Revised: May 11, 2012

THERE IS NO STRICTLY SINGULAR CENTRALIZER ON L_p

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Abstract. We prove that if Φ is a centralizer on L_p, where $0 < p < \infty$, then there is a copy of ℓ_2 inside L_p where Φ is bounded. If Φ is symmetric then it is also bounded on a copy of ℓ_q, provided $0 < p < q < 2$. This shows that for a wide class of exact sequences $0 \to L_p \to Z \to L_p \to 0$ the quotient map is not strictly singular and generalizes a recent result of Jesús Suárez.

1. Introduction

An operator acting between Banach or quasi-Banach spaces is said to be strictly singular if it is not an isomorphism on any infinite dimensional subspace of its domain.

Exact sequences of Banach or quasi-Banach spaces $0 \to Y \to Z \to X \to 0$ in which the quotient map $\pi : Z \to X$ is strictly singular spurred a moderate interest since the early studies on the 'three space problem'. Let us call them 'strictly singular sequences'. In some sense, if one has an strictly singular sequence in which the spaces X and Y are 'nice', the middle space Z must be 'exotic'.

Amongst the most striking examples of this phenomenon one finds that for each $p \in (0, \infty)$ there is a strictly singular sequence

$$0 \longrightarrow \ell_p \longrightarrow Z_p \longrightarrow \ell_p \longrightarrow 0 \tag{1}$$

These were constructed by Kalton and Peck in [9]; see also [3].

More often than not the construction of strictly singular sequences is achieved by means of a quasilinear map from X to Y and this is certainly the case for the Kalton-Peck sequences, whose associated quasilinear maps are centralizers (a special type of quasilinear map; see Section 1.2). There is a function space analogue of (1)

$$0 \longrightarrow L_p \longrightarrow ZF_p \longrightarrow L_p \longrightarrow 0 \tag{2}$$

whose associated quasilinear map is the 'classical' centralizer

$$\Omega(f) = f \log\left(\frac{|f|}{\|f\|}\right).$$

The space ZF_p was introduced in [5], although it arises quite naturally in interpolation theory; see [10, Section 3D].

Very recently Jesús Suárez has proved the following remarkable results on the behaviour of Ω on L_p:

2000 Mathematics Subject Classification. 46B20, 46B07, 46A16.

Key words and phrases: Strictly singular operator, quasilinear centralizer, Rademacher functions, q-stable random variable.

Supported in part by MTM2010-20190-C02-01 and Junta de Extremadura CR10113 "IV Plan Regional I+D+i, Ayudas a Grupos de Investigación".

(a) For every $0 < p < \infty$, there is a copy of ℓ_2 in L_p where the restriction of Ω is bounded.
(b) If $0 < p < q < 2$, then Ω is bounded on a copy ℓ_q inside L_p.

See [12, Propositions 3.1 and 4.1]. Roughly this means that the sequence (2) is not strictly singular because the quotient map is invertible on an isomorphic copy of ℓ_2 (or ℓ_q) inside L_p.

The purpose of this short note is to prove that (a) holds for all centralizers and (b) holds (at least) for symmetric centralizers. Our approach is based on a result by Kalton that describes centralizers as differentials of interpolation scales of Köthe function spaces from [7]. We also use results from [6] and a recent result of the author on the behaviour of centralizers acting between two different Lebesgue spaces [2].

1.1. **Function spaces.** Let L_0 denote the space of all real or complex measurable functions on the unit interval $\mathbb{I}$, where we identify two functions if they agree almost everywhere with respect to Lebesgue measure. A function space X is a linear subspace of L_0, together with a quasi-norm $\|\cdot\|$ having the following properties:

- The unit ball $B_X = \{f \in X : \|f\| \leq 1\}$ is closed in L_0 for the topology of convergence in measure.
- If $f \in X, g \in L_0$ and $|g| \leq |f|$, then $g \in X$ and $\|g\| \leq \|f\|$.

Important examples of function spaces are the spaces L_p for $0 < p \leq \infty$.

Given a function space X and $A \subset \mathbb{I}$ we write $X(A)$ for the space of those functions in X vanishing outside A.

We consider Köthe function spaces in the sense of [7]. Thus they are Banach function spaces whose norm satisfies the inequalities $\|hx\|_1 \leq \|x\|_X \leq \|kx\|_\infty$ for some everywhere positive functions h, k and for every $x \in X$.

1.2. **Centralizers and extensions.** Let X and Y be function spaces. A centralizer from X to Y is a homogeneous mapping $\Phi : X \to L_0$ satisfying the following condition: there is a constant C such that, for every $a \in L_\infty$ and for every $f \in X$ the difference $\Phi(af) - a\Phi(f)$ belongs to Y and

$$\|\Phi(af) - a\Phi(f)\|_Y \leq C\|a\|_\infty \|f\|_X.$$

When $Y = X$ we say that Φ is a centralizer on X.

Although we will not use it, we remark that every centralizer is quasilinear, that is, there is a constant Q such that for every $f, g \in X$ the difference $\Phi(f+g) - \Phi f - \Phi g$ falls in Y and one has $\|\Phi(f+g) - \Phi f - \Phi g\|_Y \leq Q(\|f\|_X + \|g\|_X)$.

A centralizer from X to Y gives rise to an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y \oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0$$

as follows:

- The middle space is $Y \oplus_\Phi X = \{(g, f) \in L_0 \times X : g - \Phi(f) \in Y\}$ with the quasi-norm given by $\|(g, f)\|_\Phi = \|g - \Phi f\|_Y + \|f\|_X$.
- $\imath(g) = (g, 0)$ and $\pi(g, f) = f$.

Actually only quasilinearity of Φ is required here.

We say that two centralizers Φ and Ψ are equivalent, and we write $\Phi \approx \Psi$, if the difference takes values in Y and is bounded in the sense that $\|\Phi(f) - \Psi(f)\|_Y \leq B\|f\|_X$ for some B and every $f \in X$.

Let U be a subspace of X and suppose Φ is bounded on U in the sense that Φ maps U into Y (not L_0) and $\|\Phi(u)\|_Y \leq B\|u\|_X$ for some constant B and every $u \in U$. Then the map $s : U \to Y \oplus_\Phi X$ defined by $s(u) = (0, u)$ is a bounded linear operator and $\pi \circ s = I_U$. Thus π cannot be strictly singular if Φ is bounded on some infinite-dimensional subspace of X.

Important examples of centralizers are the following (see [6], Section 3 and specially Theorem 3.1). Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function. Then the map $L_p \to L_0$ given by

$$f \longmapsto f\varphi\left(\log\frac{|f|}{\|f\|_p}, \log\frac{|r_f|}{\|f\|_p}\right).$$

is a (real, symmetric) centralizer on L_p. Here r_g is the so called rank-function of $g \in L_0$ defined by

$$r_g(t) = \lambda\{s \in \mathbb{R}^+ : |g(s)| > |g(t)| \text{ or } s \leq t \text{ and } |g(s)| = |g(t)|\},$$

which arises in real interpolation.

1.3. **Real centralizers.** Let X be a complex function space and let $X^{\mathbb{R}} = \Re(X)$ be corresponding real function space. A centralizer on X is said to be real if it sends real functions into real functions. Clearly, every real centralizer on X induces a centralizer on $X^{\mathbb{R}}$ by restriction. On the other hand each centralizer Φ on $X^{\mathbb{R}}$ extends to a real centralizer on X by the formula $\Phi^{\mathbb{C}} f = \Phi(u) + i\Phi(v)$, where $u = \Re f$ and $v = \Im f$. These processes are each inverse of the other, up to equivalence.

Moreover, if Φ is any centralizer on X there are real centralizers Φ_1 and Φ_2 such that $\Phi \approx \Phi_1 + i\Phi_2$; see [7, Lemma 7.1].

2. Results

Let A be a Borel subset of $\mathbb{I}$. A Rademacher sequence in A is a sequence (r_n) in $L_0(A)$ such that $\lambda(\{t \in A : r_n(t) = 1\}) = \lambda(\{t \in A : r_n(t) = -1\}) = \frac{1}{2}\lambda(A)$ for all n and $\mathbb{E}[r_n r_m | A] = 0$ for $n \neq m$.

Khintchine's inequality states that if (r_n) is a Rademacher sequence and (t_n) is in ℓ_2, then $f = \sum_n t_n r_n$ belongs to L_s for every $s \in (0, \infty)$ and, moreover, there is a constant M, depending only on s and $\lambda(A)$ such that

$$M^{-1}\|(t_n)\|_{\ell_2} \leq \|f\|_s \leq M\|(t_n)\|_{\ell_2}.$$

Thus a Rademacher sequence spans a subspace isomorphic to ℓ_2 in L_s for any $s \in (0, \infty)$.

Our first result is based on certain ideas from complex interpolation. Let us indicate the minimal background one needs to understand the proof.

Let X_0 and X_1 be (complex) Köthe function spaces on the unit interval. Consider the closed strip $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$ and let $\mathcal{F}(X_0, X_1)$ denote the space of bounded, continuous functions $F : \mathbb{S} \to X_0 + X_1$ having the following properties:

- F is analytic on the interior of $\mathbb{S}$;
- $F(k + it) \in X_k$ for each $k = 0, 1$ and all $t \in \mathbb{R}$.
- For $k = 0, 1$, the map $t \in \mathbb{R} \mapsto F(k + it) \in X_k$ is bounded and continuous.

Then $\mathcal{F} = \mathcal{F}(X_0, X_1)$ is a Banach space under the norm

$$\|F\|_{\mathcal{F}} = \sup\{\|F(k + it)\|_{X_k} : t \in \mathbb{R}, k = 0, 1\}.$$

For $\theta \in [0, 1]$ we define the interpolation space

$$X_\theta = [X_0, X_1]_\theta = \{f \in L_0 : f = F(\theta) \text{ for some } F \in \mathcal{F}\}$$

with the (quotient) norm $\|f\|_{X_\theta} = \inf\{\|F\|_{\mathcal{F}} : f = F(\theta)\}$.

The equation $[X_0, X_1]_\theta = X$ induces a 'derivation' on X as follows. We fix a small $\epsilon > 0$ and for each $f \in X$ we choose $F \in \mathcal{F}(X_0, X_1)$ such that $F(\theta) = f$, with $\|F\|_{\mathcal{F}} \leq (1+\epsilon)\|f\|_X$. Then we put $\Omega(f) = F'(\theta)$. The map $\Omega : X \to L_0$ is a centralizer on X and two centralizers obtained with different choices of F are equivalent.

An important result by Kalton [7, Theorem 7.6] states that if Φ is a real centralizer on L_p, with $p > 1$, then there is a constant $c > 0$ and a couple of Köthe functions spaces such that $L_p = [X_0, X_1]_{\theta=1/2}$ with equivalent norms, in the sense that both spaces contain the same functions and there is M such that

$$M^{-1}\|f\|_p \leq \inf_{f=F(\frac{1}{2})} \|F\|_{\mathcal{F}} \leq M\|f\|_p$$

for all $f \in L_p$, and $\Phi \approx c\Omega$, where Ω is the corresponding derivation on $X_{1/2} = L_p$.

Proposition 1. *Let Φ be a centralizer on L_p, where $0 < p < \infty$. Then for each $\delta > 0$ there is a set $B \subset \mathbb{I}$ with $\lambda(B) \geq 1 - \delta$ such that, for each $A \subset B$ of positive measure, Φ is bounded on the closed subspace spanned by any Rademacher sequence in A. In particular, the sequence $0 \to L_p \to L_p \oplus_\Phi L_p \to L_p \to 0$ is not strictly singular.*

Proof. It should be clear from the remarks in Section 1.3 that it suffices to prove the Proposition assuming that Φ is a real centralizer on the complex L_p.

First suppose $p > 1$. Then, by the result of Kalton quoted above, we know that there is a couple of Köthe spaces (X_0, X_1) and $c > 0$ such that $L_p = [X_0, X_1]_{1/2}$ and $\Phi \approx c\Omega$.

Let us take a look at Ω. First, by iteration, we have $L_p = [X_{1/4}, X_{3/4}]_{1/2}$ where $X_{k/4} = [X_0, X_1]_{k/4}$ for $k = 1, 3$ and both $X_{1/4}$ and $X_{3/4}$ are super-reflexive by [8, Theorem 5.8]. On the other hand, if $F \in \mathcal{F}(X_0, X_1)$, then the function G defined by $G(z) = F(\frac{1}{2}(z + \frac{1}{2}))$ belongs to $\mathcal{F}(X_{1/4}, X_{3/4})$ and one has $\|G\|_{\mathcal{F}} \leq \|F\|_{\mathcal{F}}, G(\frac{1}{2}) = F(\frac{1}{2})$ and $G'(\frac{1}{2}) = \frac{1}{2}F(\frac{1}{2})$.

Thus replacing the couple (X_0, X_1) by $(X_{1/4}, X_{3/4})$ preserves the induced centralizer, up to a constant factor, and so we may assume X_0 and X_1 are super-reflexive Köthe spaces.

Now, for $i = 0, 1$, take everywhere positive functions h_i and k_i so that $\|h_i f\|_1 \leq \|f\|_{X_i} \leq \|k_i f\|_\infty$ for all $f \in X_i$ and observe that for fixed $\delta > 0$ there is M large enough and a subset $B \subset \mathbb{I}$ with $\lambda(B) > 1 - \delta$ where $k_i \leq M$ and $h_i \geq 1/M$ for $i = 0, 1$.

It follows that $L_\infty(B) \subset X_i(B) \subset L_1(B)$, with continuous inclusions and since X_i is super-reflexive it is also s_i-concave for some finite s_i and so we have a continuous inclusion $L_{s_i}(B) \subset X_i(B)$ (see [4, p. 14]). Taking now $s = \max s_i$ we conclude that $L_s(B)$ embeds continuously into X_i and so there is a constant M such that $\|f\|_{X_i} \leq M\|f\|_s$ for every $f \in L_s(B)$ and $i = 0, 1$.

Now, let (r_n) be a Rademacher sequence in $L_s(A)$, where $A \subset B$ and let R the closed linear span of (r_n) in $L_s(A)$. Then, for $(\lambda_n) \in \ell_2$ the sum $\sum_n \lambda_n r_n$ is in $L_s(A)$ hence in $X_0(A) \cap X_1(A)$ and $\|f\|_{X_i} \leq M\|f\|_s \leq M'\|f\|_p$, by Khintchine's inequality. Actually the restriction of the norm of the spaces X_0, X_1, L_p and L_s to R is equivalent to the norm of (λ_n) in ℓ_2.

Hence for $f \in R$ we may take $F(z) = f$ for all $z \in \mathbb{S}$ since $\|F\|_{\mathcal{F}} \leq M'\|f\|_p$ and so $\Omega(f) = F'(\frac{1}{2}) = 0$. As $\Phi \approx c\Omega$ we see that Φ is bounded on R.

Suppose now $p \leq 1$ and let Φ be a centralizer on L_p. We define r by the identity $p^{-1} = r^{-1} + 2^{-1}$. Then there is a centralizer Ψ on L_2 and a constant M such that

$$\|\Phi(gf) - g\Psi(f)\|_p \leq M\|g\|_r\|f\|_2 \qquad (g \in L_r, f \in L_2),$$

(see [6, Theorem 8.1] for the case $p = 1$ and [2, Corollary 3] for $p < 1$). We know from the first part of the proof that there is a set B with measure arbitrarily close to 1 such that if $A \subset B$ and R is a subspace of $L_2(A)$ spanned by a Rademacher sequence in A, then Ψ is bounded on R: $\|\Psi(f)\|_2 \leq M'\|f\|_2$ for $f \in R$. Taking now $g = 1$ and $f \in R$ we have

$$\|\Phi(f) - \Psi(f)\|_p \leq M\|1\|_r\|f\|_2 \leq M''\|f\|_p$$

and so Φ is also bounded on R. □

A centralizer Φ on L_p is said to be symmetric if there is a constant S such that

$$\|\Phi(f \circ \sigma) - \Phi(f) \circ \sigma\|_p \leq S\|f\|_p$$

for every $f \in L_p$ and every measure preserving Borel automorphism σ of $\mathbb{I}$.

The decreasing rearrangement of a real-valued $f \in L_0$ is defined by the formula

$$f^*(t) = \inf_{\lambda(B)=t} \sup_{s \in A \setminus B} f(s) \qquad (0 \leq t \leq 1)$$

where B runs over the Borel subsets of $\mathbb{I}$. That is, f^* is the only decreasing, right-continuous function having the same distribution as f. It is a basic fact from measure theory that for each $f \in L_0$, there is an measure preserving Borel automorphism σ of $\mathbb{I}$ (depending on f) such that $f^* = f \circ \sigma$ (almost everywhere) and so f^* is true rearrangement of f; see [11, Lemma 2].

Note that if Φ is a symmetric centralizer on L_p and $f^* = f \circ \sigma$, then $\|\Phi(f) - (\Phi(f^*)) \circ \sigma^{-1}\|_p \leq S\|f\|_p$ and so the map $\Phi_s(f) = (\Phi(f^*)) \circ \sigma^{-1}$ is a symmetric centralizer equivalent to Φ with the additional property that the distribution of $\Phi_s(f)$ depends only on the distribution of f.

We emphasize that, in general, centralizers take values in L_0. For symmetric centralizers we have, however, the following.

Lemma 1. *Suppose* $0 < p < r < \infty$ *and let* Φ *be a symmetric centralizer on* L_p*. If* $f \in L_r$*, then* $\Phi f \in L_p$*.*

Proof. It suffices to prove the Lemma for real spaces. Let $\Phi_r : L_r \to L_0$ be the restriction of Φ to L_r. This is a centralizer from L_r to L_p so by the main result in [2] Φ_r must be trivial and there is $\phi \in L_0$ and a constant M such that

$$\|\Phi_r(f) - \phi f\|_p \leq M\|f\|_r \qquad (f \in L_r).$$

We claim that $\phi \in L_s$, where $s^{-1} + r^{-1} = p^{-1}$. By the Hölder inequality this implies that $\phi f \in L_p$ and the same occurs to $\Phi(f) = \Phi_r(f)$. To see this, observe that since $f \mapsto \phi f$ is equivalent to Φ_r it is a symmetric centralizer from L_r to L_p and so there is a constant S such that

$$\|(\phi \circ \sigma)(f \circ \sigma) - \phi(f \circ \sigma)\|_p \leq S\|f\|_r \qquad (f \in L_r)$$

whenever σ is a measure preserving automorphism of the unit interval. Now, since for every $g \in L_s$ one has $\|g\|_s = \sup_{\|f\|_r \leq 1} \|gf\|_p$, we see that $\|\phi \circ \sigma - \phi\|_s \leq M'$ for some M' independent on σ. By symmetry one also has $\|\phi^* \circ \sigma - \phi^*\|_s \leq M'$, where ϕ^* is the decreasing arrangement of ϕ and σ is as before. In particular $\|\phi^* \circ \sigma - \phi^*\|_s$

is finite when $\sigma(t) = 1 - t$. Let $m = \phi^*(\frac{1}{2})$ be the median of ϕ. Now, since ϕ^* is decreasing, $\phi^* \circ \sigma$ is increasing and both agree with m at $t = \frac{1}{2}$ we see that

$$\|\phi - m1\|_s = \|\phi^* - m1\|_s \leq \|\phi^* \circ \sigma - \phi^*\|_s$$

is finite and $\phi \in L_s$. □

We are now ready to prove the following.

Proposition 2. *Let $0 < p < q < 2$. There is a subspace U of L_p isomorphic to ℓ_q where the restriction of any symmetric centralizer is bounded.*

Proof. It suffices to prove the result for real spaces. Moreover, we may and do assume that the distribution of $\Phi(f)$ depends only on that of f.

We proceed as in [12, Proof of Proposition 4.1]. For fixed $q \in (p, 2)$ we consider a q-stable random variable $\vartheta \in L_p$ and a sequence of independent copies (ϑ_n). We recall that a random variable is said to be q-stable if its characteristic function (= Fourier transform) is $e^{-|t|^q/q}$. We refer the reader to [1, Chapter 6, Section 4] for basic information on stable variables. Here we use the following facts:

- If ϑ is q-stable, then $\mathbb{E}[|\vartheta|^r] < \infty$ for $p < r < q$.
- If (ϑ_n) is a sequence on independent copies of a q-stable random variable ϑ, and (λ_n) is a sequence normalized in ℓ_q, then $\sum_n \lambda_n \vartheta_n$ has the same distribution as ϑ.

Therefore the map $(\lambda_n) \in \ell_q \mapsto \sum_n \lambda_n \vartheta_n \in L_p$ is well defined and it is an isometric embedding whose image we denote by U. Moreover, by the Lemma, $\Phi(\vartheta)$ belongs to L_p, and then for (λ_n) normalized ℓ_q we have $\|\Phi(\sum \lambda_n \vartheta_n)\|_p = \|\Phi(\vartheta)\|_p$ and so Φ is bounded on U. □

Problem. Is there a strictly singular sequence $0 \to L_p \to Z \to L_p \to 0$ for $0 < p < 2$? (See [3, Theorem 2(c)] for the case $2 \leq p < \infty$.)

Acknowledgement. I thank Paco Hernández for helping me with Köthe spaces. I'm grateful to the referee for some remarks that improved the paper.

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