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Nonlinear centralizers in homology

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Abstract. It is shown that every nonlinear centralizer from L_p to L_q is trivial unless $q = p$. This means that if $q \neq p$, the only exact sequence of quasi-Banach L_∞-modules and homomorphisms $0 \to L_q \to Z \to L_p \to 0$ is the trivial one where $Z = L_q \oplus L_p$. From this it follows that the space of centralizers on L_p is essentially independent on $p \in (0, \infty)$, which confirms a conjecture by Kalton.

1. Introduction

1.1. Background. Quasilinear maps burst onto Banach space theory in [**4**], where Enflo, Lindenstrauss and Pisier based their solution to the "three-space" problem for Hilbert spaces on the construction of a quasilinear map on ℓ_2.

Then Kalton [**7**] and Ribe [**16**] developed a rather satisfactory theory showing that extensions of quasi-Banach spaces $0 \to Y \to Z \to X \to 0$ are in correspondence with quasilinear maps $\Phi : X \to Y$. Both Kalton and Ribe gave examples of non-trivial quasilinear maps $\Phi : \ell_1 \to \mathbb{R}$ thus producing nontrivial extensions $0 \to \mathbb{R} \to Z \to \ell_1 \to 0$ and solving the "three-space" problem for local convexity.

Another counterexample was obtained independently and more or less simultaneously by Roberts in [**17**].

Soon afterwards the admirable [**14**] appeared. In it, Kalton and Peck use a kind of vector-valued version of Ribe's map to construct quasilinear maps on any quasi-Banach space with unconditional basis. Let us consider this point in more detail as it is one of the main motivations of the present paper. What was proved in [**14**] is that given any Lipschitz function $\theta : \mathbb{R}_+ \to \mathbb{R}$, with $\theta(0) = 0$, the mapping Ω_θ defined by $\Omega_\theta(f) = f\theta(-\log(|f|/\|f\|_X))$ is quasilinear on every quasi-Banach space with unconditional basis X —we invariably regard the elements of such an X as functions $f : \mathbb{N} \to \mathbb{K}$— and non-trivial as long as the basis of X contains no subsequence equivalent to the usual basis of c_0 and θ is unbounded on $\mathbb{R}_+$. Taking $X = \ell_2$, and θ as the identity on $\mathbb{R}_+$ one obtains another solution to the three space problem —nowadays called the Kalton-Peck space Z_2. All this can be seen in [**3**, Chapter 1] or [**1**, Chapter 16].

The crucial observation here is that the maps appearing in [**14**] are more than quasilinear: they also are ℓ_∞-centralizers, that is, they satisfy an estimate of the form

$$\|\Omega(af) - a\Omega(f)\| \leq C\|a\|_\infty\|f\| \qquad (a \in \ell_\infty, f \in X).$$

Equivalently, the induced extension $0 \to Y \to Z \to X \to 0$ lives in the category of ℓ_∞-modules.

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These examples suggested a correspondence between centralizers defined on different (sequence) spaces.

This issue was pursued by Kalton himself in the more general setting of function spaces in [**10**]. Fix a measure μ and consider the corresponding L_p spaces for $0 < p < \infty$. It is proved in [**10**, Theorem 5.1] that, when $p > 1$, every L_∞-centralizer Φ on L_p can be pushed to a centralizer Ψ on L_1 defined by

$$\Psi(f) = u|f|^{1/q}\Phi(|f|^{1/p}),$$

where $u|f|$ is the polar decomposition of f and $1 = p^{-1} + q^{-1}$. Moreover, all centralizers on L_1 arise in this form [**10**, Theorem 8.1]. As self extensions of L_p spaces in the category of quasi-Banach L_∞-modules are all induced by centralizers we have

$$\mathrm{Ext}(L_p) = \mathrm{Ext}(L_1) \qquad (1 < p < \infty).$$

Actually [**10**] contains much more general results for Banach function spaces, but in this paper we will focus on the Lebesgue spaces L_p.

The basic problem left open in [**10**, p. 83] was to determine whether this correspondence extends to $0 < p < 1$ or not.

This paper solves this problem in the affirmative.

We approach the problem by studying first centralizers acting between two different Lebesgue spaces, say L_p and L_q with $p, q \in (0, \infty)$ and we prove that they are all trivial unless $q = p$. This means that $\mathrm{Ext}(L_p, L_q) = 0$ for $q \neq p$, that is, the only exact sequence of quasi-Banach L_∞-modules $0 \to L_q \to Z \to L_p \to 0$ is the trivial one where $Z = L_q \oplus L_p$. This is the main result of the paper, and its proof occupies the entire Section 2. Sections 3 and 4 contain a number of applications including the proof of Kalton's conjecture. Section 1 contains, apart from this general introduction, some preliminaries on extensions and centralizers.

Some parts of this paper were written by Nigel Kalton, namely Theorem 2, Lemma 3, and the proof for Step 2. Nigel sent this material to me to improve an earlier version of the paper. I never suggested to him to compose a joint paper and so I have to present the paper authored by me only.

1.2. Quasi-Banach modules. Let A be a (real or complex) Banach algebra that for all purposes in this paper will be L_∞. A quasi-normed module over A is a quasi-normed space X together with a jointly continuous outer multiplication $A \times X \to X$ satisfying the traditional algebraic requirements. If the underlying space is complete (that is, a quasi-Banach space) we speak of a quasi-Banach module. Given quasi-normed modules X and Y, a homomorphism $T : X \to Y$ is an operator such that $T(ax) = aT(x)$ for all $a \in A$ and $x \in X$. Operators and homomorphisms are assumed to be continuous unless otherwise stated. If no continuity is assumed, we speak of linear maps and morphisms. We use $\mathrm{Hom}_A(X, Y)$ for the space of homomorphisms from X to Y. If there is no possible confusion about the underlying algebra A, we omit the subscript.

1.3. Extensions. An extension of X by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$0 \longrightarrow Y \stackrel{\imath}{\longrightarrow} Z \stackrel{\pi}{\longrightarrow} X \longrightarrow 0 \tag{1}$$

The open mapping theorem guarantees that $\imath$ embeds Y as a closed submodule of Z in such a way that the corresponding quotient $Z/\imath(Y)$ is isomorphic to X. Two extensions

$0 \to Y \to Z_i \to X \to 0$ ($i = 1, 2$) are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle u} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z_2 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

By the five-lemma [**6**, Lemma 1.1], and the open mapping theorem, u must be an isomorphism. We say that (1) splits if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus X \to X \to 0$. This just means that Y is a complemented submodule of Z (that is, there is a homomorphism $\varpi : Z \to Y$ such that $\varpi \circ \imath = \mathbf{I}_Y$; equivalently, there is a homomorphism $\jmath : X \to Z$ such that $\pi \circ \jmath = \mathbf{I}_X$) and implies that Z is isomorphic to the direct sum $Y \oplus X$ (the converse is not true in general). Given quasi-Banach modules X and Y, we denote by $\operatorname{Ext}_A(X, Y)$ (or just $\operatorname{Ext}_A(X)$ when $Y = X$) the set of all possible A-module extensions (1) modulo equivalence. By using pull-back and push-out constructions, it can be proved (see [**2**] for the details in the F-space setting) that $\operatorname{Ext}_A(X, Y)$ carries a "natural" structure of A-bimodule (without topology) in such a way that trivial extensions correspond to 0. (The usual approach using injective or projective representations completely fails dealing with quasi-Banach modules since there are neither injective nor projective objects.) Thus, $\operatorname{Ext}_A(X, Y) = 0$ means "every extension of X by Y is trivial".

Taking A as the underlying field $\mathbb{K}$, one recovers extensions in the quasi-Banach space setting.

1.4. Function spaces. From now on, μ will denote a fixed countably additive measure on $\mathcal{S}$. We will assume μ sigma-finite or, at least, decomposable. Our ambient space will be L_0, the space of all measurable functions on $\mathcal{S}$ with the topology of convergence in measure on sets of finite measure and we apply the usual convention about identifying functions equal almost everywhere. L_∞ denotes the Banach algebra of all essentially bounded measurable functions on $\mathcal{S}$ equipped with the essential supremum norm and "pointwise" operations.

A function space is a linear subspace X of L_0 containing the simple integrable functions and equipped with a quasi-norm $\|\cdot\|_X$ such that if $f \in L_0, g \in X$ and $|f| \leq |g|$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. A function space X is said to be:

- Minimal if simple integrable functions are dense.
- Maximal if whenever (f_n) is an increasing sequence of non-negative functions in X converging almost everywhere to f and $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$.

The only function spaces we shall deal with in this paper are the popular L_p spaces for $p \in (0, \infty)$. They are both minimal and maximal.

Every function space is a (quasi-normed) L_∞-module under pointwise multiplication. Our "default" category will be that of quasi-Banach modules over L_∞. Accordingly, we use minimal notation in this setting and so $\operatorname{Hom}(X, Y)$ and $\operatorname{Ext}(X, Y)$ always refer to the algebra L_∞ for a measure that should be clear for the context —and will never be the ground field $\mathbb{K}$.

1.5. Centralizers and the extensions they induce. We now define our main object of study (Cf. [**10**, Chapter 3], [**11**, p. 480], [**13**, p. 1165]).

Definition 1. Let X and Y be quasi-normed function spaces and $\Phi : X \to L_0$ a homogeneous mapping. (Homogeneous means that $\Phi(\lambda f) = \lambda \Phi f$ for every $\lambda \in \mathbb{K}$ and $f \in X$.)

(a) We say that Φ is a centralizer from X to Y if there is a constant C such that for every $a \in L_\infty, f \in X$ the difference $\Phi(af) - a\Phi(f)$ belongs to Y and
$$\|\Phi(af) - a\Phi(f)\|_Y \leq C\|a\|_\infty \|f\|_X.$$
We write $\mathcal{C}(X, Y)$ (or just $\mathcal{C}(X)$ if $Y = X$) for the set of all centralizers from X to Y and we denote by $C[\Phi]$ the least constant for which the preceding inequality holds.

(b) We say that Φ is quasilinear from X to Y if there is a constant Q such that for every $f, g \in X$ the difference $\Phi(f + g) - \Phi f - \Phi g$ belongs to Y and
$$\|\Phi(f + g) - \Phi(f) - \Phi(g)\|_Y \leq Q(\|f\|_X + \|g\|_X)$$
The least constant for which the preceding inequality holds is denoted $Q[\Phi]$.

Before going further let us state the following simple remark. The proof is the same as [**10**, Lemma 4.2].

Lemma 1. *Every centralizer is quasilinear. More precisely, if $\Phi : X \to L_0$ is a centralizer from X to Y, then Φ is quasilinear and one has $Q[\Phi] \leq 3\Delta_Y^2 C[\Phi]$, where Δ_Y is the modulus of concavity of Y.* □

(The modulus of concavity of a quasi-normed space X is the smallest positive constant Δ such that $\|x + y\| \leq \Delta(\|x\| + \|y\|)$ for all $x, y \in X$. When $X = L_p$ we just write Δ_p. Clearly, $\Delta_p = 2^{1/p-1}$ for $p \in (0, 1)$ and $\Delta_p = 1$ for $p \geq 1$.)

We now indicate the connection between centralizers and extensions. Suppose Φ is a centralizer from X to Y. Define $Y \oplus_\Phi X = \{(g, f) \in L_0 \times X : g - \Phi f \in Y\}$ quasi-normed by $\|(g, f)\|_\Phi = \|g - \Phi f\|_Y + \|f\|_X$. Clearly, the map $\imath : Y \to Y \oplus_\Phi X$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $\pi : Y \oplus_\Phi X \to X$ given as $\pi(g, f) = f$ is open, so that we have an extension of quasi-Banach spaces

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y \oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0 \tag{2}$$

Actually only the quasi-linearity of Φ is necessary here. The condition that Φ is a centralizer implies that the multiplication $a(g, f) = (ag, af)$ makes $Y \oplus_\Phi X$ into an L_∞-module in such a way that (2) becomes an extension of modules.

It is proved in [**10**, Theorem 4.5] that if X is minimal and Y maximal, then every extension of modules (1) comes from a centralizer, up to equivalence. This applies, in particular, when $X = L_p$ and $Y = L_q$ for $0 < p, q < \infty$. It is easily seen that two centralizers Ψ and Φ (from X to Y) induce equivalent extensions if and only if there is an L_∞-morphism $h : X \to L^0$ such that $\|\Psi(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_X$ for some constant K and all $f \in X$. We write $\Psi \sim \Phi$ in this case and $\Psi \approx \Phi$ if the preceding inequality holds for $h = 0$. In particular Φ induces a trivial extension if and only if $\|\Phi(f) - h(f)\|_Y \leq K\|f\|_X$ for some morphism $h : X \to L_0$. We then say that Φ is a trivial centralizer and we write $\operatorname{dist}(\Phi, h)$ for the least possible constant K in the preceding inequality.

We shall write $\mathcal{C}_\sim(X, Y)$ and $\mathcal{C}_\approx(X, Y)$ to denote the set $\mathcal{C}(X, Y)$ factored by $\sim$ and $\approx$, respectively. Note that, while $\mathcal{C}(X, Y)$ has two natural outer multiplications, namely $(a\Phi)f = a\Phi f$ and $(\Phi a)f = \Phi(af)$, these agree on the quotient $\mathcal{C}_\approx(X, Y)$ and so on $\mathcal{C}_\sim(X, Y)$.

1.6. Morphisms, homomorphisms and multiplication operators. Next we identify the morphisms between function spaces and some spaces of homomorphisms.

LEMMA 2. *Let X be a function space on a localizable measure μ.*

(a) *If $h : X \to L_0$ is any mapping satisfying $h(af) = ah(f)$ for every $a \in L_\infty$ and $f \in X$, then there is $\phi \in L_0$ such that $h(f) = \phi f$ for all $f \in X$.*

(b) *If $0 < q < p < \infty$, then $\operatorname{Hom}(L_p, L_q) = L_r$, where $r^{-1} + p^{-1} = q^{-1}$. That is, if $\phi \in L_r$, then the map $f \in L_p \longmapsto \phi f \in L_q$ is a homomorphism and every homomorphism arises in this way. Moreover, $\|\phi\|_r = \|\phi : L_p \to L_q\|$.*

PROOF. (a) First we prove the Lemma assuming finite the underlying measure, so that X contains L_∞. Set $\phi = h(1)$. We want to see that $h(f) = \phi f$ for every $f \in X$. Pick $f \in X$ and put $g = 1 + |f|$. Then $1 = (1/g)g$ and since $1/g \in L_\infty$ we have $\phi = (1/g)h(g)$ and so $h(g) = g\phi$. But $f = (f/g)g$, with $f/g \in L_\infty$. Hence

$$h(f) = h\left((f/g)g\right) = (f/g)h(g) = (f/g)\phi g = \phi f,$$

as required.

Finally, assuming μ decomposable we fix a decomposition $\mathcal{S} = \oplus_i S_i$ into sets of finite measure and we set $\phi = \sum_i h(1_{S_i})$, where the sum is performed in the pointwise sense. As $1_{S_i} h(f) = h(1_{S_i} f)$ we have $h(f) = \sum_i 1_{S_i} h(f) = \sum_i h(1_{S_i} f) = \sum_i h(1_{S_i}) f = \phi f$.

Part (b) obviously follows from (a) and Hölder inequality. □

2. The main result

In this Section we prove that $\operatorname{Ext}(L_p, L_q) = 0$ whenever $q \neq p$. We already know that every extension of L_p by L_q arises from a centralizer and, as the reader can imagine, what we shall prove is that every centralizer from L_p to L_q is trivial, that is, at finite distance to a morphism. We have the following slightly stronger statement, which is the main result of the paper.

THEOREM 1. *Let μ be a σ-finite measure on $\mathcal{S}$. Suppose p and q are different numbers in $(0, \infty)$. Then there is a constant $M = M(p, q)$ so that for each $\Phi \in \mathcal{C}(L_p, L_q)$ there is $\phi \in L_0$ such that $\|\Phi(f) - \phi f\|_q \le MC[\Phi]\|f\|_p$ for every $f \in L_p$.*

We will assume that the ground field is $\mathbb{R}$. The complex case then follows quickly by using real centralizers and [**11**, Lemma 7.1].

We break the proof up into several steps. The first one, based on old results by Kalton, is rather easy.

STEP 1. Theorem 1 holds for bounded centralizers and $q = 1$.

PROOF. In this case $p \neq 1$ and L_p is a $\mathcal{K}$-space. This means that there is a constant K such that for each quasilinear functional $\varphi : L_p \to \mathbb{R}$ there is a true linear function $\ell : L_p \to \mathbb{R}$ such that $|\varphi(f) - \ell(f)| \le KQ[\varphi]\|f\|_p$. (See [**7**, Theorem 4.3]. An obvious amalgamation argument shows that K depends only on p.)

Let $\Phi \in \mathcal{C}(L_p, L_1)$ be bounded, so that $\|\Phi(f)\|_1 \le B\|f\|_p$ for some B (depending on Φ) and all $f \in L_p$. Consider the (bounded) functional $\varphi : L_p \to \mathbb{R}$ given by $\varphi(f) = \int_{\mathcal{S}} \Phi(f) d\mu$. By Lemma 1, φ is quasi-linear, with $Q[\varphi] \le Q[\Phi] \le 3C[\Phi]$ and so there is a (bounded) linear functional ℓ on L_p such that $|\varphi(f)-\ell(f)|\leq KQ[\Phi]\|f\|_p$ for all $f\in L_p$. By Riesz representation Theorem one has $\ell(f)=\int_{\mathcal{S}}\phi f d\mu$ for some $\phi\in L_0$ and, therefore,

$$\left|\int_{\mathcal{S}}\Phi(f)d\mu-\int_{\mathcal{S}}\phi f d\mu\right|\leq KQ[\Phi]\|f\|_p \qquad (f\in L_p).$$

Let us estimate $\|\Phi(f)-\phi f\|_1$. One has

$$\begin{aligned}\|\Phi(f)-\phi f\|_1&=\sup_{\|a\|_\infty\leq 1}\left|\int_{\mathcal{S}}a(\Phi(f)-\phi f)d\mu\right|\\&\leq\sup_{\|a\|_\infty\leq 1}\left|\int_{\mathcal{S}}(a\Phi(f)-\Phi(af))d\mu\right|+\sup_{\|a\|_\infty\leq 1}\left|\int_{\mathcal{S}}(\Phi(af)-\phi\cdot af)d\mu\right|\\&\leq C[\Phi]\|f\|_p+KQ[\Phi]\|f\|_p\\&\leq M_0C[\Phi]\|f\|_p,\end{aligned}$$

where $M_0=M_0(p,1)=1+3K(p)$. □

Every measurable function f can be written as $f=u|f|$, where u has the same support as f. This is often called the polar decomposition of f. Given $r\in(0,\infty)$ we define the Mazur map on L_0 by $S_r(f)=u|f|^r$, where $u|f|$ is the polar decomposition of f.

Lemma 3. *If $\Phi\in\mathcal{C}(L_p,L_q)$ and $r\in(0,1]$, then the composition $\Phi^{(r)}=S_r\circ\Phi\circ S_{1/r}$ is a centralizer from $L_{p/r}$ to $L_{q/r}$, with $C[\Phi^{(r)}]\leq 2^{1/r}C[\Phi]^r$.*

Proof. The key point is that, when $r\in(0,1]$, one has the pointwise estimate $|S_r(f)-S_r(g)|\leq 2^{1/r}S_r(|f-g|)$ for $f,g\in L_0$. Let Φ be in $\mathcal{C}(L_p,L_q)$ and take $a\in L_\infty$ and $f\in L_{p/r}$ with polar decompositions $a=v|a|$ and $f=u|f|$, respectively. Then

$$\begin{aligned}\|\Phi^{(r)}(af)-a\Phi^{(r)}(f)\|_{q/r}&=\|S_r(\Phi(uv|a|^{1/r}|f|^{1/r}))-aS_r(\Phi(u|f|^{1/r})\|_{q/r}\\&=\|S_r(\Phi(uv|a|^{1/r}|f|^{1/r}))-S_r(v|a|^{1/r}\Phi(u|f|^{1/r})\|_{q/r}\\&\leq 2^{1/r}\|S_r(|\Phi(uv|a|^{1/r}|f|^{1/r}))-v|a|^{1/r}\Phi(u|f|^{1/r}|)\|_{q/r}\\&=2^{1/r}\|\Phi(uv|a|^{1/r}|f|^{1/r}))-v|a|^{1/r}\Phi(u|f|^{1/r})\|_q^r\\&\leq 2^{1/r}C[\Phi]^r\||a|^{1/r}\|_\infty^r\||f|^{1/r}\|_p^r\\&=2^{1/r}C[\Phi]^r\|a\|_\infty\|f\|_{p/r},\end{aligned}$$

as required. □

Step 2. Theorem 1 holds for bounded centralizers and $q<1$.

Proof. Let $\Phi:L_p\to L_q$ be a bounded homogeneous map with centralizer constant 1. We start by picking $\phi\in L_0$ so that $f\mapsto\Phi(f)-\phi f$ has a bound M which is nearly optimal in the sense that for any $v\in L_0$ one has

$$\sup_{\|f\|_p\leq 1}\|\Phi(f)-vf\|_q\geq\frac{M}{2}.$$

Without loss of generality we can assume $\phi = 0$. Applying Lemma 3 to Φ with $r = q$ we see that the map $\Phi^{(q)} = S_q \circ \Phi \circ S_{1/q}$ is a (bounded) centralizer from $L_{p/q}$ to L_1, with $C[\Phi^{(q)}] \leq 2^{1/q}$ and so there is $v \in L^0$ such that

$$\|\Omega^{(q)}(f) - vf\|_1 \leq N\|f\|_{p/q} \qquad (f \in L^{p/q}),$$

where $N = 1 + 3K(p/q)$ denotes the constant we got in Step 1. If $f \in L_p$, then

$$\|\Phi f - (S_{1/q}v)f\|_q = \|S_{1/q}(\Phi^{(q)}(S_q f)) - S_{1/q}(v(S_q f))\|_q.$$

Applying the Mean Value Theorem to the function $t \in \mathbb{R} \mapsto |t|^{1/q} \in \mathbb{R}$,

$$|S_{1/q}(\Phi^{(q)}(S_q f)) - S_{1/q}(v(S_q f))| \leq \frac{\max\{|\Phi^{(q)}(S_q f)|, |v(S_q f)|\}^{\frac{1}{q}-1}}{q} \cdot |\Phi^{(q)}(S_q f) - v(S_q f)|.$$

Now, applying Hölder inequality with exponents $(1/q-1)^{-1}$ and 1, we get

$$\|S_q^{-1}(\Phi^{(q)}(S_q f)) - S_q^{-1}(v(S_q f))\|_q \leq \frac{1}{q}\left\|\Big(|\Phi^{(q)}(S_q f)| \bigvee |v(S_q f)|\Big)^{\frac{1}{q}-1}\right\|_{\frac{q}{1-q}} \|\Phi^{(q)}(S_q f) - v(S_q f)\|_1$$

that is,

$$\|S_q^{-1}(\Phi^{(q)} S_q f)) - S_q^{-1}(v(S_q f))\|_q^q \leq \frac{\big\|\big(|\Phi^{(q)}(S_q f)| \bigvee |v(S_q f)|\big)\big\|_1^{1-q}}{q^q} \cdot \|\Phi^{(q)}(S_q f) - v(S_q f)\|_1^q.$$

Now since

$$\|\Phi^{(q)}(S_q f) - v(S_q f)\|_1 \leq N\|S_q f\|_{p/q} = N\|f\|_p^q$$

and

$$\|\Phi^{(q)}(S_q f)\|_1 = \|S_q(\Phi f)\|_1 \leq \|\Phi f\|_q^q \leq M^q\|f\|_p^q$$

we have

$$\|v(S_q f)\|_1 \leq C\|S_q f\|_{p/q} + M^q\|f\|_p^q \leq (N + M^q)\|f\|_p^q.$$

Combining,

$$\|\Phi(f) - (S_{1/q}v)f\|_q \leq \frac{N(N+2M^q)^{1/q-1}}{q}\|f\|_q.$$

Thus we must have

$$M \leq \frac{2N(N+2M^q)^{1/q-1}}{q}$$

and this leads to a bound on M which depends only on p (through N) and q. □

There are two simplifications concerning centralizers that are important for us. First if Φ is any centralizer, then the map defined by $\Psi(f) = u\Phi(|f|)$ is a centralizer which does not increase supports, in the sense that $\operatorname{supp}\Psi f \subset \operatorname{supp} f$ for each f, and one has $C[\Psi] \leq C[\Phi]$, and $\|\Psi f - \Phi f\| \leq C[\Phi]\|f\|$ for every f.

Also, if ν is a measure equivalent to μ, in the sense that they have the same null sets, then $L_\infty(\mu)$ equals $L_\infty(\nu)$ and for each $p \in (0,\infty)$ the map $f \mapsto (d\nu/d\mu)^{1/p} f$ is an isometry from $L_p(\nu)$ onto $L_p(\mu)$ which preserves the L_∞-module structures. This allows us to replace any σ-finite measure by a probability and vice versa.

The following trick allows us to approximate a given centralizer by a sequence of bounded centralizers, when $q \leq p$ and μ is a probability. In all what follows, for each fixed $k > 0$, we denote by τ_k the truncation function defined by $\tau_k(t) = k \bigwedge (t \bigvee -k)$ for $t \in \mathbb{R}$.

Lemma 4. *Let μ be a probability and let $\Phi \in \mathcal{C}(L_p, L_q)$ a centralizer that does not increase supports, with $0 < q \leq p$. Given a positive constant k, the map*

$$\Phi_k f = f \tau_k \left(\frac{\Phi f}{f} \right)$$

is a centralizer with $C[\Phi_k] \leq C[\Phi]$, moreover, one has $\|\Phi_k f\|_q \leq k \|f\|_p$ for every $f \in L_p$.

Proof. For the first part just observe that $|\Phi_k(af) - a\Phi_k(f)| \leq |\Phi(af) - a\Phi(f)|$ for every $a \in L_\infty$ and $f \in L_p$. The second part follows from the inequality $\| \cdot \|_q \leq \| \cdot \|_p$. □

Theorem 2. *Let μ be a probability on $\mathcal{S}$ and let (v_n) be a sequence in L_0. Then there exists a measurable function v with the property that if $f, g \in L_0$ and (g_n) is a sequence in L_0 converging to g almost everywhere, then for $0 < p < \infty$,*

$$\int_{\mathcal{S}} |g - vf|^p d\mu \leq 2 \limsup_{n \to \infty} \int_{\mathcal{S}} |g_n - v_n f|^p d\mu.$$

Proof. We can consider the case when $\mathcal{S}$ is a Stonean space and μ is a normal measure on $\mathcal{S}$, that is, $\mu(E) = 0$ for every nowhere dense set E. In this case every function in L_0 can be replaced by a function in the same equivalence class which is continuous into $\overline{\mathbb{R}} = [-\infty, \infty]$, the two-point compactification of $\mathbb{R}$. See [**18**, Example 11.13(f)] for details.

We proceed by considering the measure ν_n defined on $\mathcal{S} \times \overline{\mathbb{R}}$ by the formula

$$\int_{\mathcal{S} \times \overline{\mathbb{R}}} F(s,t) d\nu_n = \int_{\mathcal{S}} F(s, v_n(s)) d\mu \qquad (F \in C(\mathcal{S} \times \overline{\mathbb{R}})).$$

Quite clearly, each ν_n is a probability. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and set $\nu = \lim_{\mathcal{U}} \nu_n$, where the limit is taken with respect to the weak* topology in $M(\mathcal{S} \times \overline{\mathbb{R}})$ regarded as the dual of $C(\mathcal{S} \times \overline{\mathbb{R}})$. Note that if $f \in C(\mathcal{S})$, then

$$\int_{\mathcal{S} \times \overline{\mathbb{R}}} f(s) d\nu(s,t) = \int_{\mathcal{S}} f(s) d\mu(s).$$

Given $f \in L^\infty(\mathcal{S} \times \overline{\mathbb{R}}, \nu)$, we denote by $\mathbb{E}f$ the function in $L^\infty(\mu)$ such that

$$\int_{\mathcal{S}} g(s) \mathbb{E}f(s) d\mu(s) = \int_{\mathcal{S} \times \overline{\mathbb{R}}} g(s) f(s,t) d\nu(s,t) \qquad (g \in L_1(\mathcal{S}, \mu)).$$

(Basically $\mathbb{E}$ is the conditional expectation operator.)

Let $\mathcal{F}$ denote the set of continuous functions $h : \mathcal{S} \to \overline{\mathbb{R}}$ such that $\mathbb{E}(1_{A(h)}) \leq \frac{1}{2}$ in $L^\infty(\mu)$, where $A(h) = \{(s,t) \in \mathcal{S} \times \overline{\mathbb{R}} : t < h(s)\}$. It is easy to show that there is a maximal function u with this property and that then

$$\mathbb{E}(1_A), \mathbb{E}(1_B) \leq \frac{1}{2} \qquad (\text{in } L^\infty(\mu)),$$

where $A = \{(s,t) \in \mathcal{S} \times \overline{\mathbb{R}} : t < u(s)\}$ and $B = \{(s,t) \in \mathcal{S} \times \overline{\mathbb{R}} : t > u(s)\}$.

We let

$$v(s) = \begin{cases} u(s) & \text{if } u(s) \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose f, g and (g_n) are as in the statement. We may assume that they are continuous into $\overline{\mathbb{R}}$ and assume the values $\pm\infty$ on sets of measure zero. We also may assume the existence of a similar function G with $G \geq |g_n|$ everywhere. Suppose

$$\lim_{\mathcal{U}} \int_{\mathcal{S}} |g_n - v_n f|^p < \infty.$$

First, let

$$L_1 = \lim_{\mathcal{U}} \int_{|f|>0} |g_n - v_n f|^p \quad \text{and} \quad L_2 = \lim_{\mathcal{U}} \int_{|f|=0} |g_n - v_n f|^p = \lim_{\mathcal{U}} \int_{|f|=0} |g_n|^p d\mu.$$

Consider a clopen set $E \subset \mathcal{S}$ such that, for some $M \in \mathbb{R}$, we have $1/M \leq |f| \leq M$ and $G \leq M$ on E. For each positive $k \in \mathbb{R}$, let $\tau_k : \overline{\mathbb{R}} \to \mathbb{R}$ be as before, that is, $\tau_k(t) = k \bigwedge (t \bigvee -k)$ for $t \in \overline{\mathbb{R}}$. Then, for $k > M^2$, the function $(s,t) \mapsto (g(s) - \tau_k(t) f(s)) 1_E(s)$ is continuous on $\mathcal{S} \times \overline{\mathbb{R}}$ and so

$$\begin{aligned}\int_{E\times\overline{\mathbb{R}}} |g(s) - \tau_k(t) f(s)|^p d\nu(s,t) &= \lim_{\mathcal{U}} \int_{E\times\overline{\mathbb{R}}} |g(s) - \tau_k(t) f(s)|^p d\nu_n(s,t) \\ &= \lim_{\mathcal{U}} \int_E |g(s) - \tau_k(v_n(s)) f(s)|^p d\mu(s) \leq \lim_{\mathcal{U}} \int_E |g(s) - v_n(s) f(s)|^p d\mu(s) = L_1.\end{aligned}$$

Since this is true for all $k > M^2$ we conclude that $\nu(E \times \{\pm\infty\}) = 0$. In particular u is finite almost everywhere on E and so $v = u$ almost everywhere on E. Moreover,

$$\int_{E\times\overline{\mathbb{R}}} |g(s) - t f(s)|^p d\nu(s,t) \leq L_1.$$

Now let $E_+ = \{s \in E : g(s) > v(s) f(s)\}$. Then if $\tilde{B}$ is the complement of B in $\mathcal{S} \times \overline{\mathbb{R}}$ one has $\mathbb{E}(1_{\tilde{B}}) \geq 1/2$ and so

$$\begin{aligned}\int_{E_+} |g - vf|^p d\mu &\leq 2 \int_{E_+} |g - vf|^p \mathbb{E}(1_{\tilde{B}}) d\mu = 2 \int_{\tilde{B}\cap(E_+\times\overline{\mathbb{R}})} |g(s) - v(s) f(s)|^p d\nu(s,t) \\ &\leq 2 \int_{\tilde{B}\cap(E_+\times\overline{\mathbb{R}})} |g(s) - t f(s)|^p d\nu(s,t) \leq 2 \int_{E_+\times\overline{\mathbb{R}}} |g(s) - t f(s)|^p d\nu(s,t).\end{aligned}$$

Arguing similarly with $E_- = \{s \in E : g(s) \leq v(s) f(s)\}$ we have

$$\int_E |g - vf|^p d\mu \leq 2 \int_{E\times\overline{\mathbb{R}}} |g(s) - t f(s)|^p d\nu(s,t) \leq 2L_1.$$

We conclude that $\int_{|f|>0} |g - vf|^p d\mu \leq 2L_1$. On the other hand, $\int_{f=0} |g - vf|^p d\mu = \int_{f=0} |g|^p \leq L_2$. The proof is complete. □

STEP 3. Theorem 1 holds if $q < p$ and $q \leq 1$. □

The next Step is also easy. We remark that this time the bound $p < 1$ is required.

STEP 4. Theorem 1 holds if $p < q$ and $p < 1$.

PROOF. We may assume that Φ does not increase supports. Let $\mu = \alpha + \nu$ be the decomposition of μ into its purely atomic and continuous parts. Then Φ sends $L_p(\alpha)$ to $L_0(\alpha)$ and $L_p(\nu)$ to $L_0(\nu)$ and after a moment's reflection one realizes that it suffices to prove the result assuming that μ is either continuous or purely atomic. Let us consider the two cases separately.

First, suppose μ non-atomic. Then there is a constant $M = M(p,q)$ such that whenever $\Phi : L_p \to L_0$ is quasi-linear from L_p to L_q, there is a linear map $\ell : L_p \to L_0$ so that $\|\Phi(f) - \ell(f)\|_q \leq MQ[\Phi]\|f\|_p$ —a specialization of [**7**, Theorem 3.6 (ii)]. We must check that ℓ is in fact a morphism of L_∞-modules. But, if we fix $a \in L_\infty$, the linear map $f \in L_p \mapsto \ell(af) - a\ell(f) \in L_q$ is bounded and since every operator from L_p to L_q is zero we have $\ell(af) = a\ell(f)$ and we are done.

Now, suppose μ is purely atomic. There is no loss of generality in assuming that each atom has mass one, so that $L_r(\mu) = \ell_r(I)$ for $r > 0$ and $L_0(\mu) = \mathbb{R}^I$. We define $\phi : I \to \mathbb{R}$ by the formula $\Phi(e_i) = \phi(i)e_i$, where (e_i) is the unit basis of $\ell_p(I)$. We claim that

$$\|\Phi(f) - \phi f\|_q \leq MC[\Phi]\|f\|_p, \tag{3}$$

where M depends only on p and q. Indeed, we known from [**7**, Lemma 3.4] that if $f = \sum_i \lambda_i e_i$ is finitely supported, then

$$\left\| \Phi\left(\sum_i \lambda_i e_i\right) - \sum_i \lambda_i \Phi(e_i) \right\|_q \leq MQ[\Phi]\|f\|_p,$$

where $M = M(p,q) = \left(\sum_{n=1}^\infty (2/n)^{r/p}\right)^{1/r}$, with $r = \min(1,q)$. Thus (3) holds for finitely supported f. If $f \in \ell_p(I)$ is arbitrary as $\operatorname{supp} f$ is countable there is an increasing sequence (u_n) of finitely supported idempotents in $\ell_\infty(I)$ such that $\operatorname{supp} f = \bigcup_n \operatorname{supp} u_n$. One has $\|\Phi(u_n f) - \phi u_n f\|_q \leq MC[\Phi]\|f\|_p$ and since $\|\Phi(u_n f) - u_n\Phi(f)\|_q \leq C[\Phi]\|f\|_p$ and $\operatorname{supp}\Phi(f) \subset \operatorname{supp} f$, we have

$$\|\Phi(f) - \phi f\|_q \leq \limsup_{n\to\infty} \|u_n(\Phi(f) - \phi f)\|_q \leq \Delta_q C[\Phi](1+K)\|f\|_p,$$

which completes the proof. □

At this point we have proved Theorem 1 for $0 < q \leq 1$. The following result allows us to shift the parameters p and q from the locally convex zone to the "Kalton zone" (cf. [**5**, Section 2]).

Lemma 5. *Let $\Phi \in \mathcal{C}(L_p, L_q)$ have centralizer constant* 1 *and let $r \in (0,\infty)$. We define $\tilde{p}$ and $\tilde{q}$ by letting $\tilde{p}^{-1} = p^{-1} + r^{-1}$ and $\tilde{q}^{-1} = q^{-1} + r^{-1}$.*

(a) *The map $\Psi : L_{\tilde{p}} \to L_0$ given by $\Psi(f) = u|f|^{\tilde{p}/r}\Phi(|f|^{\tilde{p}/p})$ is a centralizer from $L_{\tilde{p}}$ to $L_{\tilde{q}}$, with $C[\Psi] \leq 1$.*

(b) *Moreover, Ψ is trivial if and only if Φ is trivial. Precisely, if $\phi \in L_0$ and $D \geq 0$ are such that $\|\Psi f - \phi f\|_{\tilde{q}} \leq D\|f\|_{\tilde{p}}$ for $f \in L_{\tilde{p}}$, then $\|\Phi(f) - \phi f\|_q \leq \Delta_{\tilde{q}}(8\Delta_p\Delta_{\tilde{q}} + D)\|f\|_p$ for $f \in L_p$.*

Proof. (a) Take $f \in L_{\tilde{p}}$ and $a \in L_\infty$, with polar decompositions $u|f|$ and $v|a|$ respectively. Then,

$$\begin{aligned}\|\Psi(af) - a\Psi f\|_{\tilde{q}} &= \|uv|a|^{\tilde{p}/r}|f|^{\tilde{p}/r}\Phi(|a|^{\tilde{p}/p}|f|^{\tilde{p}/p}) - au|f|^{\tilde{p}/r}\Omega(|f|^{\tilde{p}/p})\|_{\tilde{q}} \\ &= \||a|^{\tilde{p}/r}|f|^{\tilde{p}/r}\Phi(|a|^{\tilde{p}/p}|f|^{\tilde{p}/p}) - |a||f|^{\tilde{p}/r}\Omega(|f|^{\tilde{p}/p})\|_{\tilde{q}} \\ &\leq \||f|^{\tilde{p}/r}\|_r\||a|^{\tilde{p}/r}\Phi(|a|^{\tilde{p}/p}|f|^{\tilde{p}/p}) - |a|\Omega(|f|^{\tilde{p}/p})\|_q \\ &\leq \||f|^{\tilde{p}/r}\|_r\||a|^{\tilde{p}/r}\|_\infty\|\Phi(|a|^{\tilde{p}/p}|f|^{\tilde{p}/p}) - |a|^{\tilde{p}/p}\Omega(|f|^{\tilde{p}/p})\|_q \\ &\leq \||f|^{\tilde{p}/r}\|_r\||a|^{\tilde{p}/r}\|_\infty\||a|^{\tilde{p}/p}\|_\infty\||f|^{\tilde{p}/p}\|_p \\ &= \|a\|_\infty\|f\|_{\tilde{p}}.\end{aligned}$$

(b) That Ψ is trivial when Φ is trivial is obvious. We show the converse. Following [**10**, Proof of Theorem 5.1] we prove first that if $f_1, f_2 \in L_p$ and $g_1, g_2 \in L_r$ are such that $f_1 g_1 = f_2 g_2$, then

$$\|\Phi(f_1)g_1 - \Phi(f_2)g_2\|_{\tilde{q}} \leq 4\Delta_p\Delta_{\tilde{q}}\left(\|f_1\|_p\|g_1\|_r + \|f_2\|_p\|g_2\|_r\right). \tag{4}$$

Indeed, let $h = f_1 g_1 = f_2 g_2$ and take $f = |f_1| + |f_2|$. Then from $\|\Phi(f_i) - f_i f^{-1}\Phi(f)\|_q \leq \|f\|_p$ we get

$$\|\Phi(f_i)g_i - hf^{-1}\Phi(f)\|_{\tilde{q}} \leq \|f\|_p\|g_i\|_r \qquad (i = 1, 2).$$

And so,

$$\|\Phi(f_1)g_1 - \Phi(f_2)g_2\|_{\tilde{q}} \leq \Delta_{\tilde{q}}\|f\|_p(\|g_1\|_r + \|g_2\|_r) \leq 2\Delta_{\tilde{q}}\Delta_p(\|f_1\|_p + \|f_2\|_p)(\|g_1\|_r + \|g_2\|_r).$$

By homogeneity of Φ we also have

$$\|\Phi(f_1)g_1 - \Phi(f_2)g_2\|_{\tilde{q}} \leq 2\Delta_{\tilde{q}}\Delta_p(\|f_1\|_p + \alpha\|f_2\|_p)(\|g_1\|_r + \alpha^{-1}\|g_2\|_r)$$

for every $\alpha > 0$.

Minimizing the right-hand side of the preceding inequality over α, we get

$$\begin{aligned}\|\Phi(f_1)g_1 - \Phi(f_2)g_2\|_{\tilde{q}} &\leq 2\Delta_{\tilde{q}}\Delta_p\left(\|f_1\|_p^{1/2}\|g_1\|_r^{1/2} + \|f_2\|_p^{1/2}\|g_2\|_r^{1/2}\right)^2 \\ &\leq 4\Delta_{\tilde{q}}\Delta_p(\|f_1\|_p\|g_1\|_r + \|f_2\|_p\|g_2\|_r)\end{aligned}$$

and (4) follows.

Now, assume $\|\Psi(h) - \phi h\|_{\tilde{q}} \leq D\|f\|_{\tilde{p}}$ for some $\phi \in L_0$ some $D \geq 0$ and every $h \in L_{\tilde{p}}$. Then

$$\|\Psi(fg) - \phi fg\|_{\tilde{q}} \leq D\|f\|_p\|g\|_r \qquad (f \in L_p, g \in L_r)$$

and, according to (4),

$$\|\Psi(fg) - \Phi(f)g\|_{\tilde{q}} \leq 8\Delta\tilde{q}\Delta_p\|f\|_p\|g\|_r.$$

Thus,

$$\|\Phi(f)g - \phi fg\|_{\tilde{q}} \leq \Delta_{\tilde{q}}(8\Delta_{\tilde{q}}\Delta_p + D)\|f\|_p\|g\|_r$$

and so

$$\|\Phi(f) - \phi f\|_q = \sup_{\|g\|_r \leq 1}\|\Phi(f)g - \phi fg\|_{\tilde{q}} \leq \Delta_{\tilde{q}}(8\Delta_{\tilde{q}}\Delta_p + D)\|f\|_p.$$

□

Step 4. Theorem 1 holds in all cases.

Proof. We known that the result is true for $q \leq 1$. If $q > 1$ the result follows from the case $q = 1$ and Lemma 5 taking r so that $1 = r^{-1} + q^{-1}$. □

3. Basic applications

3.1. Isomorphisms of spaces of centralizers. Now we turn our attention to the description of the action of the functors $\mathrm{Hom}(-, L_r)$ and $\mathrm{Hom}(L_s, -)$ on centralizers. We begin with the contravariant case.

COROLLARY 1. *Let $p, q, r \in (0, \infty)$ be such that $p^{-1} + q^{-1} = r^{-1}$ and $\Phi \in \mathcal{C}(L_p)$. Then there is $\Gamma \in \mathcal{C}(L_q)$ and a constant K such that*

$$\|g\Phi(f) + \Gamma(g)f\|_r \leq K\|g\|_q\|f\|_p \qquad (g \in L_q, f \in L_p). \tag{5}$$

Moreover $L_q \oplus_\Gamma L_q$ is isomorphic to $\mathrm{Hom}(L_p \oplus_\Phi L_p, L_r)$.

PROOF. Let Φ be a centralizer on L_p. Given $g \in L_q = \mathrm{Hom}(L_p, L_r)$, consider the mapping $f \in L_p \mapsto g\Phi(f) \in L_0$. Clearly, this is a centralizer from L_p to L_r, with constant at most $\|g\|_q C[\Phi]$ and so there is $\gamma_g \in L_0$ such that $\|g\Phi(f) + \gamma_g f\|_r \leq MC[\Phi]\|g\|_q\|f\|_p$ for all $f \in L_p$. Thus we can define a mapping $\Gamma : L_q \to L_0$ just taking $\Gamma(g) = \gamma_g$. Of course this can be done homogeneously and we have the estimate in (5). It is easily seen that Γ is a centralizer on L_q. Indeed, take $a \in L_\infty$ and $g \in L_q$. Then

$$\begin{aligned}
\|\Gamma(ag) - a\Gamma g\|_q &= \sup_{\|f\|_p \leq 1} \|(\Gamma(ag) - a\Gamma g)\|_r \\
&= \sup_{\|f\|_p \leq 1} \|\Gamma(ag)f + ag\Phi f - ag\Phi f + g\Phi(af) - g\Phi(af) - (\Gamma g)af\|_r \\
&\leq MC[\Phi]\|a\|_\infty\|g\|_q.
\end{aligned}$$

Let us identify $L_q \oplus_\Gamma L_q$ with $\mathrm{Hom}(L_p \oplus_\Phi L_p, L_r)$. If $(g', g) \in L_q \oplus_\Gamma L_q$ and $(f', f) \in L_p \oplus_\Phi L_p$, then

$$\begin{aligned}
\|g'f + gf'\|_r &= \|g'f - (\Gamma g)f + (\Gamma g)f + g\Phi f - g\Phi f + gf'\|_r \\
&\leq C(\|g' - \Gamma g\|_q\|f\|_p + \|g\|_q\|f\|_p + \|g\|_q\|f' - \Phi f\|_p), \\
&\leq M(\|(g', g)\|_\Gamma\|(f', f)\|_\Phi)
\end{aligned}$$

Thus we may define $u : L_q \oplus_\Gamma L_q \to \mathrm{Hom}(L_p \oplus_\Phi L_p, L_r)$ by letting $u(g', g)(f', f) = g'f + gf'$. We have just see that u is a continuous homomorphism and it is clear that we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_q & \longrightarrow & L_q \oplus_\Gamma L_q & \longrightarrow & L_q & \longrightarrow & 0 \\
 & & \| & & \big\downarrow u & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}(L_p, L_r) & \xrightarrow{\pi^*} & \mathrm{Hom}(L_p \oplus_\Phi L_p, L_r) & \xrightarrow{\imath^*} & \mathrm{Hom}(L_p, L_r) & \longrightarrow & 0
\end{array}$$

This implies that u is an isomorphism of quasi-Banach modules. □

Let us consider the transformation $\Phi \in \mathcal{C}(L_p) \longmapsto \Gamma \in \mathcal{C}(L_q)$ in more detail. Although there is some arbitrariness in the definition of Γ, if Γ' is another centralizer such that

$$\|g\Phi(f) + \Gamma'(g)f\|_r \leq K\|g\|_q\|f\|_p \qquad (g \in L_q, f \in L_p).$$

for some K and all $f \in L_p, g \in L_q$, then $\Gamma' \approx \Gamma$. So, after a moment's reflection we see that (5) defines a mapping $\mathcal{C}_\approx(L_p) \to \mathcal{C}_\approx(L_q)$ we may denote $\mathrm{Hom}(-, L_r)$. It should be obvious that this map is in fact a morphism over L_∞. Actually, it is an isomorphism in view or the symmetric roles of Φ and Γ in (5).

In a similar vein we have the following.

Corollary 2. *Let $p, q, r \in (0, \infty)$ be such that $q^{-1} = p^{-1} + r^{-1}$. If Ψ is a centralizer on L_q, then there is $\Phi \in \mathcal{C}(L_p)$ and a constant K such that*

$$\|\Psi(fg) - \Phi(f)g\|_q \leq K\|f\|_p\|g\|_r \qquad (f \in L_p, g \in L_r). \tag{6}$$

Moreover $L_p \oplus_\Phi L_p$ is isomorphic to $\operatorname{Hom}(L_r, L_q \oplus_\Psi L_q)$.

Proof. Suppose Ψ is a centralizer on L_q. Given $f \in L_p = \operatorname{Hom}(L_r, L_q)$ we consider the map $g \in L_r \mapsto \Psi(fg) \in L_0$. This is a centralizer from L_r to L_q with centralizer constant at most $C[\Psi]\|f\|_p$ and so we may choose $\phi_f \in L_0$ in such a way that $\|\Psi(fg) - \phi_f g\|_q \leq MC[\Psi]\|f\|_p\|g\|_r$. Letting $\Phi(f) = \phi_f$ homogeneously we obtain a map $\Phi : L_p \to L_0$ which satisfies (6). We left to the reader the verification that Φ is a centralizer. We define a mapping $v : L_p \oplus_\Phi L_p \to \operatorname{Hom}(L_r, L_q \oplus_\Psi L_q)$ by the formula

$$v(f', f)g = (f'g, fg) \qquad (f', f) \in L_p \oplus_\Phi L_p, g \in L_r).$$

This is a homomorphism, since

$$\begin{aligned}\|(f'g, fg)\|_\Psi &= \|f'g - \Psi(fg)\|_q + \|fg\|_q \\ &= \|f'g - \Phi(f)g + \Phi(f)g - \Psi(fg)\|_q + \|fg\|_q \\ &\leq M(\|f' - \Phi(f)\|_p\|g\|_r + \|f\|_p\|g\|_r) \\ &= M\|(f', f)\|_\Phi\|g\|_r.\end{aligned}$$

As before, the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_p & \longrightarrow & L_p \oplus_\Phi L_p & \longrightarrow & L_p & \longrightarrow & 0 \\ & & \| & & \downarrow v & & \| & & \\ 0 & \longrightarrow & \operatorname{Hom}(L_r, L_q) & \xrightarrow{\pi_*} & \operatorname{Hom}(L_r, L_q \oplus_\Psi L_q) & \xrightarrow{\imath_*} & \operatorname{Hom}(L_r, L_q) & \longrightarrow & 0 \end{array}$$

which implies that v is actually an isomorphism of quasi-Banach modules over L_∞. □

The solution to Kalton's problem we mentioned in the Introduction comes now.

Corollary 3. *Suppose $0 < q < p < \infty$ and let $\Psi \in \mathcal{C}(L_q)$. Then there is $\Phi \in \mathcal{C}(L_p)$ and a constant K such that*

$$\|\Psi(f) - u|f|^{q/r}\Phi(|f|^{q/p})\|_q \leq K\|f\|_q \qquad (f \in L_q),$$

where $u|f|$ is the polar decomposition of f and r is given by $q^{-1} = p^{-1} + r^{-1}$.

Proof. Apply Corollary 2 to Ψ and look at (6). □

3.2. Homology. Let us explain the homological meaning of the results we have proved so far. Needless to say, Theorem 1 means that $\operatorname{Ext}(L_p, L_q)$ vanishes unless $q = p$. Now suppose we are given an extension

$$0 \longrightarrow L_p \xrightarrow{\imath} Z \xrightarrow{\pi} L_p \longrightarrow 0 \tag{7}$$

Fixing $r < p$ and applying $\operatorname{Hom}(-, L_r)$ to (7) one obtains an exact sequence (see [**2**, Appendix 7])

$$0 \longrightarrow \operatorname{Hom}(L_p, L_r) \xrightarrow{\pi^*} \operatorname{Hom}(Z, L_r) \xrightarrow{\imath^*} \operatorname{Hom}(L_p, L_r) \longrightarrow \operatorname{Ext}(L_p, L_r) \tag{8}$$

But $\mathrm{Hom}(L_p, L_r) = L_q$, where $q^{-1} + p^{-1} = r^{-1}$ and since $\mathrm{Ext}(L_p, L_r) = 0$ the sequence (8) can be seen as a self-extension of L_q which corresponds to the centralizer Γ appearing in Corollary 1.

Similarly if $0 \longrightarrow L_q \xrightarrow{\imath} Z \xrightarrow{\pi} L_q \longrightarrow 0$ is an extension and we apply $\mathrm{Hom}(L_r, -)$ with $r > q$ we obtain another exact sequence

$$(9) \quad 0 \longrightarrow \mathrm{Hom}(L_r, L_q) \xrightarrow{\imath_*} \mathrm{Hom}(L_r, Z) \xrightarrow{\pi_*} \mathrm{Hom}(L_r, L_q) \longrightarrow \mathrm{Ext}(L_r, L_q)$$

As before, $\mathrm{Ext}(L_r, L_q) = 0$ and $\mathrm{Hom}(L_r, L_q) = L_p$ where $p^{-1} + r^{-1} = q^{-1}$, so that (9) is a self-extension of L_p which corresponds to the centralizer Φ of Corollary 2.

The construction of Lemma 5 works as a tensor product and indeed it is. Let X and Y be quasi-Banach modules over L_∞. The tensor product of X and Y (over L_∞) is a quasi-Banach module $T = X \otimes Y$ (recall that our default category is that of quasi-Banach modules over L_∞) together with a bihomomorphism $\theta : X \times Y \to T$ having the following 'universal' property: if E is a quasi-Banach space and $B : X \times Y \to E$ is a bilinear operator which is 'balanced' in the sense that $B(ax, y) = B(x, ay)$ for all $a \in L_\infty, x \in X, y \in Y$, then there is a linear operator $\beta : T \to E$ such that $B = \beta \circ \theta$. It is not hard to see that if $p^{-1} + r^{-1} = q^{-1}$, then $L_p \otimes L_r = L_q$. Moreover, if $\Phi \in \mathcal{C}(L_p)$ and $Z = L_p \oplus_\Phi L_p$, then $L_q \oplus_\Psi L_q = Z \otimes L_r$, where Ψ is related to Φ is in Lemma 5. Therefore, the functor $- \otimes L_r$ is "exact" at self-extensions of L_p and Corollaries 2 and 3 together show that $- \otimes L_r$ and $\mathrm{Hom}(L_r, -)$ are "adjoint" functors, at least on self-extensions of Lebesgue spaces. Details will appear elsewhere.

3.3. Duality. Just for fun, let us take $r = 1$ in Corollary 1 so that p and q are conjugate exponents. Fix $\Phi \in \mathcal{C}(L_p)$ and consider the centralizer $\Gamma \in \mathcal{C}(L_q)$ defined by (5). We claim that $L_q \oplus_\Gamma L_q$ is isomorphic to the dual of $L_p \oplus_\Phi L_p$ under the pairing

$$\langle (g', g), (f' f) \rangle = \int_S g' f + g f' d\nu.$$

Taking a look to the proof of Corollary 1 se see that

$$|\langle (g', g), (f' f) \rangle| \leq \|g' f + g f'\|_1 \leq M(\|(g', g)\|_\Gamma \|(f', f)\|_\Phi.$$

This yields an operator w making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_q & \longrightarrow & L_q \oplus_\Gamma L_q & \longrightarrow & L_q & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \| & & \\ 0 & \longrightarrow & L_p^* & \xrightarrow{\pi^*} & (L_p \oplus_\Phi L_p)^* & \xrightarrow{\imath^*} & L_p^* & \longrightarrow & 0 \end{array}$$

Now suposse $p = q = 2$. Then (5) becomes $\|g\Phi f + (\Gamma g) f\|_1 \leq K\|g\|_2 \|f\|_2$. But taking $f_1 = g_2 = f$ and $f_2 = g_1 = g$ in (4) we see that $\|g\Phi f - (\Phi g) f\|_1 \leq 4C[\Phi]\|g\|_2\|f\|_2$. It follows that $\Gamma \approx -\Phi$. We have proved the following generalization of [**1**, Proposition 16.12] (cf. [**14**, Theorem 5.1]).

Corollary 4. *If Φ is any centralizer on L_2, then $L_2 \oplus_\Phi L_2$ is isomorphic to its own dual under the pairing $\langle (g', g), (f' f) \rangle = \int_S g' f - g f' d\nu$, where $(g', g), (f', f) \in L_2 \oplus_\Phi L_2$.*

We do not know if all "twisted Hilbert spaces" are isomorphic to their duals.

4. Further applications

Now we give some miscellaneous applications of Corollary 3. These extend some results already known for $p \geq 1$ (or $p > 1$) to any p. First recall that $\Phi \in \mathcal{C}(L_p)$ is said to be symmetric if there is a constant S such that $\|\Phi(f \circ \sigma) - (\Phi f) \circ \sigma\|_p \leq S\|f\|_p$ for every $f \in L_p$ and all measure-preserving automorphisms σ of $\mathcal{S}$. From now on the ground field is $\mathbb{C}$.

4.1. Factorization (complex interpolation). The following result is stated here without any reference to interpolation theory. That this is really a generalization of [**11**, Theorem 7.6] needs some explanations the reader will find in [**13**, Sections 8 to 11] and [**11**, p. 487]. The key fact that Calderón's formula $[X, Y]_\theta = X^{1-\theta}Y^\theta$ survives in our (quasi-Banach function space) setting for analytically-convex spaces can be seen in [**12**, Theorem 3.4]. Analytic convexity is equivalent to lattice-convexity for quasi-Banach function spaces [**9**, Theorem 4.4].

Corollary 5. *Let Ψ be a real centralizer on L_p. Then there is a factorization $L_p = UV$ and constants c and M such that*

$$\|\Psi f - cf(\log v - \log u)\|_p \leq M\|f\|_p \qquad (f \in L_p),$$

whenever $f = uv$ is an (almost) optimal factorization in the sense that $(1+\varepsilon)\|f\|_p \geq \|u\|_U\|v\|_V$.

Proof. If $1 < p < \infty$ this was proved by Kalton in [**11**, Theorem 7.6] (see also [**13**, Theorem 11.6]), so let us assume $p < 2$. Take r so that $L_p = L_2L_r$ (that is, $p^{-1} = 2^{-1} + r^{-1}$) and use Corollary 3 to get a (real) centralizer Φ on L_2 such that

$$\left\|\Psi f - (\Phi(f^{p/2}))f^{p/r}\right\|_p \leq M\|f\|_p \qquad (f \geq 0).$$

We know that there is a factorization $L_2 = XY$ and constants M and c such that

$$\|\Phi f - cf(\log x - \log y)\|_2 \leq M\|f\|_2 \qquad (f \in L_2),$$

as long as $f = xy$ is an almost optimal factorization in the sense that $(1+\varepsilon)\|f\|_2 \geq \|x\|_X\|y\|_Y$. Take now $U = L_{2r}X$ and $V = L_{2r}Y$ and check the details. □

4.2. Commutator estimates. As explained in [**13**, Sections 8 to 11] Corollary 5 implies that, with respect to the complex interpolation method, one has $L_p = [X_0, X_1]_{1/2}$ (with equivalent norms) where $X_0 = U^2, X_1 = V^2$ and Φ is strongly equivalent to the corresponding derivation. Now, proceeding as in the proof of [**11**, Corollary 7.8], we get the following version of [**10**, Theorem 6.10] for L_p when $p < 1$, and where $[T, \Phi](f) = T(\Phi(f)) - \Phi(Tf)$. The Boyd type interpolation result (for quasi-Banach function spaces) required here can be found, for instance, in [**8**, Theorem 1.3].

Corollary 6. *Suppose μ is either Lebesgue measure on an interval or counting measure on the integers. Suppose $0 < p_0 < p < p_1 < \infty$ and that T is an operator of strong types (p_0, p_0) and (p_1, p_1). Then for any symmetric centralizer Φ on L_p there is a constant C so that $\|[T, \Phi](f)\|_p \leq C\|f\|_p$ for all $f \in L_p$.*

4.3. Hardy classes. Let $\mathbb{T}$ be the unit circle of the complex plane, with its Haar measure. As usual we denote by H_p the closed subspace of $L_p(\mathbb{T})$ generated by the polynomials.

COROLLARY 7. *Let Φ be a symmetric centralizer on $L_p(\mathbb{T})$. Then there is a constant C (depending on Φ) such that, whenever $f \in H_p$ and $\Phi(f) \in L_p(\mathbb{T})$ (which is always the case if $f \in L_r$ for some $r > p$, in particular if f is a trigonometric polynomial), one has* $\operatorname{dist}(\Phi f, H_p) \leq C\|f\|_p$.

PROOF. Combine the corresponding result for Banach spaces proved by Kalton in [**10**, Theorem 7.3] with Corollary 3. Take into account that $H_p = H_q H_r$ if $1/p = 1/q + 1/r$. □

Thus, a symmetric centralizer on L_p gives rise to a self-extension of H_p (and of L_p/H_p) in the category of quasi-Banach H_∞-modules.

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