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On strictly singular nonlinear centralizers

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ABSTRACT

We show that c_0 is the only Banach space with unconditional basis that satisfies the equation $\mathrm{Ext}(X, X) = 0$. This partially improves an old result by Kalton and Peck. We prove that the Kalton–Peck maps are strictly singular on a number of sequence spaces, including ℓ_p for $0 < p < \infty$, Tsirelson and Schlumprecht spaces and their duals, as well as certain super-reflexive variations of these spaces. In the last section, we give estimates of the projection constants of certain finite-dimensional twisted sums of Kalton–Peck type.

1. Introduction and preliminaries

1.1. Background

Exact sequences $0 \to Y \to Z \to X \to 0$ of Banach or quasi-Banach spaces for which the quotient map is a strictly singular operator are strange and somewhat mysterious objects. Let us call them *strictly singular* sequences. They have been explicitly considered in [1–3]. Other papers such as [4–6] use them to obtain nontrivial counterexamples to several (quasi-) Banach space questions. The results known so far can be summarized as follows.

- If X is a quasi-Banach space that admits a quotient map $q : \ell_p \to X$ and does not contain ℓ_p, there is a strictly singular exact sequence $0 \to \ker q \to \ell_p \to X \to 0$, where $0 < p < \infty$ (folklore).
- For every $1 \leq p < \infty$, there exists a strictly singular sequence $0 \to \ell_p \to Z_p \to \ell_p \to 0$ (the case $1 < p < \infty$ is given in [6, Theorem 6.4] while the case $p = 1$ appears in [2]). These are the so-called Kalton–Peck sequences that, in addition to solving the 3-space problem for ℓ_p-spaces, do that producing "extremal" counterexamples (see the last section of this paper).
- For every separable Banach space X not containing ℓ_1, there exists a strictly singular exact sequence $0 \to C[0, 1] \to Z_C \to X \to 0$. If, moreover, X has a shrinking unconditional finite-dimensional decomposition and contains no subspace which is isomorphic to the dual of a space with summable Szlenk index, then there exists a strictly singular sequence $0 \to C(\omega^\omega) \to Z_\omega \to X \to 0$ [4]. These sequences provide Banach spaces that cannot be renormed to be Lindenstrauss spaces but whose duals are isomorphic to $L^1(\mu)$ for some measure μ (ℓ_1 in the case of Z_ω).

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- There exists a strictly singular exact sequence $0 \to \mathbb{R} \to K \to \ell_1 \to 0$. The space K is a quasi-Banach space without basic sequences [5].
- If the density character of X is "much greater" than that of Y, then there are no strictly singular exact sequences $0 \to Y \to Z \to X \to 0$ [1,3].

It is however an open problem to determine when, given Banach or quasi-Banach spaces X and Y, there exists a strictly singular exact sequence $0 \to Y \to Z \to X \to 0$. In this paper, we address this problem when $Y = X$ has a unconditional basis. Our approach to the problem is to study exact sequences through an analysis of the associated quasi-linear map. Indeed, exact sequences $0 \to Y \to Z \to X \to 0$ of quasi-Banach spaces correspond (see Section 1.2) with quasi-linear maps $\Phi : X \to Y$. The most natural type of quasi-linear maps on quasi-Banach spaces with unconditional basis are the so-called centralizers; and the best known centralizers are the Kalton–Peck maps. Thus, this paper studies the nonlinear behavior of certain natural centralizers on a given quasi-Banach space and its infinite-dimensional subspaces.

Let us briefly indicate the main results and describe the organization of the paper. This section contains this introduction plus all the required preliminaires on exact sequences, quasi-linear maps, and centralizers. In Section 2, we prove that the Kalton–Peck map is nontrivial on every Banach sequence space different from c_0. This improves for Banach spaces a classical result by Kalton and Peck, and shows that $X = c_0$ is the only Banach space with unconditional basis for which every exact sequence $0 \to X \to Z \to X \to 0$ is trivial. Section 3 deals with strictly singular exact sequences. We exhibit here two results of independent interest: a blocking principle (Lemma 2) and a striking self-similarity property of the Kalton–Peck map (Lemma 3). The combination of both suggests that Kalton–Peck sequences should be strictly singular for all quasi-Banach sequence spaces not containing c_0. We show next that this is the case for ℓ_p with $p \in (0, \infty)$, as well as every *block-saturated* sequence space. This includes the Tsirelson space $\mathscr{T}$, its dual $\mathscr{T}^*$, and its p-convexifications $\mathscr{T}_p$ for $p \in [1, \infty)$, and all complementably minimal spaces with unconditional basis such as the Schlumprecht space $\mathscr{S}$ and the super-reflexive versions $\mathscr{S}_{p,r}$ for $1 < p < r < \infty$.

A further example is provided by L^p, for $p \in [2, \infty)$ only, when it is regarded as a sequence space by using the Haar system!

Finally, in Section 4, we compute the projection constant of ℓ_p^n inside the Kalton–Peck extensions for a wide class of Lipschitz functions.

1.2. Preliminaries

A short exact sequence of quasi-Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

formed by quasi-Banach spaces and linear continuous operators such that the image of each arrow coincides with the kernel of the next one. The middle space Z is also called a *twisted sum* of Y and X. When Y, X, Z are Banach spaces, we will say that the sequence is an exact sequence of Banach spaces. Observe that it is perfectly possible, and even interesting, that both Y and X are Banach spaces while Z is not (see [7,8]). It follows from the definition that j embeds Y as a subspace of Z and, thanks to the open mapping theorem, X is isomorphic to $Z/j(Y)$. The sequence is said to be trivial, or to split, if $j(Y)$ is complemented in Z, in which case Z is isomorphic to $Y \oplus X$. Two exact sequences $0 \to Y \to Z \to X \to 0$ and $0 \to Y \to Z_1 \to X \to 0$ are said to be *equivalent* if there exists an operator $T : Z \to Z_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Such a T is necessarily an isomorphism. We call $\mathrm{Ext}(X, Y)$ the set of exact sequences $0 \to Y \to Z \to X \to 0$ modulo the equivalence relation. The set $\mathrm{Ext}(X, Y)$ carries a natural linear structure for which (the class of all) trivial sequences correspond to zero (see [9], for instance). Thus, $\mathrm{Ext}(X, Y) = 0$ means "every exact sequence $0 \to Y \to Z \to X \to 0$ of quasi-Banach spaces splits".

Quasi-linear maps were introduced, in a rudimentary form, by Enflo et al. [10]. Then, Kalton [7] and Ribe [8] developed a rather satisfactory theory linking quasi-linear maps and twisted sums. We refer the reader to [11–13,6,14] for detailed expositions. Here, we only summarize the minimal background one needs to be operative. Let X and Y be quasi-Banach spaces and X_0 a dense subspace of X. A map $\Phi : X_0 \to Y$ is *quasi-linear* if it is homogeneous and there exists a constant K such that, for any $x, y \in X_0$, one has

$$\|\Phi(x + y) - \Phi(x) - \Phi(y)\|_Y \le K(\|x\|_X + \|y\|_X).$$

A quasi-linear map $\Phi : X_0 \to Y$ induces an exact sequence of quasi-Banach spaces $0 \to Y \to Y \oplus_\Phi X \to X \to 0$ in which $Y \oplus_\Phi X$ denotes the completion of the vector space $Y \times X_0$ with respect to the quasi-norm $\|(y, x)\|_\Phi = \|y - \Phi(x)\|_Y + \|x\|_X$. Indeed, the map $y \mapsto (y, 0)$ embeds Y isometrically into $Y \oplus_\Phi X$ and $(y, x) \mapsto x$ extends to a quotient map $q : Y \oplus_\Phi X \to X$ whose kernel is Y. Two quasi-linear maps $\Phi, \Psi : X_0 \to Y$ induce equivalent sequences if and only if the difference $\Psi - \Phi$ can be written as $B + L$, where $B : X_0 \to Y$ is a bounded map and $L : X_0 \to Y$ is linear. We then say that

Φ and Ψ are equivalent quasi-linear maps. A quasi-linear map Φ is said to be trivial if the induced sequence splits, which happens if and only if Φ is equivalent to the zero map.

By a quasi-Banach *sequence space* we mean a linear subspace X of $\mathbb{R}^{\mathbb{N}}$ together with a quasi-norm for which the unit vectors form a normalized 1-unconditional basis of X. Thus, a sequence space is just a quasi-Banach space with a distinguished 1-unconditional basis. From now on, we shall work exclusively with sequence spaces. We will write X_0 for the space of finitely supported sequences endowed with the restriction of the quasi-norm of X. As usual, ℓ_∞ denotes the algebra of all bounded sequences, equipped with the sup norm. Note that ℓ_∞ operates on every sequence space X through pointwise product; moreover, one has $\|ax\|_X \leq \|a\|_\infty \|x\|_X$.

Let X and Y be sequence spaces. A homogeneous map $\Phi : X_0 \to Y$ is said to be a centralizer if there is a constant C such that

$$\|\Phi(ax) - a\Phi(x)\|_Y \leq C\|a\|_\infty \|x\|_X \quad (a \in \ell_\infty, x \in X_0).$$

Centralizers are automatically quasi-linear (see [15, Lemma 4.2] or [16, Proposition 3.1]). The infimum of the constants C above will be called the centralizer constant $C(\Phi)$ of Φ. The most important centralizers are the so-called Kalton–Peck maps we define now. Let $\mathscr{L}$ denote the set of Lipschitz functions on $\mathbb{R}_+$ vanishing at zero. Given $\varphi \in \mathscr{L}$ and a sequence space X, put

$$\Omega_X^\varphi(x)(n) = x(n)\varphi\left(\log \frac{\|x\|}{|x(n)|}\right),$$

with the understanding that $\Omega_X^\varphi(x)(n) = 0$ if $x(n) = 0$. The map $\Omega_X^\varphi : X_0 \to X$ is a centralizer whose centralizer constant depends only on the Lipschitz constant of φ; see [15, Theorem 3.1], which is much stronger than [6, Theorem 3.5]. By $X(\varphi)$, we will denote the twisted sum of X with itself induced by Ω_X^φ. If φ is the identity on $\mathbb{R}_+$, we write Z_X instead of $X(\varphi)$. If, besides, $X = \ell_p$, we just write Z_p.

2. A homological characterization of c_0

Kalton and Peck proved in [6, Theorem 4.2(c)] that, if φ is unbounded on $\mathbb{R}_+$, then the corresponding quasi-linear map Ω^φ is nontrivial on each quasi-Banach sequence space whose canonical basis has no subsequence equivalent to the canonical basis of c_0. In this section, we prove that c_0 is the only *Banach* space where Ω^φ is trivial. The proof is based on the following result of independent interest.

Lemma 1. *Let X be a quasi-Banach sequence space, Y a Banach sequence space, and $\Phi : X_0 \to Y$ a centralizer. If Φ is trivial, then there exist a sequence $\lambda \in \mathbb{R}^{\mathbb{N}}$ and a constant $M > 0$ such that $\|\Phi(x) - \lambda x\|_Y \leq M\|x\|_X$ for all $x \in X_0$.*

Proof. Let $\Phi : X_0 \to Y$ be any centralizer. Suppose that Φ is trivial; i.e., there is a linear map $L : X_0 \to Y$ with

$$\|\Phi(x) - L(x)\|_Y \leq M\|x\|_X \tag{1}$$

for some M independent on $x \in X_0$. Let $U = \{\pm 1\}^{\mathbb{N}}$ be the unitary group of ℓ^∞, which we furnish with the product topology. Then U is compact and abelian, and we can consider the Haar measure (probability) m on U. This measure allows one to define for each continuous function $f : U \to Y$ the Bochner integral

$$\int_U f(u)dm(u) \in Y.$$

Given $u \in U$ and $x \in X_0$, one has

$$\begin{aligned} \|\Phi(x) - u^{-1}L(ux)\| &= \|u\Phi(x) - L(ux)\| \\ &\leq \|u\Phi(x) - \Phi(ux)\| + \|\Phi(ux)L(ux)\| \\ &\leq (C(\Phi) + M)\|x\|. \end{aligned}$$

The function $u \in U \mapsto u^{-1}L(ux) \in Y$ is continuous, since the orbit $U(x) = \{ux : u \in U\}$ is finite for every $x \in X_0$. We define a linear map $\ell : X_0 \to Y$ by

$$\ell(x) = \int_U u^{-1}L(ux)dm(u).$$

It then follows from the convexity of the norm of Y and the preceding inequality that

$$\|\Phi(x) - \ell(x)\| \leq (C(\Phi) + M)\|x\| \quad (x \in X_0).$$

Moreover, given $v \in U$ and $x \in X_0$, one has

$$\ell(vx) = \int_U u^{-1}L(uvx)dm(u) = \int_U v(uv)^{-1}L(uvx)dm(u) = v\int_U u^{-1}L(ux)dm(u) = v\ell(x)$$

by the invariance of m. Now, for fixed n, write $e_n = \frac{1}{2}(u+v)$, with $u, v \in U$ (e.g., $u = 1$ and $v = -1 + 2e_n$). We have

$$\ell(e_n) = \ell\left(\frac{u+v}{2} e_n\right) = \frac{u+v}{2}\ell(e_n) = e_n \ell(e_n).$$

It follows that $\ell(e_n) = \lambda_n e_n$ for some $\lambda_n \in \mathbb{R}$, and this completes the proof. $\square$

The following result provides a very simple criterion to check the triviality of centralizers on Banach sequence spaces.

Corollary 1. *Let X be a Banach sequence space and $\Phi : X_0 \to X$ a centralizer such that $\Phi(e_n) = 0$ for every n. Then Φ is a trivial quasi-linear map if and only if it is bounded in the sense of satisfying an estimate of the form $\|\Phi(x)\|_X \le M\|x\|_X$ for some M independent on $x \in X_0$.*

Proof. The "if" part is obvious. As for the other implication, if Φ is a trivial quasi-linear map, then according to Lemma 1 there is a constant M and there is a sequence λ such that $\|\Phi(x) - L(x)\| \le M\|x\|$. Taking $x = e_n$, we see that λ is bounded, with $\|\lambda\|_\infty \le M$. By the triangle inequality, we have $\|\Phi(x)\| \le 2M\|x\|_X$. $\square$

We hasten to remark that the hypothesis is not restrictive: every centralizer admits an equivalent centralizer vanishing on the unit basis, as we prove next. Observe first that, if Φ is a centralizer, then we can assume that $\operatorname{supp}\Phi(f) \subset \operatorname{supp} f$ for every $f \in X$; indeed, if u is the characteristic function of $\operatorname{supp} f$, then $\|\Phi(f) - u\Phi(f)\|_X \le C(\Phi)\|f\|_X$, and we may consider $f \mapsto u\Phi(f)$ instead of Φ. Then, assuming that property, $\Phi(e_n) = \lambda_n e_n$, and the map $x \mapsto \Phi(x) - \lambda x$ is equivalent to Φ and vanishes on each e_n.

Theorem 1. *Let X be a Banach space with unconditional basis. Then* $\operatorname{Ext}(X, X) = 0$ *if and only if X is isomorphic to c_0.*

Proof. The "if" part is as follows. Kalton and Roberts proved in [17, Theorem 6.3] that, if $0 \to Y \to Z \to c_0 \to 0$ is an exact sequence of quasi-Banach spaces and Y is a Banach space, then the middle space E is isomorphic a Banach space. If, moreover, $Y = c_0$, then Z is separable, and thus, by Sobczyk's theorem [18], there must be a bounded projection of Z onto Y. Hence $\operatorname{Ext}(c_0, c_0) = 0$.

As for the "only if", let $\varphi \in \mathscr{L}$ be unbounded, and let X be a Banach sequence space different from c_0. Then, if $s_n = \sum_{i=1}^n e_i$, one has $\|s_n\| \to \infty$, and since $\|s_n\| \le \|s_{n+1}\| \le 1 + \|s_n\|$ and φ is Lipschitz we see that the sequence of real numbers $\varphi(\log\|s_n\|)$ is not bounded. But $\Omega_X^\varphi(s_n) = \varphi(\log\|s_n\|)s_n$; hence Ω_X^φ is not bounded on X_0 and, by Corollary 1, the sequence $0 \to X \to X(\varphi) \to X \to 0$ is not trivial. Hence $\operatorname{Ext}(X, X) \ne 0$. $\square$

Let us record the following fact included in the previous proof.

Corollary 2. *If $\varphi \in \mathscr{L}$ is unbounded, then Ω_X^φ is nontrivial on every Banach sequence space X different from c_0.*

3. Some strictly singular centralizers

The following "blocking principle" will be useful to check the strict singularity of certain extensions. Let X be a sequence space. A block-sequence is a (normalized) sequence (u_n) in X_0 such that $\max\operatorname{supp} u_n < \min\operatorname{supp} u_{n+1}$ for each $n \in \mathbb{N}$. A subspace of X which is spanned by a block-sequence is often called a block-subspace.

Lemma 2. *Let $q : Z \to X$ be a quotient map, where Z is a quasi-Banach space and X a sequence space. Then either q is strictly singular or it is invertible on some infinite-dimensional block-subspace of X.*

Proof. There is no loss of generality in assuming that Z is p-normed for some $p \in (0, 1]$ and q is an isometric quotient, so that X is also p-normed. Suppose that q is not strictly singular. Then there is an infinite-dimensional subspace H of X and there is an operator $s : H \to Z$ such that $q \circ s = \mathrm{Id}_H$. A standard procedure allows one to find a normalized sequence (h_n) in H with $h_n(i) = 0$ for $i < n$; see [19, Proof of Corollary 5.3]. Then we fix $\varepsilon \in (0, \frac{1}{2})$ and apply the gliding hump argument to get a subsequence (that we do not relabel) and a block-sequence (u_n) in X such that $\|h_n - u_n\| < \varepsilon^n$.

Let U denote the subspace of X spanned by (u_n). We are about to see that the operator $v : U \to H$ defined by $v(u_n) = h_n$ is a bounded isomorphism. It suffices to check that, given a bounded sequence (t_n), the series $\sum_n t_n h_n$ converges if and only if so does the series $\sum_n t_n u_n$. This is due to the fact that

$$\sum_n t_n h_n - \sum_n t_n u_n = \sum_n t_n(h_n - u_n),$$

and the last series converges since

$$\sum_n \|t_n(h_n - u_n)\|_X^p \le \sum_n |t_n|^p \|h_n - u_n\|_X^p \le \|(t_n)\|_\infty^p \frac{\varepsilon^p}{1 - \varepsilon^p}.$$

Next, we construct an operator $S : U \to Z$ such that $q \circ S = \mathrm{Id}_U$. For each n, we choose z_n in Z in such a way that $q(z_n) = u_n$ and $\|z_n - s(h_n)\| < \varepsilon^n$, and we define $S : U_0 \to Z$ as follows. Given $u = \sum_n t_n u_n$, we put $S(u) = \sum_n t_n z_n$. To compute $\|S(u)\|$, take $h = \sum_n t_n u_n = v(u)$ to obtain

$$\begin{aligned}\|Su\|^p &\le \|s(h)\|^p + \|S(u) - s(h)\|^p \\ &\le \|s\|^p \|h\|^p + \left\| \sum_n t_n (z_n - s(h_n)) \right\| \\ &\le \|s\|^p \|v\|^p \|u\|^p + \sum_n |t_n|^p \varepsilon^{pn} \\ &\le \left(\|s\|^p \|v\|^p + \frac{\varepsilon^p}{1-\varepsilon^p} \right) \|u\|^p. \quad \square\end{aligned}$$

The preceding Lemma 2 applies to quasi-linear maps as follows. Let X be a sequence space and $\Phi : X_0 \to Y$ a quasi-linear map, where Y is any quasi-Banach space. Then either the quotient map $q : Y \oplus_\Phi X \to X$ is strictly singular or there is a block-sequence (u_n) such that the restriction of Φ to U_0 is trivial; that is, there is a linear map $L : U_0 \to Y$ and there is a constant K so that $\|\Phi(u) - L(u)\|_Y \le K\|u\|_X$ for $u \in U_0$.

Let X be a sequence space, and let $\Omega_X : X_0 \to X$ be the corresponding Kalton–Peck map. If (u_n) is a normalized block sequence in X, its closed linear span U is also a sequence space with basis (u_n), and we can consider its own Kalton–Peck map $\Omega_U : U_0 \to U$, which is defined on $u = \sum_n \lambda_n u_n$ by

$$\Omega_U(u) = \sum_n \lambda_n \log\left(\frac{\|u\|}{|\lambda_n|}\right) u_n.$$

The next result reveals a surprising "self-similarity" property of Ω.

Lemma 3. *There is a linear map $L : U_0 \to X$, depending on U, such that $\Omega_X = \Omega_U + L$ on U_0.*

Proof. We define a linear map $L : U_0 \to X$ by $L(u_n) = -u_n \log|u_n|$ and linearly on the rest. Let $u = \sum_n \lambda_n u_n$ be normalized in U_0. We have

$$\begin{aligned}-\Omega_X(u) &= u \log|u| = \sum_n \lambda_n u_n \log(|\lambda_n||u_n|) \\ &= \left(\sum_n \lambda_n u_n \log|\lambda_n| \right) + \left(\sum_n \lambda_n u_n \log|u_n| \right) \\ &= -\Omega_U(u) - L(u),\end{aligned}$$

and the result follows from homogeneity. $\square$

Thus, whenever U is a block-subspace of X, one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z_X & \longrightarrow & X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & U & \longrightarrow & Z_U & \longrightarrow & U & \longrightarrow & 0. \end{array}$$

Here, the arrows $U \to X$ are plain inclusions, while $Z_U \to Z_X$ is given by $(v, u) \mapsto (v + L(u), u)$ for $u \in U_0$. This operator is in fact an into isometry:

$$\|(v + Lu, u)\|_{Z_X} = \|y + Lu - \Omega_X(u)\| + \|u\| = \|y - \Omega_U(u)\| + \|u\| = \|(v, u)\|_{Z_U}.$$

We give one more definition and then we are ready for the main result of the paper. A sequence space X will be called *complementably block-saturated* if every infinite-dimensional subspace contains a block-subspace complemented in X. All ℓ_p spaces, $1 \le p < \infty$, are complementably block-saturated, as well as the vector sums $\ell_p(\ell_q)$ for $p, q \in [1, \infty)$, and so are the Tsirelson space $\mathscr{T}$ and its dual $\mathscr{T}^*$ [20, Corollary 9 and Remark 3 in p. 93]. (In all these examples one actually has that every block-subspace is complemented.) The product of two complementably block-saturated spaces is also complementably block-saturated.

When the complemented block-subspace can be chosen isomorphic to the whole space X then it is said that X is *complementably minimal*. (This is not the original definition, but an equivalent condition for sequence spaces.) The list of complementably minimal spaces is shorter: ℓ_p spaces for $1 \le p < \infty$, c_0, the Schlumprecht space $\mathscr{S}$ [21,22] and its dual [23]; the super-reflexive variations $\mathscr{S}_{p,r}$ for $1 \le p < r \le \infty$ constructed in [23] and their duals $\mathscr{S}^*_{p,r}$ [23]; and the p-convexified Tsirelson space $\mathscr{T}_p$ for $1 < p < \infty$ [24]. The Tsirelson space and its dual are not complementably minimal [24, pp. 54–59]. One has.

Theorem 2. *The exact sequence* $0 \to X \to Z_X \to X \to 0$ *induced by the Kalton–Peck map* $\Omega(x) = x\log(\|x\|_X/|x|)$ *is strictly singular for the following sequence spaces* X.:

(a) *All complementably block-saturated sequence spaces not containing* c_0.
(b) ℓ_p *for* $0 < p < 1$.
(c) L_p *for* $p \in [2, \infty)$ *when regarded as a sequence space through the Haar system.*

Proof. (a) Assuming the contrary, we know from Lemma 2 that Ω_X must be trivial on some block-subspace U that we may assume complemented in X by a bounded projection $P : X \to U$. Hence there is a linear map $\ell : U_0 \to X$ with $\Omega_X - \ell$ bounded on Y. Let $L : U_0 \to X$ be the linear map given by Lemma 3; i.e., $\Omega_X = \Omega_Y + L$. Since $\Omega_Y + L - \ell$ is bounded, so is the composition $P \circ (\Omega_Y + L - \ell) = \Omega_Y + P \circ (L - \ell)$. Since $P \circ (L - \ell) : U \to X$ is linear, this means that $\Omega_Y : Y \to Y$ is trivial, in contradiction with Corollary 2.

(b) The spaces ℓ_p for $p \in (0, 1)$ are not complementably minimal; see [25]. However, block-sequences are again equivalent to the unit basis of ℓ^p, and if U is a block-subspace of ℓ^p, then $\ell^p \oplus_{\Omega_p} U$ cannot be isomorphic to the direct sum $\ell^p \oplus U$ since it contains a subspace isomorphic with Z_p, which is not a p-Banach space. A similar argument could have worked also for $p > 1$ except for the fact that in this case it is not as easy to check that Z_p is not a subspace of ℓ_p.

(c) The key point is that, for $p \in [2, \infty)$, every normalized weakly null sequence in L_p contains a subsequence equivalent to the unit basis of ℓ_r, where r is either p or 2 — a classical result by Kadets and Pełczyński [26]. Hence it suffices to check that $\Omega : U_0 \to L^p$ is not trivial when U is a block subspace spanned by a sequence equivalent to the unit basis of ℓ^r, where r is either p or 2. Thus, $L^p \oplus_\Omega U$ has a subspace isomorphic to Z_r, where r is either p or 2, which cannot be embedded into L^p: Z_2 does not embed into L^p by Kalton and Peck [6, Corollary 6.8], nor does Z_p, since it fails to have cotype p. □

Remark 1. It is shown in [27] that the twisted sum of L^p induced by the map $\Phi(f) = f\log(|f|/\|f\|)$ is not strictly singular for any $0 < p < \infty$. It is important to realize that this map has very little to do with the centralizer appearing in the third part of Theorem 2. Indeed, if $f = \sum_i x_i h_i$ is a linear combination of finitely many functions of the Haar system, then

$$\Omega(f) = \sum_i x_i \log\left(\frac{\|f\|_p}{|x_i|}\right) h_i.$$

Of course, Φ and Ω are not equivalent quasi-linear maps.

4. Finite-dimensional projection constants

Let Y be a subspace of X. The projection constant of Y in X is defined as

$$\lambda(Y, X) = \inf\{\|P\| : P \text{ is a projection on } X \text{ whose range is } Y\}.$$

Our aim in this section is to estimate the projection constant of ℓ_p^n in $\ell_p^n(\varphi)$, the twisted sum induced by the Kalton–Peck map Ω^φ on ℓ_p^n, where φ is a Lipschitz function.

4.1. Lower bounds

The lower bound for $\lambda(\ell_p^n, \ell_p^n(\varphi))$ was obtained in [6, Theorem 6.3] for $p > 1$ and φ the identity on $\mathbb{R}^+$. The following result yields the corresponding bounds for every $0 < p < \infty$ and every increasing φ. Curiously enough, for $p \le 1$, we can use the same idea that Kalton and Peck used for the case $1 < p \le 2$, while the case $p \ge 1$ requires now an averaging technique similar to that of the proof of Lemma 1.

Proposition 1. *For every* $\varphi \in \mathscr{L}$, *one has*

$$\lambda(\ell_p^n, \ell_p^n(\varphi)) \ge \begin{cases} 2^{-1/p}\left|\varphi\left(\dfrac{1}{p}\log n\right)\right| & \text{for } 0 < p \le 1, \\ \dfrac{1}{2}\left|\varphi\left(\dfrac{1}{p}\log n\right)\right| & \text{for } 1 \le p < \infty. \end{cases}$$

Proof. *Case* $p \ge 1$. Let P be a projection of $\ell_p^n(\varphi)$ onto its subspace ℓ_p^n. Let G be the group witnessing the symmetry of the basis of ℓ_p^n. Each element of G is an isometry of ℓ_p^n of the form $g(x) = u(x \circ \sigma)$, where σ is a permutation of $\{1, 2, \ldots, n\}$ and u is unitary. Note that G also acts on $\ell_p^n(\varphi)$ by the rule

$$g(y, x) = (gy, gx).$$

This action is isometric since G and Ω^φ commute in the sense that $\Omega^\varphi(g(x)) = g\Omega^\varphi(x)$. Thus, as we did in the proof of Lemma 1, we can average P to get another projection

$$\pi(y,x) = \int_G g^{-1}P(g(x,y))dm(g).$$

(Here dm is again Haar measure, but as G is finite this is a counting measure divided by the order of G.) It is evident that π is a projection of $\ell_p^n(\varphi)$ onto its subspace ℓ_p^n and also that $\|\pi\| \le \|P\|$ by convexity of the norm in ℓ_p^n. Moreover, one has $\pi(g(y,x)) = g\pi(y,x)$ for every $g \in G$, $y, x \in \ell_p^n$. It is really easy to check that π must then have the form

$$\pi(y,x) = y - cx$$

for some $c \in \mathbb{R}$. Now, $\pi(0,e_i) = -ce_i$; hence $\|\pi\| \ge |c|$, while $\pi(\Omega^p(s_n), s_n) = s_n\varphi(\log n^{1/p}) - cs_n$, whence $\|\pi\| \ge |\varphi(\frac{1}{p}\log n) - c|$, and so $\|P\| \ge \frac{1}{2}\varphi(\frac{1}{p}\log n)$.

Case $p \le 1$. Let P be a projection of $\ell_p^n(\varphi)$ onto its subspace ℓ_p^n. Clearly, P has the form $P(y,x) = y - L(x)$, where $L : \ell_p^n \to \ell_p^n$ is linear. Moreover,

$$\|y - L(x)\|_p \le \|P\|(\|(y,x)\|_{\Omega^\varphi}) = \|P\|(\|y - \Omega^\varphi(x)\|_Y + \|x\|_X).$$

Taking $y = \Omega^\varphi(x)$, we see that L approximates Ω^φ in the sense that $\|\Omega^\varphi(x) - L(x)\|_p \le \|P\|\,\|x\|_p$. As $\Omega^\varphi(e_i) = 0$ for every i, we see that $\|L(e_i)\| \le \|P\|$, from which it follows (recall that $p \le 1$) that $\|L\| \le \|P\|$.

Take $s_n = \sum_{i\le n} e_i$. Since $\|s_n\|_p = n^{1/p}$, we have $\|\Omega^\varphi(s_n) - L(s_n)\|^p \le \|P\|^p n$. On the other hand, recalling again that $p \le 1$,

$$\|\Omega^\varphi(s_n) - L(s_n)\|_p^p \ge \|\Omega^\varphi(s_n)\|_p^p - \|L(s_n)\|_p^p \ge \|s_n\|^p|\varphi(\log(n^{1/p}))|^p - \|P\|^p\|s_n\|^p.$$

Therefore, $2\|P\|^p \ge |\varphi(\frac{1}{p}\log n)|^p$; that is, $\|P\| \ge 2^{-1/p}|\varphi(\frac{1}{p}\log n)|$. □

4.2. *Upper bounds*

Kalton and Peck asked in [6, Remark (2) on p. 26] whether $\lambda(\ell_p^n, X) = O(\log n)$ if X is a twisted sum of ℓ_p^n with itself. This was affirmatively solved for $p = 2$ in [28]; the case $p \ne 2$ seems to be open. It can be proved that the answer is affirmative for twisted sums induced by centralizers with bounded centralizer constants (roughly this means that the corresponding sequences live in the category of ℓ_∞^n-modules). We will not pursue this point here.

Anyway these "universal" bounds can be considerably improved for a wide class of Kalton–Peck centralizers, as we are about to see. We need the following lemma.

Lemma 4. *Suppose $\varphi \in \mathscr{L}$ to be differentiable and strictly increasing, with nonincreasing derivative. Then*

$$\max\{\|\Omega_p^\varphi(x)\|_p : \|x\|_p = 1\} = \varphi(p^{-1}\log n),$$

where $0 < p < \infty$.

Proof. We first prove that, for each fixed p, the function $s \mapsto \varphi(s)^p - \varphi(s)^{p-1}\varphi'(s)$ is injective on $\mathbb{R}^+$. Suppose on the contrary there exist $t < s$ such that

$$\varphi(t)^p - \varphi(t)^{p-1}\varphi'(t) = \varphi(s)^p - \varphi(s)^{p-1}\varphi'(s). \qquad (2)$$

Now, the argument depends on whether $p \ge 1$ or $p < 1$. First, suppose that $p \ge 1$. We have $\varphi(t)^{p-1}(\varphi(t) - \varphi'(t)) = \varphi(s)^{p-1}(\varphi(s) - \varphi'(s))$, and so

$$\frac{\varphi(t)^{p-1}}{\varphi(s)^{p-1}}(\varphi(t) - \varphi'(t)) = \varphi(s) - \varphi'(s).$$

As $p \ge 1$, $t < s$, and φ is increasing, we have $\varphi(t)^{p-1}/\varphi(s)^{p-1} \le 1$ (with equality if $p = 1$), and we find that $\varphi(t) - \varphi'(t) > \varphi(t) - \varphi'(s)$, which yields $\varphi'(s) > \varphi'(t)$, an absurdity.

As for $p < 1$, if we assume (2) with $t < s$ we have $\varphi(t)^{p-1}\varphi'(t) < \varphi(s)^{p-1}\varphi'(s)$; that is,

$$\left(\frac{\varphi(t)}{\varphi(s)}\right)^{p-1} < \frac{\varphi'(s)}{\varphi'(t)}.$$

But the left-hand side of the preceding inequality must be greater than 1, and so we would have $\varphi'(t) < \varphi'(s)$, which is not the case.

We are now ready to complete the proof. It is clear that we may compute the maximum assuming $x_k \geq 0$ for $1 \leq k \leq n$, and an obvious induction argument shows that we may restrict attention to those points whose coordinates are all nonzero. Thus we must maximize

$$\sum_{k=1}^{n} x_k^p \varphi(-\log x_k)^p, \quad \text{subject to } \sum_{k=1}^{n} x_k^p = 1 \text{ and } 0 < x_k < 1 \text{ for all } k.$$

We use Lagrange's multiplier theorem. Write

$$\Lambda(x, \lambda) = \sum_{k=1}^{n} x_k^p \varphi(-\log x_k)^p + \lambda \left(1 - \sum_{k=1}^{n} x_k^p\right).$$

From $\partial \Lambda / \partial x_k = 0$, we get $\varphi(-\log x_k)^p - \varphi(-\log x_k)^{p-1}\varphi'(-\log x_k) = \lambda$, and, using the injectivity of the function

$$s \mapsto \varphi(-\log s)^p - \varphi(-\log s)^{p-1}\varphi'(-\log s),$$

we see that x_k does not depend on k, and then it must be the case that $x_k = n^{-1/p}$ for $1 \leq k \leq n$. A simple evaluation then gives the result. □

We observe that the preceding lemma applies to every increasing Lipschitz concave (differentiable) function. Important examples are the "powers" $s \mapsto (s+1)^\alpha - 1$ for $\alpha \in (0, 1]$; see [6, Corollary 5.5]. From this we get the following.

Proposition 2. *Let $\varphi \in \mathscr{L}$ be differentiable and strictly increasing, with nonincreasing derivative, and let $0 < p < \infty$. Then*

$$\lambda(\ell_p^n, \ell_p^n(\varphi)) \leq c_p(2 + \varphi(p^{-1}\log n)),$$

where c_p is the modulus of concavity of ℓ_p; that is, $c_p = 2^{1/p-1}$ for $p \in (0, 1)$ and $c_p = 1$ for $p \in [1, \infty)$.

Proof. Obviously, the map $s : \ell_p^n \to \ell_p^n(\varphi)$ given by $s(x) = (0, x)$ is a linear selector for the quotient map $q : \ell_p^n(\varphi) \to \ell_p^n$. The norm of s is the smallest constant C for which the following holds for all x

$$\|(0, x)\|_{\Omega^\varphi} = \|\Omega^\varphi(x)\|_p + \|x\|_p \leq C\|x\|_p.$$

Hence,

$$\|s\| = 1 + \sup_{\|x\|=1} \|\Omega^\varphi(x)\|_p = 1 + \varphi(p^{-1}\log n).$$

For the projection $P = \mathrm{Id}_{\ell_p^n(\varphi)} - sq$, we have $\|P\| \leq c_p\,(1 + \|s\|) = c_p(2 + \varphi(p^{-1}\log n))$. □

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