negative parts. Notice that (14) means $\Phi \approx \langle 1_{\mathcal{S}}, \Psi \rangle$ on L^1_0. Then it is shown [**21**, Theorem 6.8] that each quasi-linear map $L^1_0 \to \mathbb{K}$ is equivalent to a semilinear one and therefore one has

$$\operatorname{Ext}(L^1, L^1) = \operatorname{Ext}_{\mathbb{K}}(L^1, \mathbb{K})$$

via evaluation at $1_{\mathcal{S}}$. This generalizes as follows. Part (a) holds for much less restrictive hypothesis on Z but the details would take us too far afield.

Proposition 5. (a) *If μ is finite and Z is a symmetric and minimal function space with nontrivial concavity, then* $\operatorname{Ext}_{\mathbb{K}}(Z,\mathbb{K})=\operatorname{Ext}(Z,L^1)$ *though evaluation at* $1_{\mathcal{S}}$.
(b) *If Z is a minimal sequence space, then* $\operatorname{Ext}_{\mathbb{K}}(Z,\mathbb{K})=\operatorname{Ext}(Z,\ell^1)$.

Proof. (a) Of course every extension of Z by L^1 induces a minimal extension of Z taking push-out with $1\in L^\infty=(L^1)^*$. Let us go in the opposite direction. We may assume $q>1$ and we have $X=L^q\cdot \mathrm{M}(L^q,X)$ by Proposition 4 (b). By symmetry Z contains L^q and so $\mathrm{M}(L^q,X)$ contains L^∞. Set M^∞ for the closure of L^∞ in $\mathrm{M}(L^q,X)$ and let us observe that $L^qM^\infty=X$. Now let $\mathbb{K}\rightarrowtail E\twoheadrightarrow Z$ be a minimal extension of Z. Applying $L(L^q,-)$ and taking into account that L^q is a K-space we get the extension of left L^∞-modules $L(L^q,\mathbb{K})\rightarrowtail L(L^q,E)\twoheadrightarrow L(L^q,Z)$. Let p be the conjugate exponent of q and consider the pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & L(L^q,\mathbb{K}) \arrow[r] & L(L^q,E) \arrow[r] & L(L^q,Z) \arrow[r] & 0\\
0 \arrow[r] & L^p \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & M^\infty \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

By our choice of M^∞ the lower extension in the preceding diagram arises (up to equivalence in $\operatorname{Ext}(M^\infty,L^p)$) from a centralizer and we can apply $-\cdot L^q$ to get an extension of $Z=L^qM^\infty$ by $L^1=L^qL^p$. Probably it is not necessary to say more.

Part (b) follows form Theorem 5, taking into account that the elements of finite support are dense in Z and that every element of $(\ell^\infty)^*$ with finite support is already in ℓ^1. □

3.6. Applications to Banach function spaces. The following result should be contrasted with the fact that $\operatorname{Ext}_{\mathbb{K}}(L^p,L^q)\neq 0$ for every choice of $p,q\in[1,\infty]$, apart from $p=q=\infty$ (see [**7**]).

Theorem 6. *Let $1\le p,q\le\infty$. Then* $\operatorname{Ext}(L^p,L^q)=0$ *unless* $p=q$.

Proof. Each element of $\operatorname{Ext}(L^p,L^q)$ comes from a centralizer Ω. If $q=1$ then Ω is trivial, by Corollary 2. Otherwise we apply Theorem 4 to Ω taking $V=L^r$, where r is the conjugate exponent of q so that $\Omega'\in\mathcal{C}(L^s,L^1)$, where $1/s=1+1/p-1/q$. A quick look at Lemma 7 ends the proof. □

Corollary 3. $\operatorname{Ext}(L^p,L^q)=0$ *for all* $0<p<q\le\infty$.

Proof. If $p\ge 1$ the result is contained in Theorem 6. We prove it for $0<p<1$. If $\mu=\alpha\oplus\gamma$ is the decomposition of μ into its (purely) atomic part and the continuous part, then

$$\begin{aligned}\operatorname{Ext}_{L^\infty}(L^p,L^q)&=\operatorname{Ext}_{L^\infty}(L^p(\alpha),L^q(\alpha))\times\operatorname{Ext}_{L^\infty}(L^p(\gamma),L^q(\gamma))\\&=\operatorname{Ext}_{L^\infty(\alpha)}(L^p(\alpha),L^q(\alpha))\times\operatorname{Ext}_{L^\infty(\gamma)}(L^p(\gamma),L^q(\gamma)).\end{aligned}$$

Hence we may assume that μ is either purely atomic or non-atomic. If μ is non-atomic, then the result follows from $\operatorname{Ext}_{\mathbb{K}}(L^p,L^q)=0$ (proved by Kalton in [**17**, Theorem 3.6(ii)]) and $L(L^p,L^q)=0$ (see Lemma 4(b)).

If μ is purely atomic, then it is (up to a renormalization which preserves all structures in consideration) counting measure on some index set and the result has been already proved in Proposition 2 —see also Proposition 6 below. □

3.7. Applications to sequence spaces. Let us begin with an ultrapower-free proof of Proposition 2 based on Theorem 4, its companion Lemma 7 and an old estimate due (of course) to Kalton.

Proposition 6. *Given $\Omega\in\mathcal{C}(\ell^p,\ell^q)$, define $\phi:\mathbb{N}\to\mathbb{K}$ by $\phi(k)=\Omega(e_k)(k)$, where (e_k) is the unit basis of ℓ^p. If $0<p<q\le\infty$, then*

$$\|\Omega f-\phi f\|_q\le MC[\Omega]\|f\|_p\qquad(f\in\ell^p),$$

for some constant M depending only on p and q.

Proof. First suppose $q \leq 1$, so that ℓ^q is q-normed. As $\Omega : \ell_0^p \to \ell^q$ is quasi-linear and one has

$$\left\| \Omega\left(\sum_{i=1}^n f_i\right) - \sum_{i=1}^n \Omega(f_i) \right\|_q \leq Q[\Omega] \cdot \left(\sum_{i=1}^n \left(\frac{2}{i}\right)^{q/p} \right)^{1/q} \left\| \sum_{i=1}^n f_i \right\|_p , \tag{15}$$

as long as (f_i) have disjoint supports (see [**17**, Lemma 3.4]). The result follows just taking $f_i = e_i$.

For $q > 1$ let r be the conjugate exponent so that $\ell^q\ell^r = \ell^1$ and $\ell^p\ell^r = \ell^s$, with $s < 1$. Now apply Theorem 4 with $V = \ell^r$, Lemma 7 and the case just proved to the resulting centralizer $\Omega' \in \mathcal{C}(\ell^s, \ell^1)$. □

Next, we show how Theorem 4 can be used to invert the action of $\operatorname{Hom}(\ell^s, -) : \operatorname{Ext}(\ell^p) \to \operatorname{Ext}(\ell^q)$. Recall that when $s^{-1} + q^{-1} = p^{-1}$ and Ω is a centralizer on ℓ^p, $\operatorname{Hom}(\ell^s, -)$ gives a centralizer Θ on ℓ^q satisfying an estimate

$$\|\Theta(f)g - \Omega(fg)\|_p \leq M\|f\|_q\|g\|_s$$

But $\ell^q\ell^s = \ell^p$ and we can apply Theorem 4 to Θ thus obtaining a centralizer Θ' on ℓ^p such that

$$\|\Theta'(fg) - (\Theta f)g\|_p \leq M'\|f\|_q\|g\|_s,$$

which implies that $\Omega \approx \Theta'$. This provides an alternative proof for the second part of Theorem 3.

Theorem 7. *Let X be a q-concave quasi-Banach sequence space. Then there is a sequence space V such that $\ell^q V = X$ and $- \cdot V : \mathcal{C}_\sim(\ell^q) \to \mathcal{C}_\sim(X)$ is an isomorphism.*

Proof. We have $X = \ell^q \operatorname{M}(\ell^q, X)$ and since X is lattice-convex one also has $\ell^p = X \cdot \operatorname{M}(X, \ell^p)$ for some $0 < p < q$. Take $V = \operatorname{M}(\ell^q, X)$ and $W = \operatorname{M}(X, \ell^p)$. Then $\ell^p = \ell^q VW$ from where it follows that $VW = \ell^r$, where $1/q = 1/p + 1/r$. As forming products is associative we have $- \cdot \ell^r = (- \cdot W) \circ (- \cdot V)$. But $- \cdot \ell^r : \mathcal{C}_\sim(\ell^q) \to \mathcal{C}_\sim(\ell^p)$ is an isomorphism and so are $- \cdot V : \mathcal{C}_\sim(\ell^q) \to \mathcal{C}_\sim(X)$ and $- \cdot W : \mathcal{C}_\sim(X) \to \mathcal{C}_\sim(\ell^p)$ □

3.8. Applications to tensor products. We now introduce tensor products of quasi-Banach modules over a commutative Banach algebra A. The case $A = L^\infty$ suffices for our present purposes. Anyway, commutativity of the underlying algebra is necessary in order to guarantee that the algebraic tensor product of two left modules makes sense.

Let X, Y and Z be quasi-Banach modules over A. A bihomomorphism $B : X \times Y \to Z$ is a (jointly continuous) bilinear operator which acts as a homomorphism in each variable: $B(ax, y) = B(x, ay) = aB(x, y)$.

The common idea behind the construction of any tensor product is that it makes bihomomorphisms into ordinary homomorphisms on a suitable (perhaps more complex) domain.

Definition 3. Let $\mathfrak{C}$ be a category of quasi-Banach modules over a commutative Banach algebra A and let X and Y be members of $\mathfrak{C}$. A tensor product of X and Y over A in $\mathfrak{C}$ is an element T of $\mathfrak{C}$ together with a bihomomorphism $\theta : X \times Y \to T$ having the following universal property: for every object Z of $\mathfrak{C}$ and every bihomomorphism $B : X \times Y \to Z$, there is a unique homomorphism $L : T \to Z$ such that $B = L \circ \theta$.

It is easily seen that all possible tensor products are essentially the same, and so, we speak of the tensor product (instead of a tensor product).

The main problem dealing with tensor products of quasi-Banach spaces (or modules) is existence. For instance it is unknown whether or not L^p and L^q have a tensor product in the category of quasi-Banach spaces, even if $p = q = 1$. Fortunately, we can explicitly construct tensor products over L^∞ for all pairs of quasi-Banach function spaces:

THEOREM 8. *Suppose X and Y are function spaces. Then the bihomomorphism $\theta : X\times Y \to XY$ given by $\theta(f,g) = fg$ is the tensor product of X and Y in the category of quasi-Banach L^∞-modules.*

PROOF. Clearly, the map θ is a bihomomorphism, with $\|\theta\| = 1$. Let $B : X \times Y \to Z$ be a bihomomorphism. Define $T : XY \to Z$ taking $T(u) = B(f,g)$, where $u = fg$, with $f \in X$ and $g \in Y$. We show that T is a well-defined map. Suppose $u = f'g'$ is another factorization. Let $f'' = |f|+|f'|, g'' = |g|+|g'|$. Write $f = af'', g = bg'', f' = a'f'', g' = b'g''$ with $\|a\|_\infty, \|a'\|_\infty, \|b\|_\infty$ and $\|b'\|_\infty$ at most one and $\operatorname{supp} a, \operatorname{supp} a' \subset \operatorname{supp} f'', \operatorname{supp} b, \operatorname{supp} b' \subset \operatorname{supp} g''$. By our choice of a, a', b, b' the condition $u = abf''g'' = a'b'f''g''$ implies that $ab = a'b'$ and we have

$$B(f,g) = B(af'', bf'') = abB(f'', g'') = a'b'B(f'', g'') = B(a'f'', b'g'') = B(f', g'),$$

which proves that the definition of T makes sense. On the other hand, it is clear that $T(au) = aT(u)$ for all $a \in L^\infty, u \in XY$, which implies that T is linear (Lemma 1) and thus, it is a homomorphism, with $\|T\| \le \|B\|$ and $B = T \circ \theta$. This completes the proof. □

It should be noted that tensor products depend, not only of the involved spaces, but on the underlying category. For instance, the tensor product of L^1 with itself in the category of quasi-Banach L^∞-modules is $L^{1/2}$ (see Proposition 3), while the tensor product of the same spaces in the (smaller) category of Banach L^∞-modules is the Banach envelope $\operatorname{co}(L^{1/2}) = L^1(\alpha)$, where α is the atomic part of μ, and in particular, reduces to zero if μ is nonatomic. We refer the reader to [**25**, Chapter 2, §4] for information on Banach envelopes.

It will be convenient to introduce some notation for tensor products. If X and Y are quasi-Banach A-modules, then $X \otimes_A Y$ will denote the corresponding quasi-Banach tensor product over A. If X and Y are Banach modules $X\hat{\otimes}_A Y$ stands for the tensor product (over A) in the category of Banach modules. This construction can be seen in [**15**]. As we have seen, if X and Y are Banach A-modules, $X\hat{\otimes}_A Y$ may be different from $X \otimes_A Y$. It is clear, however, that $X\hat{\otimes}_A Y$ equals $\operatorname{co}(X \otimes_A Y)$.

Following our general conventions we write $X \otimes Y$ for the tensor product in the category of quasi-Banach L^∞-modules. Our next result clarifies the meaning of Theorem 4.

THEOREM 9. *With the notations of Theorem 4, we have $YV \oplus_{\Omega'} ZV = (Y \oplus_\Omega Z) \otimes V$.*

PROOF. More precisely we shall prove that the map

$$\theta : (Y \oplus_\Omega Z) \times V \to (YV) \oplus_{\Omega'} (ZV)$$

given by $\theta((y,z),v) = (yv, zv)$ is a bihomomorphism with the required universal property.

That θ is a 'bimorphism' is obvious. Let us check (joint) continuity:

$$\begin{aligned}
\|(yv, zv)\|_{\Omega'} &= \|yv - \Omega'(zv)\|_{YV} + \|zv\|_{ZV} \\
&= \|yv - \Omega(z)v + \Omega(z)v - \Omega'(zv)\|_{YV} + \|zv\|_{ZV} \\
&\le M\|(y,z)\|_\Omega \|v\|_V.
\end{aligned}$$

Next we show that every $(f,g) \in YV \oplus_{\Omega'} ZV$ can be written as $(f,g) = (yv, zv)$, with

$$M\|(f,g)\|_{\Omega'} \ge \|(y,z)\|_\Omega \|v\|_V. \tag{16}$$

To see this let $g = zv'$ be the decomposition used in Theorem 4 so that $\Omega'(g) = \Omega(z)v'$ and $(1+\varepsilon)\|g\|_{ZV} \ge \|z\|_Z\|v'\|_V$. As $f - \Omega'(g)$ belongs to YV we can find a decomposition $f - \Omega'(g) = yv''$, with $(1+\varepsilon)\|f - \Omega'(g)\|_{YV} \ge \|y\|_Y\|v''\|_V$ and $\|v''\|_V = \|v'\|_V$. Now take $v = |v'| + |v''|$ so that $\|v\| \le 2\Delta_V\|v'\|_V$ and $v' = av, v'' = bv$, with $\|a\|_\infty, \|b\|_\infty \le 1$. We have $g = azv$ and

$$f = yv'' + \Omega' g = ybv + (\Omega z)av = (by + a(\Omega z))v.$$

Let us see that the factorization $(f,g) = ((by + a(\Omega z))v, azv)$ works. Indeed

$$\begin{aligned}\|(by + a\Omega z, az)\|_\Omega \|v\|_V &= (\|(by + a\Omega z - \Omega(az)\|_Y + \|az\|_Z)\|v\|_V \\ &\leq M(\|y\|_Y + \|z\|_Z)\|v\|_V \\ &\leq M(\|y\|_Y \|v''\|_V + \|z\|_Z \|v'\|_V) \\ &\leq M(\|f - \Omega' g\|_{YV} + \|g\|_{ZV}).\end{aligned}$$

Now, let $B : (Y \oplus_\Omega Z) \times V \to U$ be a bihomomorphism, where U is any quasi-Banach L^∞-module. We define $T : YV \oplus_{\Omega'} ZV \to U$ taking $T(f,g) = B((y,z),v)$ where $(f,g) = (yv, zv)$. Of course we must check that this definition makes sense. Suppose

$$(f,g) = (yv, zv) = (y'v', z'v')$$

Write once again $w = |v| + |v'|$ so that $v = aw, v' = a'w$ for suitable $a, a' \in L^\infty$. We have $f = ayw = a'y'w$ and $g = azw = a'z'w$ and obviously the support of w contains those of a and a', hence $ay = a'y'$ and also $az = a'z'$. Now, applying 'bimodularity' of B we get

$$\begin{aligned}B((y,z),v) &= B((y,z),aw) = B(a(y,z),w) = B((ay,az),w) \\ &= B((a'y',a'z'),w) = B((y',z'),aw) = B((y',z'),v').\end{aligned}$$

In particular T preserves the outer multiplication. Anyway one needs to prove additivity, so take (f,g) and (f',g') in $YV \oplus_{\Omega'} ZV$ and write $(f,g) = (yv,zv)$ and $(f',g') = (y'v', z'v')$. Set $w = |v| + |v'|$ so that $v = aw, v' = a'w$ for certain $a, a' \in L^\infty$. Then $(f + f', g + g') = (yv + y'v', zv + z'v') = (ayw + a'y'w, azw + a'z'w)$ and thus

$$\begin{aligned}T(f + f', g + g') &= B((ay + a'y', az + a'z'), w) = B((ay,az),w) + B((a'y',a'z'),w) \\ &= B((y,z),aw) + B((y',z'),a'w) = T(f,g) + T(f',g').\end{aligned}$$

All this shows that T is a morphism satisfying $T \circ \theta = B$. Continuity follows from the very definition, taking a factorization as in (16). □

We can summarize Theorems 4, 8 and 9 and Lemma 7 as follows.

THEOREM 10. *Let Z and Y be quasi-Banach function spaces, $\Omega \in \mathcal{C}(Z,Y)$, and $Y \rightarrowtail X \twoheadrightarrow Z$ the corresponding extension. If V is another function space, then the sequence*

$$0 \longrightarrow Y \otimes V \longrightarrow X \otimes V \longrightarrow Z \otimes V \longrightarrow 0 \tag{17}$$

makes sense and is exact. If, in addition, Y and V are maximal and lattice-convex, then (17) is trivial if and only if so is the starting extension. □

In other words, $-\otimes V$ defines a map (actually a morphism) from $\operatorname{Ext}(Z,Y)$ to $\operatorname{Ext}(Z \otimes V, Y \otimes V)$, at least if Z is minimal. It is injective at least when Y and V are maximal and lattice-convex. Let us remark some consequences which are now easy.

COROLLARY 4. *Let Z be a minimal quasi-Banach function module and Y be a maximal Köthe space over the same measure.*

(a) *If $\operatorname{Ext}(ZY', L_1) = 0$ (in particular if ZY' is a K-space), then $\operatorname{Ext}(Z,Y) = 0$.*
(b) *If $\operatorname{Ext}(Y', L_1) = 0$ (in particular, if Y' is a K-space), then $\operatorname{Ext}(Z,Y) = \operatorname{Ext}(ZY', L_1)$.*

PROOF. Part (a) follows from Theorem 10, taking $V = Y'$ which is always maximal. Let us derive (b). We already know that the transformation $-\otimes Y' : \operatorname{Ext}(Z,Y) \to \operatorname{Ext}(ZY', L_1)$ is injective. We show it is onto.

Let $L^1 \rightarrowtail X \twoheadrightarrow ZY'$ be an extension of modules. Taking homomorphisms from Y' we obtain an exact sequence

$$0 \longrightarrow \operatorname{Hom}(Y', L_1) \longrightarrow \operatorname{Hom}(Y', X) \longrightarrow \operatorname{Hom}(Y', ZY') \longrightarrow \operatorname{Ext}(Y', L_1) \longrightarrow \cdots$$

But $\mathrm{Ext}_{L^\infty}(Y', L_1) = 0$ and $\mathrm{Hom}(Y', L_1) = Y'' = Y$ and we have in fact an extension of $\mathrm{Hom}(Y', ZY')$ by Y. Even if we cannot identify $\mathrm{Hom}(Y', ZY')$ with Z we can form the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \mathrm{Hom}(Y', X) & \longrightarrow & \mathrm{Hom}(Y', ZY') & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & \mathrm{PB} & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

It is easily seen that tensorizing the lower row with Y' one obtains the starting extension $L^1 \rightarrowtail X \twoheadrightarrow ZY'$. □

Taking $Y = Z$ in part (b) we get the following result of Kalton [**20**, Theorem 8.1]. The hypothesis on Z' is necessary (consider $Z = L^\infty$).

Corollary 5. *If Z is a maximal Köthe space such that Z' is a K-space, then* $\mathrm{Ext}(Z) = \mathrm{Ext}(L_1)$. □

Remark 3 (From Banach to quasi-Banach). An we mentioned in the Introduction Theorem 10 has no counterpart for Banach modules. To see this, consider any extension of modules $L^2 \rightarrowtail X \twoheadrightarrow L^2$. As L^2 is a K-space, X must be a Banach space, by a result of Dierolf [**11**, Satz 2.4.1] (perhaps the reader will find [**17**, Theorem 4.10] more accessible). Taking tensor products with L^2 in the category of Banach L^∞-modules we obtain the complex

$$0 \longrightarrow L^2 \hat{\otimes} L^2 \longrightarrow X \hat{\otimes} L^2 \longrightarrow L^2 \hat{\otimes} L^2 \longrightarrow 0 \tag{18}$$

We have $L^2 \hat{\otimes} L^2 = L^2 \otimes L^2 = L^1$ and since $X \hat{\otimes} L^2$ is the Banach envelope of $X \otimes L^2$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^1 & \longrightarrow & X \hat{\otimes} L^2 & \longrightarrow & L^1 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \| & & \\ 0 & \longrightarrow & L^1 & \longrightarrow & X \otimes L^2 & \longrightarrow & L^1 & \longrightarrow & 0 \end{array}$$

Now, if (18) were exact, the middle ascending arrow in the preceding diagram would be an isomorphism and $X \otimes L^2$ locally convex. By Lindenstrauss' lifting $L^1 \rightarrowtail X \otimes L^2 \twoheadrightarrow L^1$ is a trivial extension of Banach spaces. By the amenability of L^∞ it splits as an extension of (quasi-) Banach L^∞-modules. By Lemma 7 the starting extension $L^2 \rightarrowtail X \twoheadrightarrow L^2$ is trivial. Let us stop here.

4. Symmetric centralizers

Although we leave open the main problems we addressed in this work we have complete solutions for symmetric centralizers. To introduce them, recall that a function module X is symmetric if every measure preserving automorphism σ of $\mathcal{S}$ induces an isometry on X in the sense that $\|f \circ \sigma\|_X = \|f\|_X$ for every $f \in X$.

For the sake of clarity, from now on we will consider symmetric spaces associated either to counting measures on discrete sets or to Lebesgue measure (denoted by λ) on Borel subsets of the line. In fact we are interested in the cases $[0, 1]$ and $[0, \infty)$ only, but we need some flexibility at certain intermediate stages of the proofs.

Definition 4. Let Z and Y be symmetric function spaces. A centralizer $\Omega \in \mathcal{C}(Z, Y)$ is said to be symmetric if there is a constant S such that

$$\|\Omega(f \circ \sigma) - (\Omega f) \circ \sigma\|_Y \le S\|f\|_Z \qquad (f \in Z), \tag{19}$$

whenever σ is a measure preserving automorphism of the underlying measure space. The least possible constant S for which the above inequality holds will be denoted by $S[\Omega]$.

THEOREM 11. *If $p \neq q$, each symmetric centralizer in $\mathcal{C}(L^p, L^q)$ is trivial.*

The case $p < q$ has been already proved in Corollary 3, so let us focus on the case $p > q$. The proof consists in several steps. To begin with, let us prove the following minimax result.

LEMMA 8. *Let X be a symmetric function space and let $\mathcal{S} = \oplus_{k=1}^{\infty} B_k$ be a partition of the underlying measure space into Borel sets of finite measure. Then, for every $f \in L^0$, one has*

$$\inf_{c_k \in \mathbb{R}} \left\| f - \sum_k c_k 1_{B_k} \right\|_X \leq \sup_\sigma \|f - f \circ \sigma\|_X,$$

where σ runs over all measure preserving automorphisms of $\mathcal{S}$ leaving every B_k invariant.

PROOF. We write the proof only in the continuous case. The discrete case is easier and we leave it to the reader. Applying a suitable measure preserving automorphism to $\mathcal{S}$ we may and do assume that B_k are adjacent intervals so that $B_k = [b_k, b_{k+1})$ for $k \in \mathbb{N}$. Suppose g is any measurable function defined on a set of finite measure A. Then we say that $m \in \mathbb{R}$ is a median for g over A provided

$$\lambda\{t \in A : g(t) < m\} \leq \frac{\lambda(A)}{2} \quad \text{and} \quad \lambda\{t \in A : g(t) \leq m\} \geq \frac{\lambda(A)}{2}.$$

Elementary considerations show that every measurable function admits at least one median on each set of finite measure. We want to see that $\|f - \sum_k c_k 1_{B_k}\|_X \leq \sup_\sigma \|f - f \circ \sigma\|_X$ when c_k is a median for f on B_k.

Let A be a bounded interval and $g \in L^0(A)$. The decreasing rearrangement of g is defined as

$$g^\star(t) = \inf_{\lambda(B)=t} \sup_{s \in A \setminus B} f(s) \qquad (0 \leq t \leq \lambda(A))$$

where B runs over the Borel subsets of A. That is, $g^\star$ is the only decreasing, right-continuous function having the same distribution as g. It is clear that we can regard $g^\star$ as a function defined on A and we will do it without relabelling. It is a basic fact from measure theory that there is a measure preserving automorphism σ of A for which $g^\star = g \circ \sigma$, so that $g^\star$ is true rearrangement of g. We can define the corresponding increasing rearrangement as $g_\star = -(-g)^\star$. As before, there is a measure preserving automorphism such that $g_\star = g \circ \eta$.

After this preparation it suffices to prove the inequality

$$\left\| f - \sum_k c_k 1_{B_k} \right\|_X \leq \left\| f - \bigoplus_k f_k^\star \right\|_X$$

when c_k is a median for f on B_k, f_k is the restriction of f to B_k, and assuming f increasing on each B_k. Indeed in this case we have

$$f(t) \leq c_k \leq f_k^\star(t) \qquad \text{for } t \in B_k^- = \left[b_k, \frac{b_k + b_{k+1}}{2}\right),$$

while

$$f(t) \geq c_k \geq f_k^\star(t) \qquad \text{for } t \in B_k^+ = \left(\frac{b_k + b_{k+1}}{2}, b_{k+1}\right).$$

Hence, with a slight abuse of notation,

$$\begin{aligned} \left| f - \sum_k c_k 1_{B_k} \right| &= \sum_k (1_{B_k^-}(c_k - f) + 1_{B_k^+}(f - c_k)) \\ &\leq \sum_k (1_{B_k^-}(f_k^\star - f) + 1_{B_k^+}(f - f_k^\star)) = u\left(f - \bigoplus_k f_k^\star\right), \end{aligned}$$

where $u = \sum_k (1_{B_k^+} - 1_{B_k^-})$ is an unitary of L^∞. Therefore,

$$\left\| f - \sum_k c_k 1_{B_k} \right\|_X \leq \left\| u(f - \bigoplus_k f_k^\star) \right\|_X \leq \left\| f - \bigoplus_k f_k^\star \right\|_X$$

which completes the proof. □

The preceding result shows that the 'local' behaviour of symmetric centralizers is pretty simple, as we now state.

PROPOSITION 7. *Let Ω be a symmetric centralizer from Z to Y. Suppose $f = \sum_k \lambda_k 1_{B_k}$, where (B_k) is a disjoint sequence of Borel sets of finite measure. Then there exist numbers η_k such that*

$$\left\| \Omega(f) - \sum_k \eta_k 1_{B_k} \right\|_Y \leq S[\Omega] \|f\|_Z. \tag{20}$$

PROOF. If $f = \sum_k \lambda_k 1_{B_k}$, then $f = f \circ \sigma$ if σ is a measure preserving automorphism of $\mathcal{S}$ leaving every B_k invariant. Thus, for these σ we get

$$\|\Omega f - (\Omega f) \circ \sigma\|_Y = \|\Omega(f \circ \sigma) - (\Omega f) \circ \sigma\|_Y \leq S[\Omega] \|f\|_Z.$$

Now (20) follows from Lemma 8, taking η_k as a median of Ωf on B_k. □

PROOF OF THEOREM 11. We consider the case $p > q$ and we will get the stronger conclusion that $\Omega \approx 0$. We write the proof for $\mathcal{S} = [0, 1]$. The case $\mathcal{S} = [0, \infty)$ requires only minor modifications we leave again to the reader. For each $n \in \mathbb{N}$ let us partition the unit interval into the corresponding dyadic intervals

$$[0, 1] = \bigoplus_{k=1}^{2^n} \left(\frac{k-1}{2^n}, \frac{k}{2^n} \right]$$

and, given a function space X on $[0, 1]$, we write X_n for the subspace of simple functions associated to that partition —but we keep the notation $X_0 = X \cap L^\infty$.

Now, if $\Omega \in \mathcal{C}(L^p, L^q)$ is symmetric, there is no loss of generality in assuming that Ω maps L_n^p to L_n^q, as an L_n^∞ centralizer. By Proposition 1 there is $h_n \in L_n^r$ such that

$$\|\Omega f - h_n f\|_q \leq M \|f\|_p \qquad (f \in L_n^p), \tag{21}$$

with M independent on n. If σ is a permutation of $\{1, 2, \ldots, 2^n\}$, then σ° operates on every X_n in the obvious way and since

$$\|\sigma^\circ(\Omega f) - \Omega(\sigma^\circ(f))\|_q \leq M \|f\|_p,$$

one also has

$$\|\sigma^\circ(h_n f) - h_n \sigma^\circ(f)\|_q = \|\sigma^\circ(h_n)\sigma^\circ(f) - h_n \sigma^\circ(f)\|_q \leq M \|f\|_p \qquad (f \in L_n^p).$$

And since functions in L_n^r have the same quasi-norm as the corresponding multiplication operator from L_n^p to L_n^q we have $\|\sigma^\circ(h_n) - h_n\|_r \leq M$ for all permutations of $\{1, 2, \ldots, 2^n\}$ and an obvious application of Proposition 8 yields $\|h_n - c_n\|_r \leq M$, where c_n is any median for h_n on $[0, 1]$. Therefore we may assume that the functions h_n appearing in (21) are constant. Taking now $f = 1$ and comparing (21) for different values of n we see that the sequence (h_n) is bounded in $\mathbb{R}$, so one actually has

$$\|\Omega f\|_q \leq M \|f\|_p \qquad (f \in \bigcup_n L_n^p).$$

Suppose now $f \in L_0^p$ (this is a dense submodule of L^p). Then for n large enough there is $g \in L_n^p$ such that $|f| \leq g$ and $\|g\|_p \leq 2\|f\|_p$ and we can write $f = ag$, where $\|a\|_\infty \leq 1$. Hence,

$$\|\Omega f - a\Omega g\|_q \leq C\|a\|_\infty \|g\|_p \leq 2C\|f\|_p,$$

and since $\|a\Omega g\|_q \leq \|\Omega g\|_q \leq 2M\|f\|_p$ the result obtains on account of Lemmas 2 and 7. □

The same ideas can be used to 'lift' symmetric centralizers, as we now show. Notice, however, that we cannot proceed as in the proof of Theorem 3 because multiplication operators do not preserve the symmetry of centralizers.

THEOREM 12. *Let Ω be a symmetric centralizer on L^p. Then, for every $q > p$, there is a symmetric centralizer $\Omega^{[q]}$ on L^q such that*

$$\|\Omega(fg) - (\Omega^{[q]} f)g\|_p \leq M\|f\|_q\|g\|_s \qquad (f \in L^q, g \in L^s), \tag{22}$$

where $p^{-1} = q^{-1} + s^{-1}$. Moreover, if Θ is another centralizer on L^q satisfying the corresponding estimate, then $\Theta \approx \Omega^{[q]}$.

PROOF. As we did in the proof of Theorem 11 we write the proof for $\mathcal{S} = [0,1]$ only. In view of Proposition 7 we may assume Ω maps every L_n^p to itself. Now the idea is that the self-extension of L^p induced by Ω is (isometrically) the direct limit of the finite-dimensional extensions $L_n^p \rightarrowtail L_n^p \oplus_\Omega L_n^p \twoheadrightarrow L_n^p$ that live in the category of quasi-Banach L_n^∞-modules. Writing $E_n = L_n^p \oplus_\Omega L_n^p$ and taking L_n^∞-homomorphisms from L_n^s we get

$$\mathrm{Hom}(L_n^s, L_n^p) \rightarrowtail \mathrm{Hom}(L_n^s, E_n) \twoheadrightarrow \mathrm{Hom}(L_n^s, L_n^p),$$

which is in fact a self-extension of L_n^q, say $L_n^q \rightarrowtail F_n \twoheadrightarrow L_n^q$. The point is to show that these extensions admit a direct limit $L^q \rightarrowtail F \twoheadrightarrow L^q$. Then the centralizer we are looking for is the quasi-linear map inducing the later sequence.

In any case some details need explanation and so we present a slightly different proof that uses Ω as a pure function. We have $L^q L^s = L^p$ and also $L_n^q L_n^s = L_n^p$ for all $n \in \mathbb{N}$. We define $\Omega_n^{[q]} : L_n^q \to L_n^q$ as follows. Given $f \in L_n^q$ we consider the map $L_n^s \to L_n^p$ defined as $g \mapsto \Omega(fg)$. This an L_n^∞-centralizer with constant at most $C[\Omega]\|f\|_q$. As $s > p$ there is a homomorphism $\phi : L_n^s \to L_n^p$ such that

$$\|\Omega(fg) - \phi(g)\|_p \leq M\|f\|_q\|g\|_s.$$

Regarding ϕ as a function in L_n^q, the correspondence $f \mapsto \phi$ defines an L_n^∞-centralizer $\Omega_n^{[q]} : L_n^q \to L_n^q$. What happens with $\Omega_m^{[q]}(f)$ if $f \in L_n^q$ and $m > n$? We may apply Proposition 7 to get

$$\|\Omega_m^{[q]}(f) - \Omega_n^{[q]}(f)\|_q \leq M\|f\|_q \qquad (m > n, f \in L_n^q).$$

Thus, we can define a mapping

$$\Omega^{[q]} : \bigcup_n L_n^q \longrightarrow \bigcup_n L_n^q$$

by the rule $\Omega^{[q]}(f) = \Omega_n^{[q]}(f)$, where $n = \min\{m : f \in L_m^q\}$. We then have

$$\|\Omega(fg) - (\Omega^{[q]} f)g\|_p \leq M\|f\|_q\|g\|_s, \qquad (f \in \bigcup_n L_n^q, g \in \bigcup_n L_n^s).$$

Let us extend $\Omega^{[q]}$ to L_0^q. For each $f \in L_0^q \backslash \cup_n L_n^q$, pick $f' \in \cup_n L_n^q$ with $|f| \leq f'$ and $\|f'\|_q \leq 2\|f\|_q$, so that $f = af'$, with $\|a\|_\infty \leq 1$. This selection can be done homogeneous. Then set $\Omega^{[q]}(f) = a\Omega^{[q]}(f')$. It is easily checked that the estimate in (22) now holds for every $f \in L_0^q, g \in L_0^s$, increasing the value of M if necessary. It follows that $\Omega^{[q]} : L_0^q \to L^q$ is a centralizer. Now, an obvious application of Lemma 2 shows that estimate in (22) remains true for all $g \in L^s$, with f still in L_0^p. Then we can use the argument in [**20**, Proposition 4.4] to extend $\Omega^{[q]}$ to a centralizer defined on the whole L^q (but L^0-valued), while [**20**, Lemma 4.3] is exactly what we need to get (22) in general. Finally (22) and the symmetry of Ω imply that $\Omega^{[q]}$ is a symmetric centralizer. This completes the proof. □

REMARK 4. I don't known whether $\mathrm{Ext}(L^p, L^q)$ vanishes for $0 < q < q < \infty$. An affirmative answer would imply that $\mathrm{Ext}(L^p) = \mathrm{Ext}(L^1)$ for $0 < p < \infty$. It is more or less clear from the proof of Proposition 1 that there is a constant $K = K(p, q)$ such that if $\Omega \in \mathcal{C}(L^p, L^q)$ is trivial, then there is $\phi \in L^0$ satisfying $\mathrm{dist}(\Omega, \phi) \le KC[\Omega]$. It seems interesting to known if given $\Omega \in \mathcal{C}(Z, Y)$ one can find a sequence of trivial centralizers (Ω_n), with $C[\Omega_n]$ uniformly bounded and such that $\|\Omega f - \Omega_n f\|_Y \to 0$ for each fixed $f \in Z$.

4.1. An application to H^p when $0 < p \le 1$. Theorem 12 can be applied to extend some results already known for $p > 1$ to any p. Let us mention the following. The ground field is now $\mathbb{C}$, and H^p stands for the corresponding Hardy class.

COROLLARY 6. *Let Ω be a symmetric centralizer on $L^p(\mathbb{T})$. Then there is a constant M such that, whenever $f \in H^p$ and $\Omega f \in L^p(\mathbb{T})$ (which is always the case if f is a trigonometric polynomial), one has* $\mathrm{dist}(\Omega f, H^p) \le MS[\Omega]\|f\|_p$.

PROOF. Combine the corresponding result for Banach spaces proved by Kalton in [**20**, Theorem 7.3] with Theorem 12. Take into account that $H^p = H^q H^r$ if $1/p = 1/q + 1/r$. □

Thus, a symmetric centralizer on L^p gives rise to a self extension of H^p (and of L^p/H^p) in the category of quasi-Banach H^∞-modules.

4.2. Centralizers and derivations. The following result is stated without any reference to interpolation theory. That this is really a generalization of [**21**, Theorem 7.6] needs some explanations the reader will find in [**23**, Sections 8 to 11] and [**21**, p. 487]. The key fact that Calderón formula $[X, Y]_\theta = X^{1-\theta} Y^\theta$ survives in our (quasi-Banach function space) setting for analytically-convex spaces can be seen in [**22**, Theorem 3.4]. Analytic convexity is equivalent to lattice-convexity for quasi-Banach function spaces [**19**, Theorem 4.4].

COROLLARY 7. *Let Ω be a real centralizer on $X = \ell^p$ (or a symmetric real centralizer on $X = L^p$). Then there is a factorization $X = UV$ and constants c and M such that*

$$\|\Omega f - cf(\log v - \log u)\|_p \le M\|f\|_p \qquad (f \in X),$$

where $f = uv$ is an (almost) optimal factorization in the sense that $(1 + \varepsilon)\|f\|_p \ge \|u\|_U \|v\|_V$.

PROOF. If $1 < p < \infty$ this was proved by Kalton in [**21**, Theorem 7.6] (see also [**23**, Theorem 11.6]), so let us assume $p < 2$ and $X = \ell^p$. The continuous case is almost the same, using Theorem 12. Take s so that $\ell^p = \ell^2 \ell^s$ and use Theorem 3 to get a (real) centralizer $\Omega^{[2]}$ on ℓ^2 such that

$$\left\|\Omega f - (\Omega^{[2]}(f^{\frac{p}{2}}))f^{\frac{p}{s}}\right\|_p \le M\|f\|_p \qquad (f \ge 0).$$

Dividing $\Omega^{[2]}$ by 400 times $C[\Omega^{[2]}]$ if necessary we may apply [**21**, Theorem 7.6] to get a couple of super-reflexive sequence spaces (Y_0, Y_1) such that $[Y_0, Y_1]_{1/2} = \ell^2$ and $\Omega^{[2]}$ is equivalent to the corresponding derivation. As explained in [**21**, p. 487] this means that

$$\|\Omega^{[2]}(g) - g(\log y_1 - \log y_0)\|_2 \le M\|g\|_2 \qquad (g \ge 0),$$

where $g = y_0 y_1$ is an almost optimal factorization in the sense that

$$(1 + \varepsilon)\|g\|_2 \ge \|y_0\|_{Y_0^{1/2}} \|y_1\|_{Y_1^{1/2}}. \tag{23}$$

We have

$$\ell^p = \ell^s \ell^2 = \ell^{2s} \ell^{2s} Y_0^{\frac{1}{2}} Y_1^{\frac{1}{2}} = UV,$$

where $U = \ell^{2s} Y_0^{\frac{1}{2}}$ and $V = \ell^{2s} Y_1^{1/2}$.

Take $f \in \ell_+^p$ and write $f = f^{p/2} f^{p/s}$ Let $f^{p/2} = y_0 y_1$ be an almost optimal factorization, as in (23). We have $f = (y_0 f^{p/2s})(y_1 f^{p/2s})$ and

$$\left\| y_0 f^{\frac{p}{2s}} \right\|_U \left\| y_1 f^{\frac{p}{2s}} \right\|_V \leq \|y_0\|_{Y_0^{1/2}} \left\| f^{\frac{p}{2s}} \right\|_{\ell^{2s}} \|y_1\|_{Y_1^{1/2}} \left\| f^{\frac{p}{2s}} \right\|_{\ell^{2s}} \leq (1+\varepsilon) \left\| f^{\frac{p}{2}} \right\|_2 \left\| f^{\frac{p}{s}} \right\|_s = (1+\varepsilon)\|f\|_p,$$

so that if $u = (y_0 f^{p/2s})$ and $v = (y_1 f^{p/2s})$, then $f = uv$ is almost optimal and moreover,

$$\begin{aligned} \|\Omega(f) - f(\log v - \log u)\|_p &\leq M \left\| f^{\frac{p}{s}} \Omega^{[2]}(f^{\frac{p}{2}}) - f\left(\log(y_1 f^{p/2s}) - \log(y_0 f^{p/2s})\right) \right\|_p \\ &= M \left\| f^{\frac{p}{s}} \left(\Omega^{[2]}(f^{\frac{p}{2}}) - f^{\frac{p}{2}}(\log y_1 - \log y_0) \right) \right\|_p \leq M \|f^{\frac{p}{s}}\|_s \|f^{\frac{p}{2}}\|_2 = M\|f\|_p, \end{aligned}$$

as desired. □

5. Hom versus Ext

Some results in this paper seem to support the rather vague idea that if Z and Y are function modules and $\operatorname{Hom}(Z, Y)$ is 'small', then $\operatorname{Ext}(Z, Y)$ 'must' vanish. Even if I cannot precise what should 'small' mean here we will present examples where $\operatorname{Hom}(Z, Y)$ vanishes and $\operatorname{Ext}(Z, Y)$ not. It is perhaps a little ironic that this is in fact an ubiquitous phenomenon.

Let Z and Y be function spaces and $\Omega \in \mathcal{C}(Z, Y)$. We put $X = Y \oplus_\Omega Z$. Consider the set $S = \{f \in Z : \Omega f \in Y\}$ quasi-normed by

$$\|f\|_S = \|(0, f)\|_\Omega = \|\Omega f\|_Y + \|f\|_Z.$$

It is easily seen that S is a quasi-normed function space. Less obvious is the fact that S is indeed complete. To check this it clearly suffices to see that the set of elements of the form $(0, f)$ is closed in X. Suppose $(0, f_n)$ converges to (g, f). Then

$$\|(g, f - f_n)\|_\Omega = \|g - \Omega(f - f_n)\|_Y + \|f - f_n\|_Z \to 0.$$

But Ω is continuous at the origin as a function from Z to L_0 (see Theorem 13 in the Appendix below) and since the inclusion $Y \to L^0$ is continuous we have $g = 0$.

Considering the isometric inclusion $\jmath : S \to X$ sending f to $(0, f)$ and the corresponding quotient $Q = X/\jmath S$ we have a diagram

$$\begin{array}{ccccc} & & S & & \\ & & \downarrow{\scriptstyle \jmath} & & \\ Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z \\ & & \downarrow{\scriptstyle \varpi} & & \\ & & Q & & \end{array} \tag{24}$$

where the vertical arrows in (24) represent an extension of Q by S. The compositions $\pi \circ \jmath : S \to Z$ and $\varpi \circ \imath : Y \to Q$ are homomorphisms. Both are injective, since $\jmath S \cap \imath Y = 0$.

We may identify Q with the set of those $g \in L^0$ such that $(g, f) \in X$ for some $f \in Z$ with the quasi-norm

$$\|g\|_Q = \inf_{f \in Z} \|(g, f)\|_\Omega.$$

It is easily seen that Q is a quasi-Banach function space. That Q embeds continuously in L^0 is as follows: if $\|g_n\|_Q \to 0$, then we can find a sequence (f_n) in Z such that $\|g_n\|_Q \leq \|g_n - \Omega f_n\|_Y + \|f_n\|_Z \to 0$. We have $\|f_n\|_Z \to 0$, so $\Omega f_n \to 0$ in L^0 and since including Y into L^0 is continuous we have $g_n \to 0$ in L^0.

There is a centralizer $\mho \in \mathcal{C}(Q,S)$ inducing the vertical extension in (24) whose description is an amusing exercise: given $g \in Q$ we select (homogeneously) $f \in Z$ (almost-) minimizing $\|(g,f)\|_\Omega$ and we put $\mho g = f$. Notice the perfectly symmetric rôles of Ω and $\mho$. As pure maps Ω goes from Z to Q, while $\mho$ goes from Q to Z. Moreover, if $g = \Omega f$ one may take $f = \mho g$. Also, $\{f \in Q : \mho f \in S\} = Y$ and $\{g \in L^0 : g - \mho f \in S \text{ for some } f \in Q\} = Z$. We will not spoil the reader's fun going into further details, but let us remark that $SQ \subset YZ$ and, by symmetry, $SQ = YZ$ with equivalent quasi-norms. When $Z = Y$ we then have $SQ = Z^2$, so $Z = Q^{1/2}S^{1/2} = [Q,S]_{1/2}$ and we can obtain another centralizer $\Phi \in \mathcal{C}(Z)$ by interpolation, at least when Z has non-trivial concavity. I don't known how are Φ and Ω related —simple examples show that they are not equivalent in general.

Let us be more specific taking a symmetric centralizer $\Omega \in \mathcal{C}(Z,Y)$, where Z and Y are symmetric spaces on $[0,1]$, both maximal and minimal —so that either $\operatorname{Hom}(Z,Y)$ vanishes or Z embeds continuously into Y. In this case both S and Q are (after re-quasi-norming) symmetric function spaces (both maximal and minimal) and $\mho$ is symmetric. Referring to (24) we always have $Y \subset Q$ and $S \subset Z$. Hence if $\operatorname{Hom}(Z,Y) \neq 0$ and $\operatorname{Hom}(Q,S) \neq 0$ we have continuous embeddings

$$Z \subset Y \subset Q \subset S \subset Z,$$

so $Z = S$ with equivalent (quasi-) norms and $\Omega \approx 0$.

Thus, if Ω is nontrivial and $\operatorname{Hom}(Z,Y) \neq 0$ then $\operatorname{Hom}(Q,S)$ vanishes. Let us see that in this case $\mho$ cannot be trivial. Suppose $\mho \sim 0$ so that there is a unique $u \in \operatorname{Hom}(X,S)$ such that $u \circ \jmath = I_S$ —uniqueness follows from $\operatorname{Hom}(Q,S) = 0$. Consider the composition

$$v : X \xrightarrow{u} S \xrightarrow{\jmath} X \xrightarrow{\pi} Z \xrightarrow{I} Y.$$

It is easily seen that $v \circ \imath = I_Y$, hence Ω is trivial —a contradiction.

Finally, let us consider even a concrete example. Let $\Omega \in \mathcal{C}(L^2)$ be the popular Kalton-Peck centralizer given by $\Omega f = f \log(|f|/\|f\|_2)$. It is easily seen that S agrees with the Orlicz space L_N, where $N(t) = t^2 \log^2 t$. Let us identify the quotient $Q = (L^2 \oplus_\Omega L^2)/S$. It is not hard to check that Ω is self-dual in the sense that $L^2 \oplus_\Omega L^2$ is isomorphic to its own dual under the pairing

$$\langle (g', f'), (f,g) \rangle = \int_0^1 (g'g - f'f)dt.$$

This can be seen as in [**24**, Theorem 5.1], but notice that our pairing is slightly different. It follows that Q is isomorphic to $S^* = S' = L_N'$ and so it agrees with the Orlicz space associated to the complementary function of N. (That $Q = S' = S^*$ for every $\Omega \in \mathcal{C}(L^2)$ follows from the identity $SQ = L^2L^2 = L^1$ and the fact that super-reflexivity is a three-space property [**9**].) So we have the diagram

$$\begin{tikzcd} & L_N \arrow[d, "\jmath"] & \\ L^2 \arrow[r, "\imath"] & L^2 \oplus_\Omega L^2 \arrow[r, "\pi"] \arrow[d, "\varpi"] & L^2 \\ & L_N' & \end{tikzcd}$$

which shows that $\operatorname{Hom}(L_N', L_N) = 0$, while $\operatorname{Ext}(L_N', L_N) \neq 0$.

As the vertical extension is associated to $\mho$ we can apply $- \otimes L_N'$ thus obtaining a nontrivial extension of $F = L_N' L_N'$ by $L_N' L_N = L^1$. It follows that $\operatorname{Ext}_{\mathbb{K}}(F, \mathbb{K}) \neq 0$. However F has dual trivial since F is minimal and so

$$F^* = F' = \mathrm{M}(F, L^1) = \mathrm{M}(L_N' L_N', L^2 L^2) = \mathrm{M}(L_N', L^2)^2 = 0.$$

Clearly, F is an Orlicz function space and we have the following.

EXAMPLE 1. There is a locally bounded Orlicz function space F such that $\operatorname{Hom}(F, L^1) = L(F, L^1) = 0$ but $\operatorname{Ext}(F, L^1) \neq 0$.

This is related to the fact that the ultrapowers of F have nontrivial dual, which follows from the results in [**10**, Section 4]: ultrapowers of Orlicz spaces associated of Orlicz functions of regular variation are, in general, Musielak-Orlicz spaces. In particular,

$$(L_N)_U = L_N(\mu) \oplus L^2(\nu),$$

for certain (large) measures μ and ν. This already implies that F_U contains $L^1(\nu)$ as a direct summand.

6. Appendix. Nearly morphisms with values in L^0

From now on we consider only σ-finite measures. A metric function space is a linear subspace X of L^0 equipped with an F-norm $|\cdot|_X$ such that if $|f| \leq g$ and $g \in X$, then $f \in X$ and $|f|_X \leq |g|_X$. Besides, we assume X saturated and the inclusion $X \subset L^0$ continuous. Apart form the quasi-normed ones, the most interesting example is L^0 itself with the F-norm given by

$$|f|_0 = \int_S \frac{|f|}{1+|f|} d\mu.$$

The formula makes sense only for finite μ. If μ is σ-finite, just replace it by an equivalent finite measure. We shall prove the following result already used in Section 5. Observe that homogeneity is no longer required.

THEOREM 13. *Let Z be a (not necessarily complete) metric function space and $\Omega : Z \to L^0$ any mapping such that $\Omega(af) - a\Omega f \to 0$ in L^0 as $(a, f) \to 0$ in $L^\infty \times Z$. Then Ω is continuous at the origin of Z.*

PROOF. We write the proof for probability measures only. First suppose $Z = L^\infty$ and $|\cdot|_Z = \|\cdot\|_\infty$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|\Omega(cu) - c\Omega(u)|_0 \leq \varepsilon$ for $|c|, \|u\|_\infty \leq \delta$. Now, take $\gamma > 0$ such that $|u\delta^{-1}\Omega(\delta 1)|_0 \leq \varepsilon$ for all $\|u\|_\infty < \gamma$. Suppose $\|u\|_\infty \leq \min\{\delta^2, \gamma\}$. Then,

$$|\Omega(u)|_0 = |\Omega(\delta^{-1}u\delta 1)|_0 \leq |\Omega(\delta^{-1}u\delta 1) - \delta^{-1}u\Omega(\delta 1)|_0 + |\delta^{-1}u\Omega(\delta 1)|_0 \leq \varepsilon + \varepsilon,$$

as required.

Returning to the general case let us show that Ω must be 'quasi-additive' in the sense that

$$|\Omega(x+y) - \Omega(x) - \Omega(y)|_0 \to 0$$

as $(x, y) \to 0$ in Z. This part of the proof borrows from [**20**, Lemma 4.2]. Let $\varepsilon > 0$. Fix $\gamma > 0$ such that $|\Omega(az) - a\Omega(z)|_0 \leq \varepsilon$ for $\|a\|_\infty, |z|_Z \leq \gamma$. Now, choose $\delta > 0$ so that $|\gamma^{-1}z|_Z < \delta$ provided $z = |x| + |y|$, and $|x|_Z, |y|_Z < \delta$. Suppose $x, y \in Z$ are such that $|x|_Z, |y|_Z \leq \delta$. Take $z = |x| + |y|$ and write $x = uz, y = vz$ and $x + y = (u + v)z$, where $\|u\|_\infty, \|v\|_\infty, \|u + v\|_\infty \leq 1$. One then has

$$\begin{aligned} &|\Omega(x+y) - \Omega(x) - \Omega(y)|_0 = |\Omega(\gamma(u+v)\gamma^{-1}z) - \Omega(\gamma u\gamma^{-1}z) - \Omega(\gamma v\gamma^{-1}z)|_0 \\ &\leq |\Omega(\gamma(u+v)z/\gamma) - \gamma(u+v)\Omega(z/\gamma)|_0 + |\Omega(\gamma uz/\gamma) - \gamma u\Omega(z/\gamma)|_0 + |\Omega(\gamma vz/\gamma) - \gamma v\Omega(z/\gamma)|_0 \\ &\leq \varepsilon + \varepsilon + \varepsilon. \end{aligned}$$

We are now ready to complete the proof. Fix $\varepsilon > 0$. Choose $\gamma > 0$ in such a way that $|f|_Z, |g|_Z, \|a\|_\infty \leq \gamma$ implies

(a) $|\Omega(f + g) - \Omega(f) - \Omega(g)|_0 \leq \varepsilon$;
(b) $|\Omega(a)|_0 \leq \varepsilon$;
(c) $|\Omega(af) - a\Omega(f)|_0 \leq \varepsilon$.

As the set $V = \{f \in L_0 : \mu(\{t \in \mathcal{S} : |f(t)| > \gamma\}) < \varepsilon\}$ is open in L_0, we can take $0 < \delta \leq \gamma$, such that every $f \in Z$ with $|f|_Z \leq \delta$ belongs to V.

Suppose now that $|f|_Z \leq \delta$ and let us estimate $|\Omega(f)|_0$. Put $A = \{t \in \mathcal{S} : |f(t)| > \gamma\}$ and write $f = s + b$, where $b = 1_A f$ and $s = (1 - 1_A)f$. Then $|s|_Z, |b|_Z \leq |f|_Z < \delta \leq \gamma$ and

$$|\Omega f|_0 \leq |\Omega(s + b) - \Omega s - \Omega b|_0 + |\Omega s|_0 + |\Omega b|_0 \leq \varepsilon + |\Omega s|_0 + |\Omega b|_0,$$

by (a). On the other hand, $|\Omega(s)|_0 \leq \varepsilon$, by (b), since $\|s\|_\infty \leq \gamma$. Finally,

$$|\Omega(b)|_0 = |\Omega(\gamma 1_A \gamma^{-1} f)|_0 \leq |\Omega(\gamma 1_A \gamma^{-1} f) - \gamma 1_A \Omega(\gamma^{-1} f)|_0 + |\gamma 1_A \Omega(\gamma^{-1} f)|_0 \leq \varepsilon + \lambda(A) \leq 2\varepsilon.$$

Hence $|\Omega f|_0 \leq 4\varepsilon$ and we are done. □

The following result applies to $Z = L^p$ for $0 \leq p \leq \infty$.

Corollary 8. *Let Z be a minimal, complete function space. Every short exact sequence of F-modules over L^∞ and homomorphisms $L^0 \rightarrowtail X \twoheadrightarrow Z$ splits.*

Proof. As $\pi : X \to Z$ is open there is a mapping $s : Z \to X$ such that $\pi \circ s = I_Z$, with $s(0) = 0$ and continuous at the origin (see [**12**, Lemma 2.2(a)]). We define Ω on $Z_0 = Z \cap L^\infty$ taking $\Omega f = s(f) - f s(1)$. Then $\pi(\Omega(f)) = 0$, so Ω takes values in $L^0 = \ker \pi$. On the other hand,

$$\Omega(af) - a\Omega f = s(af) - af s(1) \to 0$$

in L^0 as $(a, f) \to 0$ in $L^\infty \times Z_0$. According to Theorem 13 Ω is continuous at zero and therefore $f \longmapsto f s(1)$ defines a homomorphism $Z_0 \to X$. Extending it to Z ends the proof. □

We hasten to remark that L^0 is not injective amongst (topological) L^∞-modules [**2**].

Corollary 9. *Let Z and Y be quasi-Banach function spaces and $\Omega \in \mathcal{C}(Z, Y)$ a homogeneous centralizer. If Ωf belongs to Y for all $f \in Z$, then $\Omega \approx 0$. Consequently if Z and Y are both maximal and minimal, every extension of L^∞-modules $Y \rightarrowtail X \twoheadrightarrow Z$ which is trivial in the pure algebraic sense is null in* $\operatorname{Ext}(Z, Y)$.

Proof. Referring to Section 5 we have $S = Z$ (as sets) and the Open Mapping theorem gives that the expression $\|\Omega f\|_Y + \|f\|_Z$ is equivalent to the quasi-norm of Z. □

References

[1] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. Vol. 1. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[2] D. Berg, N.T. Peck, and H. Porta, There is no continuous projection from the measurable functions on the square onto the measurable functions on the interval, Israel Journal of Mathematics 14 (1973) 199–204.

[3] J. Bretagnolle, D. Dacunha-Castelle and J-L. Krivine, Lois stables et espaces L^p. Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques, 2 no. 3 (1966) 231–259

[4] F. Cabello Sánchez, Uniform boundedness theorems for nearly additive mappings. Abh. Math. Sem. Univ. Hamburg 69 (1999), 167–173.

[5] F. Cabello Sánchez, A simple proof that super-reflexive spaces are K-spaces. Proc. Amer. Math. Soc. 132 (2004), no. 3, 697–698

[6] F. Cabello Sánchez and J.M.F. Castillo, The long homology sequence for quasi-Banach spaces, with applications, Positivity 8 (2004), no. 4, 379–394

[7] F. Cabello Sánchez and J.M.F. Castillo, Uniform boundedness and twisted sums of Banach spaces, Houston J. Math. 30 (2004), no. 2, 523–536

[8] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190

[9] J.M.F. Castillo and M. González, Three-space problems in Banach space theory, Lecture Notes in Mathematics 1667, Springer, 1997.

[10] D. Dacunha-Castelle, J.L. Krivine, Applications des ultraproduits à l'étude des espaces et des algèbres de Banach. (French) Studia Math. 41 (1972), 315–334

[11] S. Dierolf, *Über Vererbbarkeitseigenschaften in topologischen Vektorräumen*, Dissertation, Ludwing-Maximilians-Universität, München, 1973.

[12] P. Domański, On the splitting of twisted sums, and the three space problem for local convexity. Studia Mathematica 82 (1985) 155–189.
[13] P. Enflo, J. Lindenstrauss, and G. Pisier, On the three space problem. Mathematica Scandinavica 36 (1975) 199–210.
[14] G. Godefroy, A glimpse at Nigel Kalton's work. Banach spaces and their applications in analysis, 1–35, Walter de Gruyter, Berlin, 2007
[15] A.Ya. Helemskii, *Banach and locally convex algebras.* Oxford Science Publications 1993.
[16] E. Hilton and K. Stammbach, *A course in homological algebra.* Graduate Texts in Mathematics 4, Springer-Verlag 1970.
[17] N.J. Kalton, The three space problem for locally bounded F-spaces, Compositio Mathematica 37 (1978) 243–276.
[18] N.J. Kalton, Convexity conditions for nonlocally convex lattices. Glasgow Math. J. 25 (1984), no. 2, 141–152
[19] N.J. Kalton, Plurisubharmonic functions on quasi-Banach spaces. Studia Math. 84 (1986), no. 3, 297–324.
[20] N.J. Kalton, Nonlinear commutators in interpolation theory, Memoirs of the American Mathematical Society 385, 1988.
[21] N.J. Kalton, Differentials of complex interpolation processes for Köthe function spaces, Trans. Amer. Math. Soc. 333 (1992) 479–529
[22] N.J. Kalton and M. Mitrea, Stability results on interpolation scales of quasi-Banach spaces and applications, Trans. Amer. Math. Soc. 350 (1998) 3903–3922
[23] N.J. Kalton and S. Montgomery-Smith, Interpolation of Banach spaces. Handbook of the Geometry of Banach Spaces II, Edited by W.B. Johnson and J. Lindenstrauss. Elsevier 2003, 1131–1175
[24] N.J. Kalton and N.T. Peck, Twisted sums of sequence spaces and the three-space problem, Transactions of the American Mathematical Society 255 (1979) 1–30.
[25] N.J. Kalton, N.T. Peck, and W. Roberts, *An F-space sampler,* London Mathematical Society Lecture Note Series 89.
[26] N.J. Kalton and J.W. Roberts, Uniformly exhaustive submeasures and nearly additive set functions, Transactions of the American Mathematical Society 278 (1983) 803–816.
[27] G. Ya. Lozanovskiĭ, On some Banach lattices, Siberian Mathematical Journal 10 (1969) 419–430.
[28] Y. Raynaud, On ultrapowers on non commutative L_p spaces, J. Operator Theory 48 (2002) 41–68
[29] M. Ribe, Examples for the nonlocally convex three space problem. Proceedings of the American Mathematical Society 237 (1979) 351–355.
[30] J.W. Roberts, A non-locally convex F-space with the Hanh-Banach approximation property, Proceedings Kent State Conference, Lecture Notes in Mathematics 604, Springer, 1977.
[31] A.R. Schep, Products and factors of Banach function spaces, Preprint (arXiv:0803.4336v1).
[32] M. Schreiber, Quelques remarques sur les caractérisations des espaces $L^p, 0 \leq p < 1$. Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques, 8 no. 1 (1972), 83–92
[33] B. Sims, Ultra-techniques in Banach space theory,

Departamento de Matemáticas, Universidad de Extremadura

Avenida de Elvas, 06071 Badajoz, Spain.
E-mail address: fcabello@unex.es