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Certain homological properties of Schatten classes

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Abstract. An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms $0 \to Y \to X \to Z \to 0$. When properly organized all these extensions constitute a linear space denoted by $\operatorname{Ext}_B(Z, Y)$, where B is the underlying (Banach) algebra. In this paper we compute the spaces of extensions for the Schatten classes when they are regarded in its natural (left) module structure over $B = B(\mathcal{H})$, the algebra of all operators on the ground Hilbert space. Our main results can be summarized as follows:

$$\operatorname{Ext}_B(S^p, S^q) = \begin{cases} 0 & \text{if } 0 < q < p \leq \infty \text{ or } p = q = \infty, \\ \operatorname{Ext}_{\mathbb{C}}(S^1, \mathbb{C}) & \text{if } q = p \text{ is finite,} \\ \operatorname{Ext}_{\mathbb{C}}(\mathcal{H}) & \text{if } 0 < p < q \leq \infty. \end{cases}$$

In the first case, every extension $0 \to S^q \to X \to S^p \to 0$ splits and so $X = S^q \oplus S^p$. In the second case, every self-extension of S^p arises (and gives rise) to a minimal extension of S^1 in the quasi-Banach category, that is, a short exact sequence $0 \to \mathbb{C} \to M \to S^1 \to 0$. In the third case, each extension corresponds to a "twisted Hilbert space", that is, a short exact sequence $0 \to \mathcal{H} \to T \to \mathcal{H} \to 0$. Thus, the subject of the paper is closely connected to the early "three-space" problems studied (and solved) in the seventies by Enflo, Lindenstrauss, Pisier, Kalton, Peck, Ribe, Roberts, and others.

1. Introduction

1.1. Purpose. Let Z and Y be quasi-Banach modules over a fixed Banach algebra A. An extension (of Z by Y) is a short exact sequences of (quasi-) Banach modules and homomorphisms

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0.$$

Less technically we may think of X as a module containing Y as a closed submodule in such a way that X/Y is (isomorphic to) Z. The extension is said to be trivial (or to split) if Y is complemented in X through a homomorphism.

When properly classified and organized the extensions of Z by Y constitute a linear space denoted by $\operatorname{Ext}_A(Z, Y)$.

While the homomorphisms between a given couple of modules display the most basic links between them, extensions reflect much more subtle connections, often in a encrypted or disguised form.

In this paper we study extensions between Schatten classes when these are regarded as modules over $B = B(\mathcal{H})$, the algebra of all (linear, bounded) operators on the underlying Hilbert space $\mathcal{H}$. Thus we are concerned with short exact sequences of (say left) B-modules

$$0 \longrightarrow S^q \longrightarrow X \longrightarrow S^p \longrightarrow 0. \tag{1}$$

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Our main results can be summarized as follows, according to the relative position between p and q:

$$\operatorname{Ext}_B(S^p,S^q)=\begin{cases}0 & \text{if } 0<q<p\leq\infty \text{ or } p=q=\infty,\\ \operatorname{Ext}_{\mathbb{C}}(S^1,\mathbb{C}) & \text{if } q=p \text{ is finite,}\\ \operatorname{Ext}_{\mathbb{C}}(\mathcal{H}) & \text{if } 0<p<q\leq\infty.\end{cases}$$

(S^∞ is the ideal of compact operators, with the operator norm.) In the first case every extension is trivial and we have $X=S^p\oplus S^q$ (Corollary 2). In the second case we see that $\operatorname{Ext}_B(S^p)$ does not depend on $p\in(0,\infty)$ (see Corollary 3) and, in fact, each self-extension of S^p corresponds to a minimal extension of S^1, that is, an exact sequence of quasi-Banach spaces and operators

$$0\longrightarrow\mathbb{C}\longrightarrow M\longrightarrow S^1\longrightarrow 0$$

(Proposition 2). Notice that such an extension is nontrivial precisely when M is not locally convex, despite the fact that both S^1 and $\mathbb{C}$ are. In the third case, each extension of S^p by S^q gives rise to (and arises from) a "twisted Hilbert space", that is, a short exact sequence of Banach spaces and operators

$$0\longrightarrow\mathcal{H}\longrightarrow T\longrightarrow\mathcal{H}\longrightarrow 0$$

which arises as its "spatial part". By the well-known projection property of Hilbert spaces such an extension is (non-) trivial if and only if T is (not) isomorphic to a Hilbert space.

It is remarkable that the results of the present paper are so cleanly connected with the early "three space" problems. We refer the reader to [**21**, Chapter 5], [**6**, Chapter 3], [**1**, Chapter 14], [**18**, Section 4] or [**19**, Sections 8 and 9] for basic information on the topic.

1.2. Background. The study of the modular structure of noncommutative L^p spaces built over a general von Neumann algebra $\mathcal{M}$ goes back to their inception. However, the computation of the spaces of homomorphisms, which plays a rôle in this paper, is very recent [**13**].

Not much is known about the corresponding spaces of extensions $\operatorname{Ext}_{\mathcal{M}}(L^p,L^q)$ for general $\mathcal{M}$. By following ideas of Kalton [**17**] it is proved in [**5**] that $\operatorname{Ext}_{\mathcal{M}}(L^p)\neq 0$ for every (infinite-dimensional) $\mathcal{M}$ and other related results.

The approach of this paper also originates in Kalton's work. Indeed, the idea of representing extensions by centralizers is already in [**15**]. Even if the connection between centralizers and extensions is deliberately neglected in both [**16**] and [**17**], these papers should be considered as the first serious studies on self-extensions of the Schatten classes within the category of quasi-Banach bimodules over B.

The commutative situation is settled in [**3**] with quite different techniques. Considering the usual Lebesgue spaces $L^p=L^p(\mu)$ for an arbitrary measure μ as L^∞-modules with "pointwise" multiplication we have $\operatorname{Ext}_{L^\infty}(L^q,L^p)=0$ when $p\neq q$ and $\operatorname{Ext}_{L^\infty}(L^p)=\operatorname{Ext}_{L^\infty}(L^1)$ for every $p\in(0,\infty)$. The preceding identity had been proved by Kalton in [**15**] for $p\in(1,\infty)$.

Some authors consider a more restrictive notion of extension by requiring the splitting in the (quasi-) Banach category. This leads to the study of the amenability of the underlying algebra, a major theme in homology of Banach algebras [**10**]. Although we will not pursue this point here, the results of this paper imply that if (1) splits as an extension of quasi-Banach spaces, then so it does as an extension of quasi-Banach modules over B, a result which is easy to prove when $q\geq 1$.

1.3. Some general conventions and notations.

- The ground field is $\mathbb{C}$, the complex numbers.
- $\mathcal{H}$ is the underlying separable Hilbert space where our operators act and $\langle\cdot|\cdot\rangle$ is the scalar product in $\mathcal{H}$.

- $B = B(\mathcal{H})$ is the Banach algebra of all (linear, bounded) operators on $\mathcal{H}$. The ideal of finite rank operators is denoted by $\mathfrak{F}$. The ideal of compact operators is denoted by K.
- $L(\mathcal{H})$ is the algebra of all (not necessarily continuous) linear endomorphisms of $\mathcal{H}$.
- $x \otimes y$ is the rank-one operator given by $h \mapsto \langle h|x\rangle y$, where $x, y, h \in \mathcal{H}$.
- The weak operator topology (WOT) in B is that generated by the seminorms $u \mapsto |\langle y|u(x)\rangle|$, with $x, y \in \mathcal{H}$.
- If V is any linear (respectively, quasi-normed) space, then $V^\star$ (respectively, V') denotes the space of linear functionals (respectively, bounded linear functionals) on V. The symbol * is reserved for the Hilbert space adjoint.
- Let U, V and W be arbitrary sets and $\varphi : U \to V$ any mapping. We define $\varphi_\circ : U^W \to V^W$ by $\varphi_\circ(f) = \varphi \circ f$. Similarly, $\varphi^\circ : W^V \to W^U$ is defined as $\varphi^\circ(f) = f \circ \varphi$. The identity on U is denoted by $\mathbf{I}_U$.
- Let v be a finite rank endomorphism of the linear space V (no topology is assumed). Then the trace of v is given by $\operatorname{tr} u = \sum_{i=1}^n v_i^\star(v_i)$ provided $v = \sum_{i=1}^n v_i^\star \otimes v_i$, with $v_i^\star \in V^\star, v_i \in V$. The trace does not depend on the given representation since, after the identification of the finite rank endomorphisms of V with $V^\star \otimes V$, the trace is nothing different from the linearization of the obvious bilinear function $V^\star \times V \to \mathbb{C}$. If u is any endomorphism of V and v has finite rank, one has $\operatorname{tr}(u \circ v) = \operatorname{tr}(v \circ u)$.
- We use M for a constant independent on operators and vectors but perhaps depending on the involved spaces and centralizers and which may vary from line to line.
- The distance between two maps ϕ and ψ (acting between the same quasi-normed spaces) is the least constant D for which one has $\|\phi(x) - \psi(x)\| \leq D\|x\|$ for every x in the common domain.
- A mapping $\phi : U \to V$ acting between linear spaces is said to be homogeneous if $\phi(tu) = t\phi(u)$ for every $t \in \mathbb{C}$ and $u \in U$.

2. Centralizers and extensions

In this Section we consider modules on the left unless otherwise stated. Let A be a Banach algebra that for all purposes in this paper will be a C*-algebra. A quasi-normed module over A is a quasi-normed space X together with a jointly continuous outer multiplication $A \times X \to X$ satisfying the traditional algebraic requirements. If the underlying space is complete (that is, a quasi-Banach space) we speak of a quasi-Banach module. Given quasi-normed modules X and Y, a homomorphism $u : X \to Y$ is an operator such that $u(ax) = au(x)$ for all $a \in A$ and $x \in X$. Operators and homomorphisms are assumed to be continuous unless otherwise stated. If no continuity is assumed, we speak of linear maps and morphisms. We use $\operatorname{Hom}_A(X, Y)$ for the space of homomorphisms and $\mathcal{M}_A(X, Y)$ for the morphisms. If there is no possible confusion about the underlying algebra A, we omit the subscript.

Quasi-normed right modules and bimodules and their homomorphisms are defined in the obvious way.

In general, $\operatorname{Hom}_A(X, Y)$ carries no module structure. However, if X is a bimodule instead of a mere left module, then $\operatorname{Hom}_A(X, Y)$ can be given a structure of left module letting $(ah)(x) = h(xa)$, where $h \in \operatorname{Hom}_A(X, Y), x \in X, a \in A$. Similarly, if Y is a bimodule, then the multiplication $ha(x) = h(x)a$ makes $\operatorname{Hom}_A(X, Y)$ into a right module.

These structures are functorial in the obvious sense.

2.1. Extensions. An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0. \tag{2}$$

The open mapping theorem guarantees that $\imath$ embeds Y as a closed submodule of X in such a way that the corresponding quotient is isomorphic to Z. Two extensions $0 \to Y \to X_i \to Z \to 0$ $(i = 1, 2)$ are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle u} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

By the five-lemma [**11**, Lemma 1.1], and the open mapping theorem, u must be an isomorphism. We say that (2) splits if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This just means that Y is a complemented submodule of X, that is, there is a homomorphism $X \to Y$ which is a left inverse for the inclusion $\imath : Y \to X$; equivalently, there is a homomorphism $Z \to X$ which is a right inverse for the quotient $\pi : X \to Z$.

Given quasi-Banach modules Y and Z, we denote by $\mathrm{Ext}_A(Z, Y)$ the set of all possible module extensions (2) modulo equivalence. When $Y = Z$ we just write $\mathrm{Ext}_A(Z)$. By using pull-back and push-out constructions, it can be proved (see [**4**] for the details in the F-space setting) that $\mathrm{Ext}_A(Z, Y)$ carries a "natural" linear structure (without topology) in such a way that trivial extensions correspond to 0. (The usual approach using injective or projective representations completely fails dealing with quasi-Banach modules since there are neither injective nor projective objects.) Thus, $\mathrm{Ext}_A(Z, Y) = 0$ means "every extension $0 \to Y \to X \to Z \to 0$ splits".

Taking A as the ground field one recovers extensions in the quasi-Banach space setting.

2.2. Centralizers and the extensions they induce. Let us introduce the main tool in our study of extensions.

Definition 1 (Kalton). Let Z and Y be quasi-normed modules over the Banach algebra A and let $\tilde{Y}$ be another module containing Y in the purely algebraic sense. A centralizer from Z to Y with ambient space $\tilde{Y}$ is a homogeneous mapping $\Omega : Z \to \tilde{Y}$ having the following properties.

(a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f+g)-\Omega(f)-\Omega(g) \in Y$ and $\|\Omega(f + g) - \Omega(f) - \Omega(g)\|_Y \le Q(\|f\|_Z + \|g\|_Z)$.

(b) There is a constant C so that if $a \in A$ and $f \in Z$, then $\Omega(af) - a\Omega(f) \in Y$ and $\|\Omega(af) - a\Omega(f)\|_Y \le C\|a\|_A\|f\|_Z$.

We denote by $Q[\Omega]$ the least constant for which (a) holds and by $C[\Omega]$ the least constant for which (b) holds. We refer to the number $\Delta[\Omega] = \max\{Q[\Omega], C[\Omega]\}$ as the centralizer constant of Ω.

We now indicate the connection between centralizers and extensions. Let Z and Y be quasi-Banach modules. Suppose $\Omega : Z_0 \to \tilde{Y}$ is a centralizer from Z_0 to Y, where Z_0 is a dense submodule of Z. Then

$$Y \oplus_\Omega Z_0 = \{(g, f) \in \tilde{Y} \times Z_0 : g - \Omega f \in Y\}$$

is a linear subspace of $\tilde{Y} \times Z_0$ and the functional $\|(g, f)\|_\Omega = \|g - \Omega f\|_Y + \|f\|_Z$ is a quasi-norm on it. Moreover, the map $\imath : Y \to Y \oplus_\Omega Z_0$ sending g to $(g, 0)$ preserves the quasi-norm, while the map

$\pi : Y \oplus_\Omega Z_0 \to Z_0$ given as $\pi(g, f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces and relatively open operators

$$0 \longrightarrow Y \xrightarrow{\imath} Y \oplus_\Omega Z_0 \xrightarrow{\pi} Z_0 \longrightarrow 0 \tag{3}$$

Actually only quasi-linearity (a) is necessary here. The estimate in (b) implies that the multiplication $a(g, f) = (ag, af)$ makes $Y \oplus_\Omega Z_0$ into a quasi-normed module over A in such a way that the arrows in (3) become homomorphisms. Indeed,

$$\|a(g, f)\|_\Omega = \|ag - \Omega(af)\|_Y + \|af\|_Z = \|ag - a\Omega f + a\Omega f - \Omega(af)\|_Y + \|af\|_Z \leq M\|a\|_A\|(g, f)\|_\Omega.$$

Let X_Ω be the completion of $Y \oplus_\Omega Z_0$. This is a quasi-Banach module and there is a unique homomorphism $X_\Omega \to Z$ extending the quotient in (3) we still denote π. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega Z_0 & \longrightarrow & Z_0 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_\Omega & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array} \tag{4}$$

in which the vertical arrows are inclusions and the horizontal rows are exact. We will always refer to the lower row in this diagram as the extension (of Z by Y) induced by Ω.

It is easily seen that two centralizers Ω and Φ (acting between the same sets, say Z_0 and $\tilde{Y}$) induce equivalent extensions if and only if there is a morphism $h : Z_0 \to \tilde{Y}$ such that $\|\Omega(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_Z$. We write $\Omega \sim \Phi$ in this case and $\Omega \approx \Phi$ if the preceding inequality holds for $h = 0$. In particular Ω induces a trivial extension if and only if $\|\Omega(f) - h(f)\|_Y \leq K\|f\|_Z$ for some morphism $h : Z_0 \to \tilde{Y}$ (that is, $\text{dist}(\Omega, h) < \infty$). In this case we say that Ω is a trivial centralizer.

2.3. The Schatten classes S^p. We now move to the concrete modules we shall deal with. For $p \in (0, \infty)$, let ℓ^p denote quasi-Banach space of (complex) sequences (t_n) for which the quasi-norm $|(t_n)|_p = (\sum_n |t_n|^p)^{1/p}$ is finite.

Let f be a compact operator on the Hilbert space $\mathcal{H}$. The singular numbers of f are the sequence of eigenvalues of $|f| = (f^*f)^{1/2}$ arranged in decreasing order and counting multiplicity. The Schatten class S^p consists of those operators on $\mathcal{H}$ whose singular numbers $(s_n(f))$ are in ℓ^p. It is a quasi-Banach space under the quasi-norm $\|f\|_p = |(s_n(f))|_p$. Each $f \in S^p$ has an expansion $f = \sum_n s_n x_n \otimes y_n$, where (s_n) are its singular numbers and (x_n) and (y_n) are orthonormal sequences in $\mathcal{H}$. This is called an Schmidt representation of f. S^p is a quasi-Banach bimodule over B in the obvious way: given $f \in S^p$ and $a, b \in B$ one has $afb \in S^p$ and $\|afb\|_p \leq \|a\|_B\|f\|_p\|b\|_B$. The submodule of finite rank operators is denoted by S^p_0. The structure of homomorphisms between Schatten classes is fairly simple. Indeed, one has

$$\text{Hom}_B(S^p, S^q) = \begin{cases} S^r & \text{if } 0 < q < p < \infty, \text{ where } p^{-1} + r^{-1} = q^{-1}; \\ B & \text{if } p \leq q. \end{cases} \tag{5}$$

This should be understood as follows: each operator g in the left-hand side defines a homomorphism $\gamma : S^p \to S^q$ by multiplication on the right $\gamma(f) = fg$. Moreover, the norm of g in in the corresponding space equals $\|\gamma : S^p \to S^q\|$ and every homomorphism arises in this way. All this can be seen in Simon's monograph [**28**].

It will be convenient at some places to consider right module structures. We indicate this just by putting the (algebra) subscript on the right. Thus, for instance, $\text{Hom}(Z, Y)_A$ is the space of homomorphisms of right modules from Z to Y, which are assumed to be (quasi-normed) right modules over A. The meaning of $\mathcal{M}(Z,Y)_A, \mathrm{Ext}(Z,Y)_A$ or "right centralizer" should be clear.

It it worth noticing that the right module structure of Schatten classes is "isomorphic" to the left one throughout the involution: $fa=(a^*f^*)^*$. Thus, for instance, if $u: S^p\to B$ is a morphism of left (respectively, right) modules, then we obtain a morphism of right (respectively, left) modules thus: $f\mapsto (u(f^*))^*$. The same formula can be used to exchange left and right homomorphisms, centralizers, and the like. We will use this fact without further mention.

Lemma 1. (a) *$\mathfrak{F}$ is a projective B-module in the pure algebraic sense: if X is any algebraic B-module and $\pi: X\to\mathfrak{F}$ is a surjective morphism, then there is another morphism $s:\mathfrak{F}\to X$ such that $\pi\circ s=\mathbf{I}_{\mathfrak{F}}$.*
(b) *$\mathcal{M}(\mathfrak{F},B)_B = L(\mathcal{H})$ in the sense that for every morphism of right modules $\alpha:\mathfrak{F}\to B$ there is a unique linear endomorphism ℓ of $\mathcal{H}$ such that $\alpha(f)=\ell\circ f$ for every $f\in\mathfrak{F}$.*
(c) *Similarly, $\mathcal{M}_B(\mathfrak{F},B)=L(\mathcal{H})$ in the sense that for every morphism of left modules $\alpha:\mathfrak{F}\to B$ there isq a unique linear endomorphism ℓ of $\mathcal{H}$ such that $\alpha(f)=(\ell\circ f^*)^*$ for every $f\in\mathfrak{F}$.*
(d) *Let $\ell:\mathfrak{F}\to\mathbb{C}$ be a linear map such that for each fixed $y\in\mathcal{H}$ one has $\ell(x\otimes y)\to 0$ as $x\to 0$ in $\mathcal{H}$. Then there is a linear endomorphism L of $\mathcal{H}$ such that $\ell(f)=\mathrm{tr}(L\circ f)$ for all $x,y\in\mathcal{H}$.*

Proof. (a) Of course, B is a projective B-module. $\mathcal{H}$ is a B-module under the obvious action $(a,h)\mapsto a(h)$. Fix any norm one $\eta\in\mathcal{H}$. Then the map $\eta\otimes - :\mathcal{H}\to B$ given by $h\mapsto \eta\otimes h$ is an injective (homo)morphism. The evaluation map $\delta_\eta: B\to\mathcal{H}$ given by $\delta_\eta(u)=u(\eta)$ provides a left inverse (homo)morphism for $\eta\otimes -$. Being a direct factor in B, $\mathcal{H}$ is projective too.

On the other hand, $\mathfrak{F}=\mathcal{H}'\otimes\mathcal{H}$ (as bimodules). If I is a Hamel basis for $\mathcal{H}'$, we have $\mathcal{H}'=\bigoplus_I\mathbb{C}$ as linear spaces. Combining, we have

$$\mathfrak{F}=\mathcal{H}'\otimes\mathcal{H}\backsimeq\left(\bigoplus_I\mathbb{C}\right)\otimes\mathcal{H}=\bigoplus_I(\mathbb{C}\otimes\mathcal{H})=\bigoplus_I\mathcal{H},$$

as (left) modules, and a direct sum of projective modules is again projective.

(b) is very easy. Take $x,y\in\mathcal{H}$, with $\|x\|=1$. Then $\alpha(x\otimes y)=\alpha((x\otimes y)(x\otimes x))=(\alpha(x\otimes y))(x\otimes x)$. Hence there is $z=z(x,y)\in\mathcal{H}$ such that $\alpha(x\otimes y)=x\otimes z$. It is easily seen that z does not depend on the first variable while it depends linearly on the second one. Thus the rule $\ell(y)=z$ is an endomorphism of $\mathcal{H}$. Quite clearly one has $\alpha(f)=\ell\circ f$ when f has rank one and the same is true for every $f\in\mathfrak{F}$.

(c) is just the left version of (b).

(d) Fix $y\in\mathcal{H}$. The hypothesis implies that $x\mapsto\ell(x\otimes y)$ is a continuous, conjugate-linear functional on $\mathcal{H}$ and by Riesz representation theorem there is $z\in\mathcal{H}$ such that $\ell(x\otimes y)=\langle z|x\rangle$. Putting $z=L(y)$ we obtain a transformation of $\mathcal{H}$ which is easily seen to be linear. And since $\ell(x\otimes y)=\langle L(y)|x\rangle=\mathrm{tr}(x\otimes L(y))=\mathrm{tr}(L\circ(x\otimes y))$ we are done. □

Corollary 1. *Up to equivalence, every extension of S^p by an arbitrary quasi-Banach module Y comes from a centralizer $\Omega: S^p_0\to Y$.*

Proof. Let $0\longrightarrow Y\longrightarrow X\xrightarrow{\pi} S^p\longrightarrow 0$ be an extension of quasi-Banach modules over B. With no serious loss of generality we may assume $Y=\ker\pi$. Putting $X_0=\pi^{-1}(S^p_0)$ we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X_0 & \longrightarrow & S^p_0 & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{\pi} & S^p & \longrightarrow & 0
\end{array}$$

where the vertical arrows are plain inclusions. We shall show there is a centralizer $\Omega: S_0^p \to Y$ and an isomorphism of quasi-normed normed modules u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y\oplus_\Omega S_0^p & \longrightarrow & S_0^p & \longrightarrow & 0\\ & & \| & & \downarrow u & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_0 & \stackrel{\pi}{\longrightarrow} & S_0^p & \longrightarrow & 0 \end{array}$$

This obviously implies that u extends to an isomorphism between X_Ω and X fitting in the corresponding diagram.

The identification of Ω is as follows. Let $b: S^p \to X$ be a homogeneous bounded section of the quotient map $\pi: X \to S^p$, that is, a map satisfying $\pi\circ b = \mathbf{I}_{S^p}$ and $\|b(f)\|_X \le M\|f\|_p$ for some M independent on $f\in S^p$. Such a section exists because π is open. Notice, moreover, that $b(f)\in X_0$ if $f\in S_0^p$.

Now we use Lemma 1(a) to get a morphism $s: S_0^p \to X_0$ such that $\pi\circ s = \mathbf{I}_{S_0^p}$ and we set $\Omega(f) = b(f) - s(f)$ for $f\in S_0^p$. Clearly, $\pi(\Omega(f)) = \pi(b(f)) - \pi(s(f)) = 0$ and so Ω takes values in Y. That Ω is a centralizer is nearly trivial: given $f, g\in S_0^p$ and $a\in B$ one has

$$\|\Omega(f+g) - \Omega f - \Omega g\|_Y = \|b(f+g) - b(f) - b(g)\|_X \le M(\|f\|_p + \|g\|_p),$$
$$\|\Omega(af) - a\Omega f\|_Y = \|b(af) - ab(f)\|_X \le M\|a\|_B\|f\|_p.$$

We define $u: Y\oplus_\Omega S_0^p \to X_0$ by $u(y,f) = y + s(f)$. This is a homomorphism in view of the bound

$$\|u(y,f)\|_X = \|y+s(f)\|_X \le M(\|y-b(f)+s(f)\|_X + \|b(f)\|_X) \le M(\|y-\Omega f\|_Y + \|f\|_p) \le M\|(y,f)\|_\Omega.$$

The inverse of u is given by $v(x) = (x - s(\pi(x)), \pi(x))$ for $x\in X_0$. It is continuous since

$$\|v(x)\|_\Omega = \|x - s(\pi(x)) - \Omega(\pi(x))\|_Y + \|\pi(x)\|_p = \|x - b(\pi(x))\|_Y + \|\pi(x)\|_p \le M\|x\|_X.$$

This completes the proof. □

REMARKS 1. (a) Corollary 1 and the material displayed in Section 4.1 imply that every extension of S^p by S^q is induced by a centralizer Φ from S^p to S^q with values in $L(\mathcal{H})$. This means that the corresponding twisted sum arises as $S^q\oplus_\Phi S^p$, and not only as X_Φ, which is the completion of $S^q\oplus_\Phi S_0^p$.

(b) The proof given by Kalton in [**16**, Proposition 4.1] can be easily adapted to show that if Z is S^p (or S_0^p) and $Y = S^q$ and $\Omega: Z\to L(\mathcal{H})$ satisfies the second condition in the definition of a centralizer, then it is automatically quasi-linear, with $Q[\Omega]\le 8C$. Although this applies to practically all centralizers appearing in this paper, we will not use it.

3. Centralizers between Schatten classes: splitting

In this Section we prove that $\mathrm{Ext}_B(S^p, S^q) = 0$ when $0 < p < q < \infty$. As the attentive reader will imagine, what we prove is that every centralizer $S_0^p \to S^q$ is trivial (see Theorem 1 below for a slightly more precise version). First we need to break a given centralizer into “small pieces” without losing the relevant information it encodes.

Let $\Phi: S_0^p \to S^q$ be a centralizer and e a finite rank projection. Then we can define a centralizer $\Phi_e: S^p\to S^q$ by the formula $\Phi_e(f) = \Phi(fe)$. Of course, Φ_e is trivial. Indeed taking $g = \Phi(e)$ we have

$$\|\Phi_e(f) - fg\|_q = \|\Phi(fe) - f\Phi(e)\|_q \le C[\Phi]\|f\|_B\|e\|_p \le C[\Phi]\,\mathrm{rk}(e)^{1/p}\|f\|_p,$$

where $\mathrm{rk}(e)$ is the dimension of the image of e.

LEMMA 2. *Let $\Phi : S_0^p \to S^q$ be a centralizer, with q finite. Then*

$$\operatorname{dist}(\Phi, \mathcal{M}_B(S_0^p, S^q)) = \sup_e \operatorname{dist}(\Phi_e, \mathcal{M}_B(S^p, S^q)),$$

where e runs over all finite rank projections in B.

PROOF. That $\operatorname{dist}(\Phi, \mathcal{M}_B(S_0^p, S^q)) \geq \operatorname{dist}(\Phi_e, \mathcal{M}_B(S^p, S^q))$ for every e is obvious. Let us prove the other inequality. Let D be a constant such that for every e there is a morphism ϕ_e so that

$$\|\Phi_e f - \phi_e(f)\|_q \leq D\|f\|_p \qquad (f \in S^p).$$

Let $\mathcal{U}$ be an ultrafilter refining the Fréchet filter on the set of finite rank projections in B. We define a mapping $\phi : S_0^p \to S^q$ by the formula

$$\phi(f) = \lim_{\mathcal{U}} \phi_e(fe) \tag{6}$$

where the limit is taken in the WOT. The definition makes sense because for each $f \in S_0^p$ one has $fe = e$ for sufficiently large e. For these projections we have $\|\Phi(f) - \phi_e(f)\|_q \leq D\|f\|_p$ and thus the net $(\phi_e(fe))_e$ is (essentially) bounded in S^q and so in B. As bounded subsets of B are relatively compact in the WOT we see that (6) defines a map from S_0^p to B. But $\|\cdot\|_q$ is lower semicontinuous with respect to the restriction of the WOT to S^q (see [**7**, Corollary 2.3]) and so

$$\|\Phi(f) - \phi(f)\|_q \leq \liminf_{\mathcal{U}} \|\Phi(f) - \phi_e(f)\|_q \leq D\|f\|_p \qquad (f \in S_0^p).$$

In particular $\phi(f)$ belongs to S^q. Finally that ϕ is a morphism follows from the fact that, for fixed $a \in B$, the map $b \mapsto ab$ is WOT-continuous on bounded sets of B. □

The sought-after result reads as follows.

THEOREM 1. *Given $0 < q < p < \infty$, there is a constant $K = K(p, q)$ so that, for every centralizer $\Omega : S_0^p \to S^q$ there is a morphism $\omega : S_0^p \to S^q$ satisfying $\|\Omega(f) - \omega(f)\|_q \leq K\Delta[\Omega]\|f\|_p$ for every $f \in S_0^p$.*

The proof uses a simple ultraproduct technique, but requires some noncommutative gadgetry. Here we only recall some definitions, mainly for notational purposes.

Let X be a quasi-Banach space, I an index set and $\mathcal{U}$ a countably incomplete ultrafilter on I. Let $\ell^\infty(I, X)$ be the space of bounded families of X indexed by I (furnished with the sup quasi-norm) and let $N_{\mathcal{U}}$ be the (closed) subspace of those $x \in \ell^\infty(I, X)$ such that $\|x_i\|_X \to 0$ along $\mathcal{U}$. The ultrapower of X with respect to $\mathcal{U}$ is the quotient space $\ell^\infty(I, X)/N_{\mathcal{U}}$ with the quotient quasi-norm. The class of the family (x_i) in $X_{\mathcal{U}}$ is denoted by $[(x_i)]$. Notice that if the quasi-norm of X is continuous one can compute the quasi-norm in $X_{\mathcal{U}}$ by the formula $\|[(x_i)]\| = \lim_{\mathcal{U}} \|x_i\|_X$. Clearly, if A is a Banach algebra, then so is $A_{\mathcal{U}}$ when equipped with the coordinatewise product $[(a_i)][(b_i)] = [(a_i b_i)]$. If besides X is a quasi-Banach module over A, then the multiplication $[(a_i)][(x_i)] = [(a_i x_i)]$ makes $X_{\mathcal{U}}$ into a quasi-Banach module over $A_{\mathcal{U}}$.

What we need to prove Theorem 4 is the following.

LEMMA 3. *Let $p, q, r \in (0, \infty)$ satisfy $q^{-1} = p^{-1} + r^{-1}$. If $\gamma : S_{\mathcal{U}}^p \to S_{\mathcal{U}}^q$ is a homomorphism of (left) modules over $B_{\mathcal{U}}$, then there is bounded family (g_i) in S^r such that $\gamma[(f_i)] = [(f_i g_i)]$ whenever (f_i) is bounded in S^p.*

PROOF. This can be obtained as a combination of results by Raynaud, and Junge and Sherman. Let us explain how.

(1) There is a general construction, due to Haagerup, that associates to a given von Neumann algebra $\mathcal{M}$ the so-called (Haagerup, non-commutative) L^p spaces $L^p(\mathcal{M})$ for $0 < p \leq \infty$. These spaces consist of certain (densely defined, closable, but in general discontinuous) operators acting on a common suitable Hilbert space which is related to $\mathcal{M}$ in a highly nontrivial way and $\mathcal{M}$ itself can be identified with $L^\infty(\mathcal{M})$, as von Neumann algebras. As it happens this provides the following generalization of Hölder inequality: suppose $p, q, r \in (0, \infty]$ are such that $q^{-1} = p^{-1} + r^{-1}$; if $f \in L^p(\mathcal{M})$ and $g \in L^r(\mathcal{M})$, then $fg \in L^q(\mathcal{M})$ and $\|fg\|_q \leq \|f\|_p \|g\|_r$, where the subscript indicates the quasi-norm of the corresponding Haagerup space. Letting $p = \infty$ or $r = \infty$ one gets the module structures over $L^\infty(\mathcal{M})$. See [**9, 23, 25**].

(2) After that it is clear that that every $g \in L^r(\mathcal{M})$ gives rise to a homomorphism (of left $L^\infty(\mathcal{M})$-modules) $\gamma : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ by multiplication: $\gamma(f) = fg$. Moreover, $\|\gamma : L^p(\mathcal{M}) \to L^q(\mathcal{M})\| = \|g\|_r$. Junge and Sherman proved in [**13**, Theorem 2.5] that all such homomorphisms arise in this way, which is crucial for us.

(3) The Haagerup spaces do not form any "scale". Indeed, by the very definition, one has $L^p(\mathcal{M}) \cap L^q(\mathcal{M}) = 0$ unless $p = q$. In particular, $L^p(B)$ (the Haagerup L^p space corresponding to the choice $\mathcal{M} = B$) cannot be the same as 'our' S^p. Nevertheless there is a system of isometric bimodule isomorphisms $\iota_p : S^p \to L^p(B)$ which are compatible with the product maps in the sense that $\iota_q(fg) = \iota_p(f)\iota_r(g)$ whenever $f \in S^p$ and $g \in S^q$ with $q^{-1} = p^{-1} + r^{-1}$.

The obvious consequence of this is that a map $u : S^p \to S^q$ is a homomorphisms of B-modules if and only if $\iota_q \circ u \circ \iota_p^{-1} : L^p(B) \to L^q(B)$ is a homomorphism of $L^\infty(B)$-modules. Therefore replacing Schatten classes by Haagerup spaces and B by $L^\infty(B)$ does not alter the Lemma.

(4) Raynaud proved in [**24**] that given a von Neumann algebra $\mathcal{M}$ and a countably incomplete ultrafilter $\mathcal{U}$ one can represent the ultrapowers of the whole family of Haagerup spaces $L^p(\mathcal{M})$ (for finite p) as the Haagerup spaces associated to some von Neumann algebra independent on p. Precisely: there is a von Neumann algebra $\mathcal{N}$ containing $L^\infty(\mathcal{M})_\mathcal{U}$ and a system of surjective isometries $\kappa_p : L^p(\mathcal{M})_\mathcal{U} \to L^p(\mathcal{N})$ for $0 < p < \infty$ compatible with the product maps in the following sense: $p, q, r \in (0, \infty)$ are such that $q^{-1} = p^{-1} + r^{-1}$ and (f_i) and (g_i) are bounded families in $L^p(\mathcal{M})$ and $L^r(\mathcal{M})$, respectively, then

$$(\kappa_p[(f_i)])(\kappa_r[(g_i)]) = \kappa_q[(f_i g_i)],$$

where the product in the left-hand side refers to spaces over $\mathcal{N}$ and those in the right-hand side to $\mathcal{M}$.

(5) Therefore we can regard $L^p(\mathcal{M})_\mathcal{U}$ as a module over $\mathcal{N}$ and every homomorphism of $\mathcal{N}$-modules $\gamma : L^p(\mathcal{M})_\mathcal{U} \to L^q(\mathcal{M})_\mathcal{U}$ can be represented as $\gamma[(f_i)] = [(f_i g_i)]$, where (g_i) is a bounded family in $L^r(\mathcal{M})$.

(6) The proof of the Lemma will be complete if we show that every homomorphism of $L^\infty(\mathcal{M})_\mathcal{U}$-modules $\gamma : L^p(\mathcal{M})_\mathcal{U} \to L^q(\mathcal{M})_\mathcal{U}$ is automatically a homomorphism of $\mathcal{N}$-modules. And this is so because on one hand $L^\infty(\mathcal{M})_\mathcal{U}$ is dense in $\mathcal{N}$ in the strong operator topology induced by the (module) action on $L^2(\mathcal{N})$ and, on the other hand, the restriction to bounded subsets of $\mathcal{N}$ of the strong operator topology induced by the action on $L^p(\mathcal{N})$ does not depend on $0 < p < \infty$ (see [**13**, Lemma 2.3]). □

Proof of Theorem 1. Assuming the contrary there is a sequence of centralizers $\Omega_n : S_0^p \to S^q$ with $\Delta[\Omega_n] \leq 1$ and $\operatorname{dist}(\Omega_n, \mathcal{M}_B(S_0^p, S^q)) \to \infty$. In view of Lemma 2 we may and do assume that for each n there is a finite rank projection $e_n \in B$ such that $\Omega_n(f) = \Omega_n(fe_n)$ for all $f \in S_0^p$. Thus we may assume Ω_n defined on the whole of S^p and also that $\operatorname{dist}(\Omega_n, \mathcal{M}_B(S^p, S^q))$ is finite for every n.

For each n we take a morphism $\phi_n : S^p \to S^q$ such that

$$\delta_n = \operatorname{dist}(\Omega_n, \phi_n) \leq \operatorname{dist}(\Omega_n, \mathcal{M}_B(S^p, S^q)) + 1/n.$$

Of course, $\delta_n \to \infty$ as $n \to \infty$. Put

$$v_n = \frac{\Omega_n - \phi_n}{\delta_n},$$

so that v_n is a homogeneous mapping from S^p to S^q with $\|v_n : S^p \to S^q\| \leq 1$ and $\Delta[v_n] \leq \delta_n^{-1}\Delta[\Omega] \to 0$ as $n \to \infty$.

Let $\mathcal{U}$ be a free ultrafilter on the integers and consider the corresponding ultrapowers $S^p_{\mathcal{U}}$ and $S^q_{\mathcal{U}}$. We can use the (probably nonlinear) maps v_n to define $v : S^p_{\mathcal{U}} \to S^q_{\mathcal{U}}$ by

$$v[(f_n)] = [(v_n(f_n))].$$

Let us check that v is well defined. First, suppose $[(f_n)] = 0$, that is, $\|f_n\|_p \to 0$ along $\mathcal{U}$. As $\|v_n(f_n)\|_q \leq \|f_n\|_p$ we have $[(v_n(f_n))] = 0$. Suppose now $[(f_n)] = [(g_n)]$. We must prove that $[(v_n(f_n))] = [(v_n(g_n))]$. But

$$\lim_{\mathcal{U}} \|v_n(f_n) - v_n(g_n)\|_q = \lim_{\mathcal{U}} \|v_n(f_n) - v_n(g_n) - v_n(f_n - g_n)\|_q \leq \lim_{\mathcal{U}} Q[v_n](\|g_n\|_p + \|f_n - g_n\|_p) = 0$$

and the definition of v makes sense. Now it is nearly obvious that v is a continuous homomorphism of $B_{\mathcal{U}}$-modules. By Lemma 3 there is a bounded sequence (u_n) in S^r representing v in the sense that $v[(f_n)] = [(f_n u_n)]$ whenever (f_n) is a bounded sequence in S^p, where $r^{-1} + p^{-1} = q^{-1}$. This implies that $\operatorname{dist}(v_n, u_n) \to 0$ along $\mathcal{U}$. In particular, for every $\varepsilon > 0$, the set $S = \{n \in \mathbb{N} : 0 < \operatorname{dist}(\delta_n^{-1}(\Omega_n - \phi_n), u_n) < \varepsilon\}$ belongs to $\mathcal{U}$ and it contains infinitely many indices n. For these n we get

$$\operatorname{dist}(\Omega_n, \phi_n + \delta_n u_n) < \varepsilon\delta_n < 2\varepsilon \operatorname{dist}(\Omega_n, \mathcal{M}_B(S^p, S^q)),$$

in striking contradiction with our choice of ϕ_n. □

Corollary 2. $\operatorname{Ext}_B(S^p, S^q) = 0$ *for* $0 < q < p \leq \infty$.

Proof. Corollary 1 and Theorem 1. For $p = \infty$ use Lemma 2. □

4. Isomorphisms of spaces of extensions

Once we know that Ext vanishes at certain couples, it is easy to use the functor Hom to compare different spaces of extensions. Let us begin with the covariant case. Suppose we are given an extension of modules

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0. \tag{7}$$

If E is another module we can apply $\operatorname{Hom}(E, -)$ to get an exact sequence (of linear spaces)

$$0 \longrightarrow \operatorname{Hom}(E, Y) \xrightarrow{\imath_\circ} \operatorname{Hom}(E, X) \xrightarrow{\pi_\circ} \operatorname{Hom}(E, Z) \xrightarrow{\alpha} \operatorname{Ext}(E, Y) \longrightarrow \cdots \tag{8}$$

Notice that $\imath_\circ$ is just the functorial image of $\imath$, and similarly with $\pi_\circ$. The connecting map α sends a given homomorphism ϕ into the (class of the) lower extension in the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow\phi & & \\ 0 & \longrightarrow & Y & \longrightarrow & \mathrm{PB} & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Thus, if $\operatorname{Ext}(E, Y)$ vanishes, then (8) represents an extension of $\operatorname{Hom}(E, Z)$ by $\operatorname{Hom}(E, Y)$. If, besides, E is a bimodule, then (8) is an extension of (left) modules. All this can be seen in [**4**].

In a similar vein, if we apply $\operatorname{Hom}(-, E)$ to (7) we obtain

$$0 \longrightarrow \operatorname{Hom}(Z, E) \xrightarrow{\pi^\circ} \operatorname{Hom}(X, E) \xrightarrow{\imath^\circ} \operatorname{Hom}(Y, E) \xrightarrow{\beta} \operatorname{Ext}(Z, E) \longrightarrow \cdots \tag{9}$$

Here, β sends a given homomorphism $\phi : Y \to E$ into the (class of the) lower row of the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathrm{PO} & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

Hence, if $\operatorname{Ext}(Z, E)$ vanishes, then (9) is an extension of $\operatorname{Hom}(Y, E)$ by $\operatorname{Hom}(Z, E)$ which lives in the category of right modules provided E is a bimodule.

Theorem 2. *Let $0 < r < p_1 \le p_2 < \infty$ be fixed. Then $\operatorname{Hom}_B(-, S^r)$ defines an isomorphism from $\operatorname{Ext}_B(S^{p_1}, S^{p_2})$ onto $\operatorname{Ext}(S^{q_2}, S^{q_1})_B$, where $q_i^{-1} + p_i^{-1} = r^{-1}$ for $i = 1, 2$. Similarly, $\operatorname{Hom}(-, S^r)_B$ is an isomorphism from $\operatorname{Ext}(S^{p_1}, S^{p_2})_B$ onto $\operatorname{Ext}_B(S^{q_2}, S^{q_1})$.*

Proof. Suppose we are given an extension of left modules $0 \to S^{p_2} \to X \to S^{p_1} \to 0$. Applying $\operatorname{Hom}_B(-, S^r)$ we get

$$0 \longrightarrow \operatorname{Hom}_B(S^{p_1}, S^r) \longrightarrow \operatorname{Hom}_B(X, S^r) \longrightarrow \operatorname{Hom}_B(S^{p_2}, S^r) \longrightarrow \operatorname{Ext}_B(S^{p_1}, S^r) \longrightarrow \cdots \tag{10}$$

But $\operatorname{Ext}_B(S^{p_1}, S^r) = 0$, so the above diagram is in fact an extension of $\operatorname{Hom}(S^{p_2}, S^r) = S^{q_2}$ by $\operatorname{Hom}_B(S^{p_1}, S^r) = S^{q_1}$ in the category of right modules over B. It is pretty obvious that this procedure preserves equivalences and so it defines a mapping

$$\operatorname{Hom}(-, S^r)_B : \operatorname{Ext}_B(S^{p_1}, S^{p_2}) \longrightarrow \operatorname{Ext}(S^{q_2}, S^{q_1})_B.$$

To see that it is indeed an isomorphism, consider now $\operatorname{Hom}(-, S^r)_B$ as a map from $\operatorname{Ext}(S^{q_2}, S^{q_1})_B$ to $\operatorname{Ext}_B(S^{p_1}, S^{p_2})$ – take into account that $r < q_2$ – and let us check that the two maps are inverse to each other. Indeed, if we apply $\operatorname{Hom}(-, S^r)_B$ to (10), we obtain another extension

$$0 \longrightarrow \operatorname{Hom}(\operatorname{Hom}_B(S^{p_2}, S^r), S^r)_B \longrightarrow \operatorname{Hom}(\operatorname{Hom}_B(X, S^r), S^r)_B \longrightarrow \operatorname{Hom}(\operatorname{Hom}_B(S^{p_1}, S^r), S^r)_B \longrightarrow 0.$$

But after the identification $S^{p_i} = \operatorname{Hom}(\operatorname{Hom}_B(S^{p_i}, S^r), S^r)_B$ this extension is equivalent to the starting one since the diagram

$$\begin{array}{ccccc} S^{p_2} & \longrightarrow & X & \longrightarrow & S^{p_1} \\ \big\| & & \big\downarrow{\scriptstyle\delta} & & \big\| \\ \operatorname{Hom}(\operatorname{Hom}(S^{p_2}, S^r), S^r)_B & \longrightarrow & \operatorname{Hom}(\operatorname{Hom}(X, S^r), S^r)_B & \longrightarrow & \operatorname{Hom}(\operatorname{Hom}(S^{p_1}, S^r), S^r)_B \end{array}$$

is commutative – the middle arrow is the obvious evaluation homomorphism given by $\delta(x)(\phi) = \phi(x)$. □

Theorem 3. *Let $0 < p_1 \le p_2 < s < \infty$ be fixed. Then $\operatorname{Hom}(S^s, -) : \operatorname{Ext}(S^{p_1}, S^{p_2}) \longrightarrow \operatorname{Ext}(S^{q_1}, S^{q_2})$ is an isomorphism, where $s^{-1} + q_i^{-1} = p_i^{-1}$ for $i = 1, 2$.*

Proof. (Absence of subscript indicates left module structure.) This can be proved in several ways. Perhaps the simplest one is checking that if $r^{-1} + s^{-1} = t^{-1}$, then one has $\operatorname{Hom}(-, S^r) \circ \operatorname{Hom}(S^s, -) = \operatorname{Hom}(-, S^t)$ at $\operatorname{Ext}(S^{p_1}, S^{p_2})$, that is, the composition

$$\operatorname{Ext}(S^{p_1}, S^{p_2}) \xrightarrow{\operatorname{Hom}(S^s, -)} \operatorname{Ext}(S^{q_1}, S^{q_2}) \xrightarrow{\operatorname{Hom}(-, S^r)} \operatorname{Ext}(S^{\ell_2}, S^{\ell_1})_B$$

agrees with $\operatorname{Hom}(-, S^t) : \operatorname{Ext}(S^{p_1}, S^{p_2}) \longrightarrow \operatorname{Ext}(S^{\ell_2}, S^{\ell_1})_B$, where $\ell_i^{-1} + p_i^{-1} = t^{-1}$ for $i = 1, 2$. Incidentally this will show that $\operatorname{Hom}(S^s, -) = \operatorname{Hom}(-, S^r)_B \circ \operatorname{Hom}(-, S^t)$ since $\operatorname{Hom}(-, S^r)_B$ is the inverse of $\operatorname{Hom}_B(-, S^r)$. See the proof of Theorem 2.

Recall that $S^r = \mathrm{Hom}(S^s, S^t)$, so that the composition $\mathrm{Hom}(-, S^r) \circ \mathrm{Hom}(S^s, -)$ agrees with $\mathrm{Hom}(\mathrm{Hom}(S^s, -), \mathrm{Hom}(S^s, S^t))$.

To each quasi-Banach left B-module M we attach the homomorphism of right modules

$$-_\circ : \mathrm{Hom}(M, S^t) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(S^s, M), \mathrm{Hom}(S^s, S^t))$$

sending a given homomorphism $u : M \to S^t$ into the transformation $u_\circ : \mathrm{Hom}(S^s, M) \longrightarrow \mathrm{Hom}(S^s, S^t)$ defined by $u_\circ(v) = u \circ v$.

This is in fact a natural transformation from $\mathrm{Hom}(-, S^t)$ to $\mathrm{Hom}(\mathrm{Hom}(S^s, -), \mathrm{Hom}(S^s, S^t))$ meaning that for every homomorphism of (left) modules $\alpha : M \to N$ the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}(M, S^t) & \xrightarrow{-_\circ} & \mathrm{Hom}(\mathrm{Hom}(S^s, M), \mathrm{Hom}(S^s, S^t)) \\ {\scriptstyle \alpha^\circ}\uparrow & & \uparrow{\scriptstyle (\alpha_\circ)^\circ} \\ \mathrm{Hom}(N, S^t) & \xrightarrow{-_\circ} & \mathrm{Hom}(\mathrm{Hom}(S^s, N), \mathrm{Hom}(S^s, S^t)) \end{array}$$

The point is that the preceding natural transformation behaves as a natural equivalence at S^{p_1} and S^{p_2} and so it induces an isomorphism at $\mathrm{Ext}(S^{p_1}, S^{p_2})$. Indeed, if we are given an extension

$$0 \longrightarrow S^{p_2} \xrightarrow{\ \imath\ } X \xrightarrow{\ \pi\ } S^{p_1} \longrightarrow 0$$

and we apply $\mathrm{Hom}(-, S^t)$ on one hand and $\mathrm{Hom}(\mathrm{Hom}(S^s, -), \mathrm{Hom}(S^s, S^t)) = \mathrm{Hom}(\mathrm{Hom}(S^s, -), S^r)$ on the other we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(S^{p_1}, S^t) & \xrightarrow{\imath^\circ} & \mathrm{Hom}(X, S^t) & \xrightarrow{\pi^\circ} & \mathrm{Hom}(S^{p_2}, S^t) \\ {\scriptstyle -_\circ}\downarrow & & {\scriptstyle -_\circ}\downarrow & & {\scriptstyle -_\circ}\downarrow \\ \mathrm{Hom}(\mathrm{Hom}(S^s, S^{p_1}), S^r) & \xrightarrow{(\imath_\circ)^\circ} & \mathrm{Hom}(\mathrm{Hom}(S^s, X), S^r) & \xrightarrow{(\pi_\circ)^\circ} & \mathrm{Hom}(\mathrm{Hom}(S^s, S^{p_2}), S^r) \end{array}$$

where the rows are extensions. Now, the left and right vertical arrows are isomorphisms (they are the identity after identifying both $\mathrm{Hom}(S^{p_i}, S^t)$ and $\mathrm{Hom}(\mathrm{Hom}(S^s, S^{p_i}), S^r)$ with S^{ℓ_i} for $i = 1, 2$) and so is the middle one. □

4.1. Nonlinear counterpart. Let us take a look at the actions of $\mathrm{Hom}(-, S^r)$ and $\mathrm{Hom}(S^s, -)$ on centralizers. To take advantage of the extra simplification provided by Lemma 1(b) we shall work with right centralizers.

Fix numbers $0 < r < p_1 \le p_2 \le \infty$ and let $\Omega : S_0^{p_1} \to S^{p_2}$ be a right centralizer – note we are allowing $p_2 = \infty$ here. Given $g \in \mathrm{Hom}(S^{p_2}, S^r)_B = S^{q_2}$ (isometric isomorphism of bimodules), we consider the mapping $f \in S_0^{p_1} \mapsto g(\Omega f) \in S^r$. Clearly, this is a right centralizer with constant at most $\|g\|_{q_2}\Delta[\Omega]$ and since $r < p_1$ there is a morphism $\phi_g \in \mathcal{M}(S_0^{p_1}, S^r)_B = \mathcal{M}(\mathfrak{F}, B)_B$ such that $\|\Omega(f)g + \phi_g(f)\|_r \le M\Delta[\Omega]\|f\|_{p_1}\|g\|_{q_2}$. By Lemma 1 there is $\ell \in L(\mathcal{H})$ that implements ϕ_g in the sense that $\phi_g(f) = \ell \circ f$ and we can define a mapping $\Phi : S^{q_2} \to L(\mathcal{H})$ just taking $\Phi(g) = \ell$. Of course this can be done homogeneously and we have the estimate

$$\|g(\Omega f) + (\Phi g) f\|_r \le M\Delta[\Omega]\|g\|_{q_2}\|f\|_{p_1}, \qquad (f \in S_0^{p_1}, g \in S^{q_2}). \tag{11}$$

Obviously, S^{q_1} is a left submodule of $L(\mathcal{H})$. That Φ is a left centralizer from S^{q_2} to S^{q_1} now follows from (11), taking into account that for $\ell \in L(\mathcal{H})$ one has $\|\ell\|_{q_1} = \|\ell_\circ : S^{p_1} \to S^r\| = \|\ell_\circ : S_0^{p_1} \to S^r\|$, where $\ell_\circ(f) = \ell \circ f$.

Let X_Ω denote the completion of $S^{p_2}\oplus_\Omega S_0^{p_1}$. It is possible to identify $\operatorname{Hom}(X_\Omega, S^r)_B$ and $S^{q_1}\oplus_\Phi S^{q_2}$ as follows: for $(h,g)\in S^{q_1}\oplus_\Phi S^{q_2}$ (hence $h-\Phi g\in S^{q_1}$) and $(f',f)\in S^{p_2}\oplus_\Omega S_0^{p_1}$, we put

$$(h,g)(f',f)=hf+gf'.$$

One then has

$$\begin{aligned}\|hf+gf'\|_r&=\|hf-(\Phi g)f+(\Phi g)f+g\Omega f-g\Omega f+gf'\|_r\\&\le M(\|h-\Phi g\|_{q_1}\|f\|_{p_1}+\|g\|_{q_2}\|f\|_{p_1}+\|g\|_{q_2}\|f'-\Omega f\|_{p_2}),\\&\le M(\|(h,g)\|_\Phi\|(f',f)\|_\Omega)\end{aligned}$$

and since $S^{p_2}\oplus_\Omega S_0^{p_1}$ is a dense submodule of X_Ω we see that $S^{q_1}\oplus_\Phi S^{q_2}$ embeds in $\operatorname{Hom}(X_\Omega,S^r)_B=\operatorname{Hom}(S^{p_2}\oplus_\Omega S_0^{p_1},S^r)_B$. That embedding is onto (and open) in view of the commutativity of the diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&S^{q_1}&\longrightarrow&S^{q_1}\oplus_\Phi S^{q_2}&\longrightarrow&S^{q_2}&\longrightarrow&0\\&&\|&&\downarrow&&\|&&\\0&\longrightarrow&\operatorname{Hom}(S^{p_1},S^r)_B&\longrightarrow&\operatorname{Hom}(X_\Omega,S^r)_B&\longrightarrow&\operatorname{Hom}(S^{p_2},S^r)_B&\longrightarrow&0\end{array}$$

whose rows are exact.

We next turn our attention to the covariant case. We fix $0<p_1\le p_2<s$ and for $i=1,2$, we put $q_i^{-1}+s^{-1}=p_i^{-1}$ so that $\operatorname{Hom}(S^s,S^{p_i})_B=S^{q_i}$ in the obvious way. As before, we consider a right centralizer $\Omega: S_0^{p_1}\to S^{p_2}$.

Given $g\in S^{q_1}$ we consider the map $f\in S_0^s\longmapsto\Omega(gf)\in S^{p_2}$. This is again a right centralizer, with constant at most $\|g\|_{q_1}C[\Omega]$. As $s>p_2$, there is a linear map ψ on $\mathcal{H}$ such that $\|\Omega(gf)-\psi_\circ(f)\|_{p_2}\le M\Delta[\Omega]\|g\|_{q_1}\|f\|_s$. Taking $\Psi(g)=\psi$ homogeneously we get a mapping $\Psi: S^{q_1}\to L(\mathcal{H})$ such that

$$\|\Omega(gf)-(\Psi g)f\|_{p_2}\le M\Delta[\Omega]\|g\|_{q_2}\|f\|_s,\qquad (g\in S^{q_2}, f\in S_0^s).\tag{12}$$

Let us verify that Ψ is a right centralizer from S^{q_1} to S^{q_2}. Take $g,g'\in S^{q_1}$ and $a\in B$ and recall that for $\ell\in L(\mathcal{H})$ one has $\|\ell\|_{q_2}=\|\ell_\circ: S_0^s\to S^{p_2}\|$. For $f\in S_0^s$ we have on account of (12):

$$\begin{aligned}\|(\Psi(g+g')-\Psi g-\Psi g')f\|_{p_2}&\le\|(\Omega(gf+g'f)-\Omega(gf)-\Omega(g'f)\|_{p_2}+M(\|g+g'\|_{q_2}+\|g\|_{q_2}+\|g'\|_{q_2})\|f\|_s\\&\le Q[\Omega](\|gf\|_{p_1}+\|g'f\|_{p_1})+M(\|g\|_{q_2}+\|g'\|_{q_2})\|f\|_s\\&\le M(\|g\|_{q_2}+\|g'\|_{q_2})\|f\|_s.\end{aligned}$$

The estimate $\|\Psi(ga)-(\Psi g)a\|_{q_2}\le MC[\Omega]\|g\|_{q_2}\|a\|_B$ is even easier and we leave it to the reader. As before, $S^{q_1}\oplus_\Psi S^{q_2}$ is isomorphic to $\operatorname{Hom}(S^s,X_\Omega)_B$ (as quasi-Banach right modules), where X_Ω is the completion of $S^{p_2}\oplus_\Omega S_0^{p_1}$. Indeed, take $(h,g)\in S^{q_1}\oplus_\Psi S^{q_2}$. Given $f\in S_0^s$ we define

$$(h,g)(f)=(hf,gf).$$

The definition is correct because f has finite rank and thus hf is bounded even if h is not. Moreover (hf,gf) falls in $S^{p_2}\oplus_\Omega S_0^{p_1}$ and we have

$$\begin{aligned}\|(hf,gf)\|_\Omega&=\|hf-\Omega(gf)\|_{p_2}+\|gf\|_{p_1}\\&\le\|hf-\Omega(gf)+(\Psi g)f-(\Psi g)f\|_{p_2}+\|gf\|_{p_1}\\&\le M(\|h-\Psi g\|_{q_2}+\|g\|_{q_1})\|f\|_s.\end{aligned}$$

This shows that $S^{q_1} \oplus_\Psi S^{q_2}$ embeds into $\operatorname{Hom}(S^s, X_\Omega)_B = \operatorname{Hom}(S^s_0, X_\Omega)_B$. Finally, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{q_2} & \longrightarrow & S^{q_2} \oplus_\Psi S^{q_1} & \longrightarrow & S^{q_1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Hom}(S^s, S^{p_2})_B & \longrightarrow & \operatorname{Hom}(S^s, X_\Omega)_B & \longrightarrow & \operatorname{Hom}(S^s, S^{p_1})_B & \longrightarrow & 0 \end{array}$$

provides the required isomorphism.

Next we focus on a different kind of transformation to be used in Section 5.

Proposition 1. *Let $\Psi : S_0^{q_1} \to S^{q_2}$ be a right-centralizer, with $0 < q_1 \leq q_2 \leq \infty$. Take $0 < s < \infty$ and let p_i be given by the identity $p_i^{-1} = q_i^{-1} + s^{-1}$ for $i = 1, 2$. We define a mapping $\Psi^{(s)} : S_0^{p_1} \to S^{p_2}$ by*

$$\Psi^{(s)}(h) = \Psi(u|h|^{p_1/q_1})|h|^{p_1/s},$$

where $u|h|$ is the polar decomposition of h. Then $\Psi^{(s)}$ is a right-centralizer.

Moreover, $\Psi^{(s)}$ is bounded if and only if Ψ is bounded.

Proof. Actually one can take $\Psi^{(s)}(h) = (\Psi f)g$ provided $h = fg$, with $M\|h\|_{p_1} \leq \|f\|_{q_1}\|g\|_s$.

We will show that $\Psi^{(s)}$ can be obtained as $\operatorname{Hom}_B(\operatorname{Hom}(\Psi, S^r)_B, S^t)$ for suitable r and t. We pick any $r < q_1$ and then we take t so that $t^{-1} = s^{-1} + r^{-1}$. Applying $\operatorname{Hom}(-, S^r)_B$ to Ψ we get a map $\Phi : S^{\ell_2} \to L(\mathcal{H})$ which is a left-centralizer from S^{ℓ_2} to S^{ℓ_1}, where $\ell_i^{-1} + q_i^{-1} = r^{-1}$ and satisfying an estimate

$$\|g(\Psi f) + (\Phi g)f\|_r \leq M\|g\|_{\ell_2}\|f\|_{q_1}, \qquad (g \in S^{\ell_2}, f \in S_0^{q_1}). \tag{13}$$

Now we apply $\operatorname{Hom}_B(-, S^t)$ to Φ as follows (notice that $S^{p_i} = \operatorname{Hom}_B(S^{\ell_i}, S^t)$ for $i = 1, 2$; in particular $t < \ell_2$). For each $h \in S^{p_1}$ we consider the map $g \in S^{\ell_2} \mapsto (\Phi g)h \in L(\mathcal{H})$. This is a left-centralizer from S^{ℓ_2} to S^t having constant proportional to $\|h\|_{p_1}$. Therefore there is $\Lambda(h) \in \mathcal{M}_B(S^{\ell_2}, L(\mathcal{H}))$ such that

$$\|\Lambda(h)(g) + (\Phi g)h\|_t \leq M\|h\|_{p_1}\|g\|_{\ell_2}, \qquad (h \in S^{p_1}, g \in S^{\ell_2}). \tag{14}$$

Even if we know no representation for arbitrary morphisms in $\mathcal{M}_B(S^{\ell_2}, L(\mathcal{H}))$ we claim that we may take $\Lambda(h)(g) = g(\Omega f)k$ provided $h = fk$ is the factorization appearing in the statement of the theorem. Indeed, by (13),

$$\|g(\Omega f)k + (\Phi g)h\|_t \leq M\|g\|_{\ell_2}\|f\|_{q_1}\|k\|_s \leq 2M\|g\|_{\ell_2}\|h\|_{p_1}$$

and we are done. The last statement obviously follows from the estimate (11). □

As we mentioned in the Introduction, $\operatorname{Ext}_B(S^p)$ is essentially independent on $p \in (0, \infty)$. Of course this follows from Theorem 3: indeed, if $p < q < \infty$, then $\operatorname{Hom}_B(S^s, -) : \operatorname{Ext}_B(S^p) \to \operatorname{Ext}_B(S^q)$ is an isomorphism provided s is given by $p^{-1} = q^{-1} + s^{-1}$. Let us record here the (right) centralizer version of this fact for future reference.

Corollary 3. *Let the numbers $p, q, s \in (0, \infty)$ satisfy $p^{-1} = q^{-1} + s^{-1}$. Given a right centralizer $\Psi : S_0^q \to S^q$, we define $\Psi^{(s)} : S_0^p \to S^p$ by $\Psi^{(s)}(f) = \Psi(u|f|^{p/q})|f|^{p/s}$, where $u|f|$ is the polar decomposition of f. Then $\Psi^{(s)}$ is a right centralizer and every right centralizer on S_0^p is at finite distance from one obtained in this way.*

PROOF. Everything but the last part is a particular case of the preceding Proposition. Please note that $q < \infty$ is required here, while $q_2 = \infty$ was allowed in Proposition 1.

Let Ω be a right centralizer on S_0^p and let Ψ be any centralizer obtained by applying $\mathrm{Hom}(S^s, -)_B$ to Ω. According to (12) we have $\|\Omega(gf) - (\Psi g)f\|_p \leq M\|g\|_q\|f\|_s$ for $g \in S^{q_2}$ and $f \in S_0^s$, from where is follows that $\Omega \approx \Psi^{(s)}$. □

5. The case $p < q$

In this Section we describe the extensions of S^p by S^q, with $0 < p < q \leq \infty$, by means of the so-called twisted Hilbert spaces. These are self-extensions of $\mathcal{H}$ in the category of (quasi-) Banach spaces, that is, short exact sequences of (quasi-) Banach spaces and operators

$$0 \longrightarrow \mathcal{H} \longrightarrow T \longrightarrow \mathcal{H} \longrightarrow 0. \tag{15}$$

As a matter of fact, the middle space T must be (isomorphic to) a Banach space [**14**, Theorems 4.3(iii) and 4.10]. Moreover, T is itself isomorphic to a Hilbert space if and only if (15) splits. The existence of nontrivial twisted Hilbert spaces was first established by Enflo, Lindenstrauss, and Pisier [**8**]. Later on Kalton and Peck [**20**] constructed fairly concrete examples, among them the nowadays famous Kalton-Peck space Z_2.

As it is well-known, twisted Hilbert spaces are in correspondence with quasi-linear maps on $\mathcal{H}$, that is, homogeneous maps $\phi : \mathcal{H} \to \mathcal{H}$ satisfying an estimate of the form

$$\|\phi(x+y) - \phi(x) - \phi(y)\|_{\mathcal{H}} \leq Q(\|x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}) \qquad (x, y \in \mathcal{H}).$$

(As we did in Section 2.2 we can replace the target space by a larger ambient space, or consider ϕ defined only on some dense subspace, or both. However, as linear spaces are free modules over the ground field, this is unnecessary to elaborate the theory.) All this can be seen in [**1, 6, 18, 19**].

THEOREM 4. *Let ϕ be a quasi-linear map on $\mathcal{H}$. We define a map $\tilde{\phi}$ on $\mathfrak{F}$ as follows. For each $f \in \mathfrak{F}$ we choose a Schmidt expansion $f = \sum_n s_n x_n \otimes y_n$ (homogeneously) and we put*

$$\tilde{\phi}(f) = \sum_n s_n x_n \otimes \phi(y_n). \tag{16}$$

Then $\tilde{\phi} : S_0^p \to S^q$ defines a right-centralizer whenever $0 < p < q \leq \infty$. Moreover, if $\Phi : S_0^p \to S^q$ is a right-centralizer, then $\Phi \approx \tilde{\phi}$ for some quasi-linear ϕ, where $\tilde{\phi}$ has the form given by (16).

PROOF. “Homogeneously” means that if $f = \sum_n s_n x_n \otimes y_n$ is the Schmidt expansion attached to f and $\lambda \in \mathbb{C}$ is not zero, then the expansion for λf is $\sum_n |\lambda| s_n x_n \otimes |\lambda|^{-1}\lambda y_n$. This makes $\tilde{\phi}$ homogeneous.

Let us beging by checking the first part when $q = \infty$ so that $S^q = K$, the ideal of compact operators on $\mathcal{H}$. To this end, recall that an operator $u : X \to Y$ acting between (quasi-) Banach spaces is said to be p-nuclear ($0 < p < \infty$) if it admits a representation as

$$u = \sum_n t_n x'_n \otimes y_n, \qquad (x'_n \in X', y_n \in Y) \tag{17}$$

with $\|x'_n\| = \|y_n\| = 1$ and (t_n) in ℓ^p. The class of all p-nuclear operators $X \to Y$ is denoted by $\mathfrak{N}^p(X, Y)$. The p-nuclear norm of u is then defined as the infimum of the (quasi-) norm in ℓ^p of the sequences (t_n) that can arise in (17). Notice that $S^p = \mathfrak{N}^p(\mathcal{H})$, with equal (quasi-) norms.

Now, let

$$0 \longrightarrow Y \stackrel{\imath}{\longrightarrow} X \stackrel{\pi}{\longrightarrow} Z \longrightarrow 0 \tag{18}$$

be an extension of quasi-Banach spaces and U another quasi-Banach space. Without loss of generality we assume $Y=\ker\pi$. If we fix $0<p<\infty$ and we apply $\mathfrak{N}^p(U,-)$ to the quotient map $\pi:X\to Z$ we obtain the operator $\pi_\circ:\mathfrak{N}^p(U,X)\longrightarrow\mathfrak{N}^p(U,Z)$ which is easily seen to be open.

Observe that $\ker\pi_\circ$ consists of certain Y-valued compact operators. Moreover, if $u\in\ker\pi_\circ$, then

$$\|u:U\to Y\|=\|u:U\to X\|\leq\|u\|_{\mathfrak{N}^p(U,X)},$$

so that the embedding $\ker\pi_\circ\to K(U,Y)$ is continuous and we may form the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi_\circ & \longrightarrow & \mathfrak{N}^p(U,X) & \xrightarrow{\pi_\circ} & \mathfrak{N}^p(U,Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K(U,Y) & \longrightarrow & \mathrm{PO} & \longrightarrow & \mathfrak{N}^p(U,Y) & \longrightarrow & 0 \end{array} \tag{19}$$

We recall that if we are given an arbitrary push-out diagram of quasi-Banach spaces and operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\varpi & \longrightarrow & A & \xrightarrow{\varpi} & C & \longrightarrow & 0 \\ & & {\scriptstyle s}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & \mathrm{PO} & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{20}$$

then, a quasi-linear map associated to the lower row can be constructed as follows: if $b:C\to A$ is a (homogeneous) bounded selection for the quotient $\varpi:A\to C$ and $\ell:C\to A$ a linear (surely unbounded) selection, then the difference $b-\ell:C\to\ker\varpi$ is associated to the upper extension in (20) and so $\sigma=s\circ(b-\ell):C\to D$ is the desired quasi-linear map. See [**4**] for the missing details.

This applies to the diagram (19) as follows. Suppose $X=Y\oplus_\phi Z$ arises from the quasi-linear map $\phi:Z\to Y$ and that $\phi=\beta-\lambda$, with $\beta:Z\to X$ homogeneous and bounded and $\lambda:Z\to X$ linear. Then, if $u\in\mathfrak{N}^p(U,Z)$ has finite rank and we choose (homogeneously) an expansion $u=\sum_n u_n'\otimes z_n$ with finitely many summands and $\|u\|_p\geq(1+\varepsilon)(\sum\|u_n'\|^p\|z_n\|^p)^{1/p}$ we may define

$$B(u)=\sum_n u_n'\otimes\beta(z_n)\qquad\text{and}\qquad\Lambda(u)=\sum_n u_n'\otimes\lambda(z_n).$$

Then B is homogeneous and bounded, $\Lambda=\mathbf{I}_{U'}\otimes\lambda$ is linear and, therefore, we can define a quasi-linear map $\tilde\phi:\mathfrak{N}^p(U,Z)\to K(U,Y)$ taking

$$\tilde\phi(u)=B(u)-\Lambda(u)=\sum_n u_n'\otimes\phi(z_n), \tag{21}$$

at least when u has finite rank. Notice, moreover, that if $u=\sum_m v_m'\otimes\zeta_m$ is another representation with $\|u\|_p\geq(1+\varepsilon)(\sum\|v_m'\|^p\|\zeta_m\|^p)^{1/p}$, then

$$\begin{aligned}\left\|\tilde\phi(u)-\sum_m v_m'\otimes\phi(\zeta_m)\right\|_{\mathfrak{N}^p(U,X)} &= \left\|\sum_n u_n'\otimes\beta(z_n)-\sum_m v_m'\otimes\beta(\zeta_m)\right\|_{\mathfrak{N}^p(U,X)}\\ &\leq 2(1+\varepsilon)2^{\frac1p-1}\|\beta\|\|u\|_{\mathfrak{N}^p(U,Z)}\end{aligned}$$

(with the factor $2^{\frac1p-1}$ deleted if $p\geq1$). Hence $\tilde\phi$ is essentially independent on the chosen representation of u.

Going back to the Schatten classes, consider the case where the starting extension (18) is the self-extension induced by ϕ, and we take $U = \mathcal{H}$ so that (19) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi_\circ & \longrightarrow & \mathfrak{N}^p(\mathcal{H}, X) & \xrightarrow{\pi_\circ} & S^p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathrm{PO} & \longrightarrow & S^p & \longrightarrow & 0 \end{array} \tag{22}$$

The preceding diagram lives in the category of quasi-Banach right modules over B (the multiplication in $\mathfrak{N}^p(\mathcal{H}, X)$ given by composition on the right) and, according to (21) the map $\tilde{\phi} : S_0^p \to K$ given by $\tilde{\phi}(u) = \sum t_n x_n \otimes \phi(y_n)$ is a right-centralizer inducing its lower row. Here, $u = \sum t_n x_n \otimes y_n$ is the Schmidt-expansion appearing in the statement of the Theorem and $X = \mathcal{H} \oplus_\phi \mathcal{H}$.

Notice that $\tilde{\phi}$ is essentially independent on the prescribed representations since any other choice yields a centralizer $S_0^p \to K$ at finite distance from $\tilde{\phi}$.

Next we prove that the map $\tilde{\phi}$ is still a right-centralizer when regarded as a map from S_0^p to S^q. To this end we consider p and q fixed and take r so that $p^{-1} = q^{-1} + r^{-1}$. We already know that $\tilde{\phi} : S_0^r \to K$ is a centralizer. We introduce a second choice of the Schmidt expansions on S^r as follows. Given a normalized $h \in S^r$ set $f = |h|^{r/p}$, so that if u is the phase of h, then $h = uf^{p/r}$, with f normalized in S^p. Now, if $uf = \sum_n s_n x_n \otimes y_n$ is the prescribed representation, we have

$$h = \sum_n s_n^{p/r} x_n \otimes y_n$$

and we can define a map $\Gamma : S_0^r \to K$ by the formula $\Gamma(h) = \sum_n s_n^{p/r} x_n \otimes \phi(y_n)$. This is in fact a centralizer and we even know that $\Gamma \approx \tilde{\phi}$.

Let us activate Proposition 1 with $s = q$ and $\Psi = \Gamma$ to conclude that if $u|f|$ is the polar decomposition of $f \in S_0^p$, then the formula

$$\Gamma^{(q)}(f) = \Phi(u|f|^{p/r})|f|^{p/q}$$

defines a centralizer from S_0^p to S^q. But $\Gamma^{(q)}$ agrees with our old friend $\tilde{\phi}$. Indeed, if $f = \sum_n s_n x_n \otimes y_n$ is the prescribed representation of f, then $\Gamma(u|f|^{p/r}) = \sum_n s_n^{p/r} x_n \otimes \phi(y_n)$ and since $|f|^{p/q} = \sum_n s_n^{p/q} x_n \otimes x_n$ and $p(r^{-1} + q^{-1}) = 1$ we have

$$\Gamma^{(q)}(f) = \left(\sum_n s_n^{p/r} x_n \otimes \phi(y_n) \right) \left(\sum_n s_n^{p/q} x_n \otimes x_n \right) = \sum_n s_n x_n \otimes \phi(y_n) = \tilde{\phi}(f).$$

This completes the proof of the first part.

We finally prove the 'moreover' part. Let $\Phi : S_0^p \to S^q$ be a right-centralizer for which we may assume (and do) that $\Phi(fe) = \Phi(f)e$ for every $f \in S_0^p$ and every projection $e \in B$. Here, $p, q \in (0, \infty]$ are arbitrary; in particular we are not assuming $p < q$. Fixing a norm one $\eta \in \mathcal{H}$, we see that $\Phi(\eta \otimes y) = \eta \otimes \phi$ for some $\phi \in H$ depending on y (and η). Taking $\phi = \phi_\eta(y)$ we obtain a self-map on $\mathcal{H}$ which is easily seen to be quasi-linear. Let ζ be another normalized vector in $\mathcal{H}$ and define ϕ_ζ by the identity $\Phi(\zeta \otimes y) = \zeta \otimes \phi_\zeta(y)$. Let $u \in B$ be an isometry of $\mathcal{H}$ sending ζ to η, so that $(\eta \otimes y)u = u^*(\eta) \otimes y = \zeta \otimes y$. One has

$$\|\phi_\zeta(y) - \phi_\eta(y)\| = \|\eta \otimes (\phi_\zeta(y) - \phi_\eta(y))\|_q = \|\Phi((\eta \otimes y)u) - (\Phi(\eta \otimes y)u)\|_q \leq C[\Phi]\|\eta\|\|y\|\|u\|_B \leq M\|y\|.$$

Therefore, $\phi_\eta \approx \phi_\zeta$ and so there is a quasi-linear map ϕ on $\mathcal{H}$ that could be properly called the spatial part of Φ and satisfies

$$\|\Phi(x\otimes y) - x\otimes\phi(y)\|_q \leq M\|x\|\|y\| \qquad (x,y\in\mathcal{H}). \tag{23}$$

We want to see that when $0<p<q\leq\infty$ one has $\Phi\approx\tilde{\phi}$ as long as (23) holds true. Clearly, we may and so assume that $\Phi(x\otimes y) = x\otimes\phi(y)$ for all $x,y\in\mathcal{H}$ and we must prove that if $f=\sum_n t_n x_n\otimes y_n$ is a Schmidt representation, then

$$\left\|\Phi(f) - \sum_n t_n x_n\otimes\phi(y_n)\right\|_q \leq M\|f\|_p \tag{24}$$

for some constant M depending only on Φ, p and q.

Assume first $p<1$. Then $(x_n\otimes y_n)$ is (isometrically) equivalent to the unit basis of ℓ^p and since S^q is an m-Banach space for $m=\min(1,q)$ the bound (24) follows from an inequality due to Kalton [**14**, Lemma 3.4] – indeed one may take

$$M = \left(\sum_{k=1}^{\infty}\left(\frac{2}{k}\right)^{m/p}\right)^{1/p} Q[\Phi].$$

Now, if $p\geq 1$ we can use Proposition 1 to lower Φ to a centralizer defined on $S^{1/2}$, say. So, take s such that $p^{-1}+s^{-1}=2$ and let q' be given by $1/q' = p^{-1}+s^{-1}$. We know from Proposition 1 that the map $\Phi^{(s)}: S_0^{1/2}\to S^{q'}$ defined by

$$\Phi^{(s)}(h) = \Phi(u|h|^{\frac{1}{2p}})|h|^{\frac{1}{2s}}$$

is a right-centralizer. Here, $u|h|$ is the polar decomposition of h. Notice that for $\|x\|=\|y\|=1$, the polar decomposition of $x\otimes y$ is $(x\otimes y)(x\otimes x)$ and since $(x\otimes x)^t = x\otimes x$ for all $t>0$ we have

$$\Phi^{(s)}(x\otimes y) = (\Phi(x\otimes y))(x\otimes x) = (x\otimes\phi(y))(x\otimes x) = x\otimes\phi(y)$$

and $\Phi^{(s)}\approx\tilde{\phi}$ on $S^{1/2}$. On the other hand we know that $\tilde{\phi}^{(s)}\approx\tilde{\phi}$ on $S_0^{1/2}$ and thus, $\Phi^{(s)}-\tilde{\phi}^{(s)}\approx(\Phi-\tilde{\phi})^{(s)}$ is bounded as a map from $S_0^{1/2}$ to $S^{q'}$. The 'moreover' part of Proposition 1 now yields $\Phi\approx\tilde{\phi}$ on S_0^p. □

6. Minimal extensions and $\mathcal{K}$-spaces

Recall that a (complex) quasi-Banach space Z is said to be a $\mathcal{K}$-space if every minimal extension (of quasi-Banach spaces) $0\to\mathbb{C}\to X\to Z\to 0$ splits. Equivalently, if for every dense subspace Z_0 of Z and every quasi-linear map $\varphi: Z_0\to\mathbb{C}$ there is a linear map $\ell: Z_0\to\mathbb{C}$ such that $\operatorname{dist}(\varphi,\ell)<\infty$. The main examples of $\mathcal{K}$-spaces were supplied by Kalton and coworkers: it turns out that ℓ^p (or L^p) is a $\mathcal{K}$-space if and only if $p\in(0,\infty]$ is different from 1. See [**26, 14, 27, 22**]. In contrast to the commutative situation, one has:

THEOREM 5. *S^p is a $\mathcal{K}$-space for no $p\in(0,1)$.*

PROOF. Let ϕ be quasi-linear on $\mathcal{H}$ and $\tilde{\phi}: S_0^p\to S^1$ the right centralizer given by Theorem 4. Composing with $\operatorname{tr}: S^1\to\mathbb{C}$ we get a quasi-linear function $\varphi: S_0^p\to\mathbb{C}$ such that

$$\varphi(x\otimes y) = \operatorname{tr}(\tilde{\phi}(x\otimes y)) = \operatorname{tr}(x\otimes\phi(y)) = \langle\phi(y)|x\rangle.$$

Suppose there is a linear $\ell: S_0^p\to\mathbb{C}$ at finite distance from φ. As $\varphi(x\otimes y)\to 0$ for fixed y when $x\to 0$ in $\mathcal{H}$ we see that $\ell(x\otimes y)\to 0$ for fixed y when $x\to 0$ in $\mathcal{H}$ and by Lemma 1(d) there is a linear map L on $\mathcal{H}$ such that $\ell(x\otimes y)=\langle L(y)|x\rangle$. This obviously implies $\operatorname{dist}(\phi,L)<\infty$. Starting with a non-trivial ϕ we get a non-trivial, minimal extension of S^p. □

Of course S^1 is not a $\mathcal{K}$-space as it contains a complemented subspace isomorphic to ℓ^1, while S^p is a $\mathcal{K}$-space for $p\in(1,\infty)$, as all B-convex spaces are.

We finally add a result which partially answers a question raised by Kalton and Montgomery-Smith at the end of the survey [**19**, p. 1172].

Proposition 2. *Let $\Phi: S_0^2\to L(\mathcal{H})$ be a left centralizer from S_0^2 to S^2. Then the function $\varphi: S_0^1\to\mathbb{C}$ given by*

$$\varphi(f)=\operatorname{tr}\left(u|f|^{1/2}\Phi(|f|^{1/2})\right) \tag{25}$$

is quasi-linear, where $u|f|$ is the polar decomposition of f. Every quasi-linear (complex) function on S_0^1 is at finite distance from one arising in this way.

Sketch of the Proof. Let us see the first part assuming that Φ takes values in S^2. An specialization ($q_1=q_2=s=2$) of the obvious left version of Proposition 1 shows that the map $\Phi^{(2)}: S_0^1\to S^1$ defined by $\Phi^{(2)}(f)=u|f|^{1/2}\Phi(|f|^{1/2})$ is a centralizer, hence a quasi-linear map. Since the trace is bounded and linear on S^1, the composition $\varphi=\operatorname{tr}\circ\Phi^{(2)}$ is quasi-linear, too.

In any case, we know from Corollary 1 that there is a centralizer $\Psi: S_0^2\to S^2$ that induces an extension equivalent to that induced by Φ. Hence (see Section 2.2) there exist a morphism $\alpha: S_0^2\to L(\mathcal{H})$ and a bounded homogeneous map $b: S_0^2\to S^2$ such that $\Phi=\Psi+\alpha+b$. We have

$$\varphi(f)=\operatorname{tr}\left(u|f|^{1/2}\Psi(|f|^{1/2})\right)+\operatorname{tr}\left(u|f|^{1/2}\alpha(|f|^{1/2})\right)+\operatorname{tr}\left(u|f|^{1/2}b(|f|^{1/2})\right).$$

We have just proved that the first summand in the right-hand side of the preceding equality is a quasi-linear function of f. The second one is linear since $u|f|^{1/2}\alpha(|f|^{1/2})=\alpha(u|f|^{1/2}|f|^{1/2})=\alpha(f)$. The third one is clearly bounded. Thus φ is itself quasi-linear.

As for the second one, let $\phi: S_0^1\to\mathbb{C}$ be quasi-linear. Consider the map $S_0^2\times S_0^2\to\mathbb{C}$ sending (f,g) into $\phi(fg)$. For fixed $g\in S_0^2$, the function $f\mapsto\phi(fg)$ is quasi-linear on S_0^2, with constant at most $\|g\|_2 Q[\phi]$. But S^2 is a $\mathcal{K}$-space and so there is a linear map $\ell_g: S_0^2\to\mathbb{C}$ (depending on g) such that

$$|\phi(fg)-\ell_g(f)|\leq K\|g\|_2 Q[\phi]\|f\|_2 \tag{26}$$

where $K\leq 37$ is the "$\mathcal{K}$-space constant" of S^2.

Next we want to see that $\ell(f)=\operatorname{tr}(L\circ f)=\operatorname{tr}(fL)$ for some $L\in L(\mathcal{H})$. According to Lemma 1(d) it suffices to check that for each fixed $y\in\mathcal{H}$ one has $\ell(x\otimes y)\to 0$ as $x\to 0$ in $\mathcal{H}$. As (26) must hold, it suffices to verify that for fixed $g\in S_0^2$ and $y\in\mathcal{H}$ one has

$$\phi((x\otimes y)g)\to 0 \qquad (x\to 0). \tag{27}$$

Write $g=\sum_{n=1}^m t_n x_n\otimes y_n$. Then

$$(x\otimes y)g=g^*(x)\otimes y=\sum_{n=1}^m t_n\langle x|y_n\rangle x_n\otimes y.$$

As ϕ is quasi-linear we have the estimate (see [**14**])

$$\left|\phi((x\otimes y)g)-\sum_{n=1}^m t_n\langle x|y_n\rangle\phi(x_n\otimes y)\right|\leq\sum_{n=1}^m |nt_n\langle x|y_n\rangle|\|x_n\|\|y\|$$

and (27) follows.

To sum up, there is homogeneous map $\Phi : S_0^2 \to L(\mathcal{H})$ such that

$$|\phi(fg) - \operatorname{tr}(f\Phi(g))| \leq M\|f\|_2\|g\|_2 \qquad (f, g \in S_0^2). \tag{28}$$

Clearly, $\phi \approx \varphi$, where φ is given by (25). It only remains to check that Φ is a centralizer. Take $g, f \in S_0^2, a \in B$. We have:

$$|\phi(f(ag)) - \operatorname{tr}(f\Phi(ag))| \leq M\|f\|_2\|ag\|_2$$
$$|\phi((fa)g) - \operatorname{tr}(fa\Phi(g))| \leq M\|fa\|_2\|g\|_2,$$

so

$$\|\Phi(ag) - a\Phi(g)\|_2 = \sup_{\|f\|_2 \leq 1} |\operatorname{tr}(f(\Phi(ag) - a\Phi(g))| \leq M\|a\|_B\|g\|_2$$

and we are done. □

7. Appendix: Bicentralizers

A bicentralizer is just a left centralizer which is also a right centralizer. This amounts to modifying Definition 1 by requiring Z, Y and $\tilde{Y}$ to be bimodules and replacing the estimate in (b) by

$$\|\Omega(afb) - a\Omega(f)b\|_Y \leq C_2\|a\|_A\|f\|_Z\|b\|_A \qquad (a, b \in A, f \in Z).$$

Bicentralizers on Schatten classes are the subject of [**16**] and [**17**]. It can be proved that every extension of quasi-Banach B-bimodules $0 \to S^q \to X \to S^p \to 0$ arises from a bicentralizer $\Omega : S_0^p \to S^q$ although we will refrain from entering into the details here. Let us draw some consequences of the results proved so far.

Theorem 6. *Let $\Omega : S_0^p \to S^q$ be a bicentralizer, with $p \neq q$. Then there is $t \in \mathbb{C}$ such that $\|\Omega(f) - tf\|_q \leq D\|f\|_p$ for some constant D independent on $f \in S_0^p$.*

Sketch of the proof. Case $q < p$. If $\Omega : S^p \to S^q$ is a bicentralizer, with q finite for which we may assume it preserves both left and right supports, then given a finite rank projection $e \in B$ we have that Ω maps eS^pe to eS^qe, as a bicentralizer over eBe. Proceeding as in Lemma 2 we see that the distance from Ω to the space of bimodule morphisms $S_0^p \to S^q$ equals $\sup_e \delta_e$, where δ_e is the distance from $\Phi : eS^pe \to eS^pe$ to the corresponding space of bimodule homomorphisms over the corner algebra eBe (they all given by multiplication by some constant) and e runs over all finite rank projections in B. After that one should consider the obvious version of Lemma 3 for bimodules using ultraproducts (instead of ultrapowers) of the families $(eS^pe)_e, (eS^qe)_e$ and the corresponding ultraproduct algebra $(eBe)_{\mathcal{U}}$. The remainder of the proof of Theorem 1 goes undisturbed to get the desired conclusion.

In case $q > p$, as Ω is a right-centralizer, we know from Theorem 4 that there is a quasi-linear map ϕ on $\mathcal{H}$ such that $\|\Omega(x \otimes y) - x \otimes \phi(y)\|_q \leq M\|x\|\|y\|$ for some M independent on $x, y \in \mathcal{H}$. But Ω is also a left centralizer and so $\|\Omega(a(x \otimes y)) - a\Omega(x \otimes y)\|_q \leq M\|x\|\|y\|$, which yields

$$\|x \otimes \phi(ay) - x \otimes a\phi(y)\|_q = \|x\|\|\phi(ay) - a\phi(y)\| \leq M\|a\|_B\|x\|\|y\| \qquad (a \in B, x, y \in \mathcal{H}).$$

As $\{ay : \|a\|_B \leq 1\}$ is the ball of radius $\|y\|$ in $\mathcal{H}$ we see that ϕ is bounded and so is Ω. □

As for "self-bicentralizers" on S^p, we have the following extension of a result by Kalton.

Theorem 7. *Let $\phi : \ell_0^p \to \ell^p$ be a symmetric centralizer over ℓ^∞, with $p \in (0, \infty)$. Define a self map on S_0^p as follows. Given $f \in S_0^p$ choose a Schmidt expansion $f = \sum_n s_n x_n \otimes y_n$. Let $(t_n) = \phi((s_n))$ and put $\Phi f = \sum_n t_n x_n \otimes y_n$. Then $\Phi : S_0^p \to S^p$ is a bicentralizer. Moreover, every bicentralizer is at finite distance from one obtained in this way.*

SKETCH OF THE PROOF. Symmetric means that there is a constant M such that $|\phi(f\circ\sigma)-\phi(f)\circ\sigma|_p\leq M|f|_p$ for every $f\in\ell_0^p$ whenever σ is a bijection of $\mathbb{N}$.

The proof is based on the following four facts:

(1) The statement holds for $p>1$ as proved by Kalton in [**17**, Theorem 8.3].
(2) Corollary 3 is true replacing right centralizer by bicentralizer everywhere.
(3) The commutative version of Corollary 3 holds: let $p,q,s\in(0,\infty)$ satisfy $p^{-1}=q^{-1}+s^{-1}$ and let $\omega:\ell_0^q\to\ell^q$ be a centralizer over ℓ^∞, where ℓ_0^q stands for the finitely supported sequences in ℓ^q. Define $\omega^{(s)}:\ell_0^p\to\ell^p$ taking $\omega^{(s)}(f)=\omega(u|f|^{p/q})|f|^{p/s}$, where u is the signum of f. Then $\omega^{(s)}$ is a centralizer and every centralizer on ℓ_0^p is at finite distance from one obtained in this way.
(4) Referring to the preceding statement, $\omega^{(s)}$ is symmetric if and only if ω is.

Now, let ϕ be a symmetric ℓ^∞-centralizer on ℓ_0^p, where $p\leq 1$. By (3) and (4), there is a symmetric centralizer ω on ℓ_0^2 such that $\phi\approx\omega^{(s)}$, where $p^{-1}=2^{-1}+s^{-1}$ and we may assume $\phi=\omega^{(s)}$. Applying (1) to ω we can extend it to a B-bicentralizer $\Omega:S_0^2\to S^2$ just taking

$$\Omega(f)=\sum_n t_n x_n\otimes y_n,$$

where $\sum_n s_n x_n\otimes y_n$ is the prescribed Schmidt expansion of f and $\omega((s_n))=(t_n)$. Finally, applying Corollary 3 to Ω with $q=2$ one realizes that $\Phi=\Omega^{(s)}$ from where it follows that Φ is a bicentralizer.

The "moreover" part follows from the case $p=2$ and the "moreover" part of Corollary 3. □

8. Concluding remarks

★ Most results in Sections 3 and 4 would generalize to noncommutative L^p spaces associated to arbitrary von Neumann algebras as long as one could find a good substitute for Lemma 2. More precisely, we ask if for every $\mathcal{M}$-centralizer $\Omega:L_0^p\to L^q$ with $0<q<p<\infty$ there is a system of trivial centralizers Ω_i such that $\operatorname{dist}(\Omega,\mathcal{M}_{\mathcal{M}}(L_0^p,L^q))=\sup_i\operatorname{dist}(\Omega_i,\mathcal{M}_{\mathcal{M}}(L_0^p,L^q))$. Here, $L_0^p=\{af^{1/p}:a\in\mathcal{M}\}$, where f is a normal, faithful state on $\mathcal{M}$.

★ The procedure described in Proposition 1 works as a tensor product. And indeed it is. It can be proved that if X_Ψ is the completion of $S^{q_2}\oplus_\Psi S_0^{q_1}$, then $X_{\Psi^{(s)}}$ represents the tensor product of X_Ψ and S^s in the category of quasi-Banach B-modules. This means that the bilinear operator $\theta:X_\Psi\times S^s\to X_{\Psi^{(s)}}$ defined by $\theta((g,f),h)=(gh,fh)$ has the following universal property: for every quasi-Banach space V and every bilinear operator $\beta:X_\Psi\times S^s\to V$ which is balanced in the sense of satisfying the identity $\beta(xa,h)=\beta(x,ah)$ for $a\in B,x\in X_\Psi,h\in S^s$, there is a unique linear operator $\lambda:X_{\Psi^{(s)}}\to V$ such that $\lambda(\theta(x,h))=\beta(x,h)$.

★ Concerning Theorem 5, nobody knows if K and B are $\mathcal{K}$-spaces or not. Kalton repeatedly conjectured an affirmative answer [**18**, Problem 4.2]. Also, it seems to be interesting to determine if $L^p(\mathcal{M})$ is a $\mathcal{K}$-space for $0<p<1$ if $\mathcal{M}$ is a von Neumann algebra with no minimal projection.

★ It is clear from the proof of Theorem 2 that every bicentralizer on S^p has trivial "spatial part". I don't know if the same is true for (one sided) centralizers. It can be proved that the maps $\tilde\phi$ defined in Theorem 16 are never self-centralizers on S_0^p.

★ Proposition 2 and Corollary 3 imply that if $\phi:\ell_0^p\to\ell^p$ is a (not necessarily symmetric) centralizer over ℓ^∞ and (e_n) is a fixed orthonormal basis in $\mathcal{H}$, then there is a left (or right, but not two-sided) centralizer Φ on S_0^p such that $\Phi(\sum_n s_ne_n\otimes e_n)=\sum_n t_ne_n\otimes e_n$, where $(t_n)=\phi((s_n))$.

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