

STABILITY CONSTANTS AND THE HOMOLOGY OF QUASI-BANACH SPACES

BY

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ABSTRACT

We affirmatively solve the main problems posed by Laczkovich and Paulin in *Stability constants in linear spaces*, *Constructive Approximation* **34** (2011), 89–106 (do there exist cases in which the second Whitney constant is finite while the approximation constant is infinite?) and by Cabello and Castillo in *The long homology sequence for quasi-Banach spaces, with applications*, *Positivity* **8** (2004), 379–394 (do there exist Banach spaces X, Y for which $\text{Ext}(X, Y)$ is Hausdorff and nonzero?). In fact, we show that these two problems are the same.

1. Introduction

In this paper we solve the main problems posed [2] and in [11]. This introductory section is devoted to explaining the problems and the solutions we present. The reader is referred to Section 2 for all unexplained terms and notation.

The underlying idea of [2] was to study the spaces $\text{Ext}(X, Y)$ of exact sequences of quasi-Banach spaces $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ by means of certain nonlinear maps $F : X \rightarrow Y$ called quasilinear maps. In that paper we introduced (semi-quasi-normed) linear topologies in the spaces $\text{Ext}(X, Y)$. Such topologies are completely natural in this setting since they make continuous all

* This research has been supported in part by project MTM2010-20190-C02-01 and the program Junta de Extremadura GR10113 IV Plan Regional I+D+i, Ayudas a Grupos de Investigación.

Received May 4, 2012 and in revised form September 29, 2012

the linear maps appearing in the associated homology sequences. We encountered considerable difficulties in studying the Hausdorff character of $\text{Ext}(X, Y)$ for arbitrary spaces X, Y , although we managed to produce some pairs of quasi-Banach spaces X and Y for which $\text{Ext}(X, Y)$ is (nonzero and) Hausdorff. Those examples invariably required that $\mathfrak{L}(X, Y) = 0$, which is impossible in a Banach space context. Thus the main problem left open in [2, p. 387] was to know if there are Banach spaces X and Y for which $\text{Ext}(X, Y)$ is nonzero and Hausdorff. We will show in this paper that this is indeed the case.

This paper also solves the main problem in [11]. In fact, it was this problem that suggested to us the approach of the present paper and the “form” that the examples in Section 6 had to have. In [11] Laczkovich and Paulin studied the stability of approximately affine and Jensen functions on convex sets of linear spaces. They show that, in many respects, it is enough to work the case where the convex set is the unit ball of a Banach space. Reduced to its basic elements the main problem in [11] can be stated as follows. Let B a convex set and Y a Banach space. A function $f : B \rightarrow Y$ is approximately affine if

$$\|f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)\|_Y \leq 1$$

for every $x, y \in B$ and every $t \in [0, 1]$. Suppose there is a constant $A_0 = A_0(B, Y)$ such that for every bounded approximately affine function $f : B \rightarrow Y$ there is a true affine function $a : B \rightarrow Y$ such that $\|f(x) - a(x)\| \leq A_0$ for every $x \in B$. Does it follow that for every (in general unbounded) approximately affine function $f : B \rightarrow Y$ there is an affine function a such that $\sup_{x \in B} \|f(x) - a(x)\| < \infty$?

We show in Section 8 that the Examples in Section 6 provide a negative answer to this problem. The idea is that if B is the unit ball of X , then every approximately affine function $f : B \rightarrow Y$ is (up to a bounded perturbation) the restriction of some quasilinear map $F : X \rightarrow Y$. This makes $A_0(B, Y)$ nearly proportional to $K_0(X, Y)$, a certain parameter depending only on the behavior of the bounded quasilinear maps $X \rightarrow Y$ (see Section 2.2 below); and thus it suffices to get a pair of Banach spaces with $K_0(X, Y)$ finite and admitting a quasilinear map $F : X \rightarrow Y$ that cannot be approximated by linear maps.

Let us describe the organization of the paper. Section 2 is preliminary: it contains the necessary background on quasilinear maps and extensions. In Section 3 we give a seemingly new representation of $\text{Ext}(X, Y)$, denoted by $\text{Ext}^\varpi(X, Y)$, which is based on a relatively projective presentation of X and

carries a natural topology. Section 4 is devoted to showing that the “old” functor Ext and the “new” functor Ext^ϖ are naturally equivalent, both algebraically and topologically. In Section 5 we give the basic criterion to determine when $\text{Ext}(X, Y)$ is Hausdorff. It turns out that $\text{Ext}(X, Y)$ is Hausdorff if and only if $K_0(X, Y)$ is finite. In Section 6 we arrive at the main counterexamples: we show that $\text{Ext}(X, Y)$ is nonzero and Hausdorff if, for instance, $X = \ell_p(I)$ with $1 < p < \infty$ and I uncountable and $Y = c_0$ is the space of all null sequences with the sup norm. The results of Section 7 confirm that $\text{Ext}(X, Y)$ has a quite strong tendency to not being Hausdorff.

The paper has been organized so that the interested reader can go straight to the solution of the Laczkovich–Paulin problem. The shortest path is to read first Sections 2.2 and 2.3, then Section 6 and finally Section 8.

2. Background on quasilinear maps and extensions of quasi-Banach spaces

2.1. NOTATION. Let X be a (real or complex) linear space. A **semi quasi-norm** is a function $\varrho : X \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (a) $\varrho(\lambda x) = |\lambda|\varrho(x)$ for every $x \in X$ and every scalar λ .
- (b) There is a constant Δ such that $\varrho(x + y) \leq \Delta(\varrho(x) + \varrho(y))$ for every $x, y \in X$.

A **semi quasi-normed space** is a linear space X furnished with a semi quasi-norm ϱ which gives rise to a linear topology, namely the least linear topology for which the set $B_X = \{x \in X : \varrho(x) \leq 1\}$ is a neighborhood of the origin. A **semi quasi-Banach space** is a complete semi quasi-normed space. If $\varrho(x) = 0$ implies $x = 0$, then ϱ is said to be a **quasi-norm** and X is Hausdorff.

Throughout the paper we denote by Δ_ϱ (or Δ_X if there is no risk of confusion) the **modulus of concavity** of (X, ϱ) , that is, the least constant Δ for which (b) holds.

Let X and Y be quasi-Banach spaces. A mapping $F : X \rightarrow Y$ is said to be **homogeneous** if $F(\lambda x) = \lambda F(x)$ for every scalar λ and every $x \in X$. A homogeneous map F is **bounded** if there is a constant C such that $\|F(x)\|_Y \leq C\|x\|_X$ for all $x \in X$; equivalently if it is uniformly bounded on the unit ball. We write $\|F\|$ for the least constant C for which the preceding inequality holds. Clearly, $\|F\| = \sup_{\|x\| \leq 1} \|F(x)\|$. This is coherent with the standard notation for the (quasi-) norm of a linear operator. Throughout the paper we denote by

Is this correct?

$\mathfrak{L}(X, Y)$ the space of all bounded, linear operators from X to Y endowed with the operator quasi-norm.

2.2. QUASILINEAR MAPS. Let X and Y be quasi-Banach spaces. A map $F : X \rightarrow Y$ is said to be **quasilinear** if it is homogeneous and satisfies an estimate

$$(1) \quad \|F(x + y) - F(x) - F(y)\|_Y \leq Q (\|x\|_X + \|y\|_X)$$

for some constant Q and all $x, y \in X$. The smallest constant Q above shall be denoted by $Q(F)$.

All linear maps, continuous or not, are quasilinear and so are the bounded (homogeneous) maps.

Let us say that a quasilinear map F is **trivial** if it is the sum of a linear map L and a bounded homogeneous map B . This happens if and only if F is at finite “distance” from L in the sense that $\|F - L\| < \infty$.

Let us introduce the “approximation constants” for quasilinear maps as follows.

Definition 1: Given quasi-Banach spaces X and Y we denote by $K_0(X, Y)$ the infimum of those constants C for which the following statement is true: if $F : X \rightarrow Y$ is a bounded quasilinear map, then there is a linear map $L : X \rightarrow Y$ such that $\|F - L\| \leq CQ(F)$.

We define $K(X, Y)$ analogously just omitting the word ‘bounded’.

Observe that $K_0(X, Y)$ does not vary if we replace “bounded” by “trivial” in the definition.

2.3. TWISTED SUMS. The link between quasilinear maps and the homology of (quasi-) Banach spaces is the following construction, due to Kalton [6] and Ribe [14]. Suppose $F : X \rightarrow Y$ is quasilinear. We consider the product space $Y \times X$ and we furnish it with the following quasinorm:

$$\|(y, x)\|_F = \|y - F(x)\|_Y + \|x\|_X.$$

Following a longstanding tradition, we denote the resulting quasi-Banach space by $Y \oplus_F X$. It is obvious that the operator $\iota : Y \rightarrow Y \oplus_F X$ sending y to $(y, 0)$ is an isometric embedding, so we may regard Y as a subspace of $Y \oplus_F X$. The connection between X and $Y \oplus_F X$ is less obvious. Consider the operator $\pi : Y \oplus_F X \rightarrow X$ defined by $\pi(y, x) = x$. We have $\|\pi\| \leq 1$, by the very

definition of the quasi-norm in $Y \oplus_F X$. Besides, π maps the unit ball of $Y \oplus_F X$ onto the unit ball of X since $\pi(F(x), x) = x$ and $\|(F(x), x)\|_F = \|x\|_X$. Hence π is a quotient map and, moreover, its kernel equals the image of ι , so $X = (Y \oplus_F X)/\iota(Y)$. Most of the preceding information can be rephrased by saying that the diagram

$$(2) \quad 0 \longrightarrow Y \xrightarrow{\iota} Y \oplus_F X \xrightarrow{\pi} X \longrightarrow 0$$

is a short exact sequence of quasi-Banach spaces (“exact” means that the kernel of each arrow coincides with the image of the preceding one), or else, that $Y \oplus_F X$ is a twisted sum of Y and X .

One may wonder under which conditions the sequence (2) splits, that is, there is an operator $p : Y \oplus_F X \rightarrow Y$ such that $p \circ \iota = \mathbf{I}_Y$ (equivalently, the subspace $\iota(Y)$ is complemented in $Y \oplus_F X$ through $\iota \circ p$). The answer is quite simple: (2) splits if and only if F is trivial. Indeed, every linear map, continuous or not, from $Y \oplus_F X$ to Y can be written as $p(y, x) = M(y) - L(x)$, where $M : Y \rightarrow Y$ and $L : X \rightarrow Y$ are linear. Moreover, $p \circ \iota = \mathbf{I}_Y$ if and only if $M = \mathbf{I}_Y$. Let us see that the “projection” $p(y, x) = y - L(x)$ is bounded if and only if L is at finite distance from F . Indeed, if p is bounded, we have

$$\|y - L(x)\|_Y = \|p(y, x)\|_Y \leq \|p\| \|(y, x)\|_F = \|p\| (\|y - F(x)\|_Y + \|x\|_X),$$

and taking $y = F(x)$ we see that $\|F - L\| \leq \|p\|$. The converse is also easy:

$$\begin{aligned} \|p(y, x)\|_Y = \|y - L(x)\|_Y &\leq \Delta_Y (\|y - F(x)\|_Y + \|F(x) - L(x)\|_Y) \\ &\leq \Delta_Y (\|y - F(x)\|_Y + \|F - L\| \|x\|_X) \end{aligned}$$

and so $\|p\| \leq \Delta_Y \max(1, \|F - L\|)$.

Thus quasilinear maps give rise to twisted sums (exact sequences). The converse is also true. Indeed, if we are given a pair of quasi-Banach spaces X and Y and a third space Z containing Y in such a way that $Z/Y = X$, then we can define a quasilinear map from X to Y as follows. First, we take a homogeneous bounded section of the quotient map $\pi : Z \rightarrow Y$, that is, a homogeneous map $B : X \rightarrow Z$ such that if $z = B(x)$, then $\pi(z) = x$, with $\|z\|_Z \leq 2\|x\|_X$ (note that $\|x\|_X = \inf_{\pi(z)=x} \|z\|_Z$). Of course B will not be linear or continuous in general. Then we may take a linear (but probably unbounded) section of π . Finally, we put $F = B - L$. Note that F takes values in Y ; moreover, it is really easy to check that F is quasilinear since for $x, y \in X$ one has

$$F(x + y) - F(x) - F(y) = B(x + y) - B(x) - B(y).$$

It turns out that Z is linearly isomorphic to the twisted sum $Y \oplus_F X$ through the map $u : Z \rightarrow Y \oplus_F X$ given by $u(z) = (z - L(\pi(z)), \pi(z))$ and so F is trivial if and only if Y is complemented in Z , in which case Z is isomorphic to $Y \oplus X$.

2.4. EXTENSIONS. Let X and Y be quasi-Banach spaces. An extension of X by Y is a short exact sequence of quasi-Banach spaces and operators

$$(3) \quad 0 \longrightarrow Y \xrightarrow{\iota} Z \xrightarrow{\pi} X \longrightarrow 0.$$

As $\iota(Y) = \ker \pi$ is closed in Z the operator ι embeds Y as a closed subspace of Z and X is isomorphic to the quotient $Z/\iota(Y)$, by the open mapping theorem. The extension $0 \rightarrow Y \rightarrow Z' \rightarrow X \rightarrow 0$ is said to be equivalent to (3) if there exists an operator $t : Z \rightarrow Z'$ making commutative the diagram

$$(4) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & & & \parallel & & \downarrow t & & \parallel \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

The five-lemma [12, Lemma 3.3] and the open mapping theorem imply that such a t is necessarily an isomorphism and so “being equivalent” is a true equivalence relation. The extension (3) is said to be trivial if it is equivalent to the direct sum $0 \rightarrow Y \rightarrow Y \oplus X \rightarrow X \rightarrow 0$. This happens if and only if it splits, that is, there is an operator $p : Z \rightarrow Y$ such that $p \circ \iota = \mathbf{I}_Y$ (i.e., $\iota(Y)$ is complemented in Z); equivalently, there is an operator $s : X \rightarrow Z$ such that $\pi \circ s = \mathbf{I}_X$.

For every pair of quasi-Banach spaces X and Y , we denote by $\text{Ext}(X, Y)$ the space of all exact sequences (3) modulo equivalence. The set $\text{Ext}(X, Y)$ can be given a linear structure in such a way that the class of trivial sequences corresponds to zero; see [2, Appendix 7]. Thus, $\text{Ext}(X, Y) = 0$ means that every extension of the form (3) is trivial. By [6, Proposition 3.3] $K(X, Y)$ is finite if and only if $\text{Ext}(X, Y) = 0$.

There is a correspondence between extensions and quasilinear maps (sketched in Section 2.3 and considered in full detail in [8, 3] and [2, Section 2]) which takes trivial quasilinear maps into trivial extensions. Moreover, two quasilinear maps induce equivalent extensions if and only if its difference is trivial in which case we will declare it as “equivalent”. From now on we will denote by $\mathcal{Q}(X, Y)$ the space of quasilinear maps from X to Y modulo equivalence.

3. A new construction of Ext

Two “representations” of the spaces $\text{Ext}(X, Y)$ are available so far, namely, the very definition through short exact sequences and the identification with $\mathcal{Q}(X, Y)$ provided by the construction that appears in Section 2.3. To get the results of this paper we will need a new one. To explain it, observe that when one works in Banach spaces things are somewhat simpler since there exist enough projective objects ($\ell_1(I)$ for every set of indices I) and enough injective objects ($L_\infty(\mu)$ -spaces, for instance). In the category \mathbf{Q} of quasi-Banach spaces each projective object is finite-dimensional. This easily follows from the fact, proved by Stiles in [16], that every infinite-dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p when $0 < p < 1$. (See also [9, Chapter 2, Section 3].) This could be a problem for the construction of Ext via projective objects. However, there exist spaces that act as projective, at least for a given couple of quasi-Banach spaces. We base our approach on two facts: the first one is the Aoki–Rolewicz theorem that states that every quasi-Banach space is p -normable for some $p \in (0, 1]$; see [15, Theorem 3.2.1] or [9, Theorem 1.3].

The second fact, proved by Kalton in [6, Theorem 3.5], is that $\text{Ext}(\ell_q(I), Y) = 0$ if $0 < q < 1$ and Y is p -normable for some $p > q$. A ‘formal’ consequence is that any twisted sum of two p -normable spaces is q -normable for every $0 < q < p$ (see [6, Theorem 4.1] for a more general result). Now, let X and Y be fixed quasi-Banach spaces, both p -normable. Take $0 < q < p$ and let I be a set of indices having cardinality $\text{dens}(X)$ so that we can construct a quotient map $\varpi : \ell_q(I) \rightarrow X$. Write $K = \ker \varpi$ and consider the extension

$$(5) \quad 0 \longrightarrow K \xrightarrow{\mathfrak{z}} \ell_q(I) \xrightarrow{\varpi} X \longrightarrow 0.$$

Applying $\mathfrak{L}(-, Y)$ we get the exact sequence (see Theorem 1 in [2])

$$0 \longrightarrow \mathfrak{L}(X, Y) \xrightarrow{\varpi^*} \mathfrak{L}(\ell_q(I), Y) \xrightarrow{\mathfrak{z}^*} \mathfrak{L}(K, Y) \xrightarrow{\omega} \text{Ext}(X, Y) \longrightarrow 0$$

since $\text{Ext}(\ell_q(I), Y) = 0$. The exactness of the preceding sequence encodes the following information:

- Each operator $u : K \rightarrow Y$ gives rise to an extension of X by Y , namely the lower row in the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \ell_q(I) & \longrightarrow & X & \longrightarrow & 0 \\ & & u \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & P & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

(see Section 7 in [2]).

- Two operators $u, v \in \mathfrak{L}(K, Y)$ give equivalent extensions if and only if $u - v$ extends to an operator $\mathfrak{L}(\ell_q(I), Y)$.
- All extensions $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ arise in this way, up to equivalence.

The third point is as follows: if $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ is exact, then Z is q -normable and the quotient map $\varpi : \ell_q(I) \rightarrow X$ can be lifted to an operator $u : \ell_q(I) \rightarrow Z$. The restriction of u to K takes values in Y and therefore one gets a commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & K & \longrightarrow & \ell_q(I) & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow u & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0.
 \end{array}$$

Accordingly we can define a linear space taking

$$\text{Ext}^\varpi(X, Y) = \frac{\mathfrak{L}(K, Y)}{\varpi^*(\mathfrak{L}(\ell_q(I), Y))}.$$

The algebraic part of the equivalence between the three representations: Ext (equivalence classes of exact sequences), \mathcal{Q} (equivalence classes of quasilinear maps) and Ext^ϖ are more or less straightforward and, as soon as one gets some acquaintance with the functor and natural transformation language [12, Chapter I, Section 8], can be formulated as:

PROPOSITION 1: *The functors $\mathcal{Q}, \text{Ext}_{\mathcal{Q}}$ and Ext^ϖ are naturally equivalent acting from the category $\mathbf{Q} \times \mathbf{Q}^{\text{op}}$ to the category \mathbf{V} of vector spaces. ■*

4. Natural equivalences for Ext

More interesting, and essential for our purposes, is to consider what occurs when the spaces are endowed with (natural) linear topologies. The space $\mathcal{Q}(X, Y)$ can be equipped (see Section 3 of [2]) with the semi-quasi-norm

$$Q([F]) = \inf\{Q(G) : G \text{ is equivalent to } F\}.$$

It was proved in [2, Theorem 4] that this makes $\mathcal{Q}(X, Y)$ complete, but rarely Hausdorff. The corresponding topology in $\text{Ext}(X, Y)$ can be obtained as follows. Given an extension

$$(6) \quad 0 \longrightarrow Y \xrightarrow{\iota} Z \xrightarrow{\pi} X \longrightarrow 0,$$

we put

$$\varrho(\iota, \pi) = \inf_B \sup_{x, y \in X} \frac{\|\iota^{-1}(B(x + y) - Bx - By)\|_Y}{\|x\|_X + \|y\|_X},$$

where B runs over all homogeneous bounded sections $B : X \rightarrow Z$ of π . Then set

$$\varrho([\iota, \pi]) = \inf_{(\iota', \pi')} \varrho(\iota', \pi'),$$

where the infimum is taken over those (ι', π') for which

$$0 \longrightarrow Y \xrightarrow{\iota'} Z' \xrightarrow{\pi'} X \longrightarrow 0$$

is equivalent to (6). But the representation we need for Ext is Ext^ϖ . Let us introduce a semi-quasi-norm on $\text{Ext}^\varpi(Z, Y)$ as

$$[[u]]_\varpi = \inf \{ \|u - \varkappa^*(v)\|_{\mathfrak{L}(K, Y)} : v \in \mathfrak{L}(\ell_q(I), Y) \}.$$

The image of \varkappa^* is not (as a rule) closed in $\mathfrak{L}(K, Y)$ and thus the topology induced by $|\cdot|_\varpi$ is not necessarily Hausdorff, which matches with the previous constructions. One has

THEOREM 1: *Ext, \mathcal{Q} and Ext^ϖ are naturally equivalent functors acting from the category $\mathbf{Q} \times \mathbf{Q}^{\text{op}}$ to the category $\mathbf{Q}_{1/2}$ of semi quasi-Banach spaces.*

Proof. That Ext and \mathcal{Q} still are naturally equivalent when regarded as functors $\mathbf{Q} \times \mathbf{Q} \rightsquigarrow \mathbf{Q}_{1/2}$ is nearly obvious. So let us concentrate on the natural equivalence between \mathcal{Q} and Ext^ϖ . Given quasi-Banach spaces X and Y we fix a (relatively projective) presentation for X , as in (5). Let B_0 and L_0 represent, respectively, a homogeneous bounded and a linear selection for ϖ so that $\Phi = B_0 - L_0$ is a quasilinear map associated to (5). We define a linear map from $\mathfrak{L}(K, Y)$ to the space of quasilinear maps $Z \rightarrow Y$ by $\Phi^*(u) = u \circ \Phi$. We have $Q(u \circ \Phi) \leq \|u\|Q(\Phi)$. Suppose u extends to an operator $v : \ell_q(I) \rightarrow Y$. Then the identity

$$u \circ \Phi = v \circ B_0 - v \circ L_0$$

shows that $[u \circ \Phi] = 0$ in $\mathcal{Q}(X, Y)$. Thus we can define η at the couple (X, Y) by

$$\eta_{(X, Y)}[u] = [u \circ \Phi] \quad (u \in \mathfrak{L}(K, Y)).$$

From now on, assume that Y and Z have been fixed so that we can omit the subindex and just write η . Quite clearly

$$Q[\eta([u])] \leq Q(\Phi)[u]_\varpi,$$

and so $\eta : (\text{Ext}^\varpi(X, Y), |\cdot|_\varpi) \rightarrow (\mathcal{Q}(X, Y), Q[\cdot])$ is (linear and) continuous. From now on we will identify $\ell_q(I) = K \oplus_\Phi X$ through the isomorphism $z \mapsto (z + L_0(\varpi(z)), \varpi(z))$ without further mention. Let us check the injectivity of η . Given $u \in \mathfrak{L}(K, Y)$ the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & K \oplus_\Phi X & \xrightarrow{\varpi} & X \longrightarrow 0 \\ & & u \downarrow & & U \downarrow & & \parallel \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_{\Phi^*(u)} X & \xrightarrow{\pi} & X \longrightarrow 0 \end{array}$$

where $U(k, x) = (u(k), x)$. If $u \circ \Phi$ is trivial, composing U with a bounded projection of $Y \oplus_{\Phi^*(u)} Z$ onto Y we get an extension of u to $\ell_q(I)$ and so $[u] = 0$ in $\text{Ext}^\varpi(X, Y)$. Next we show η is onto. If $F : X \rightarrow Y$ is quasilinear, consider the induced exact sequence $0 \rightarrow Y \rightarrow Y \oplus_F X \rightarrow X \rightarrow 0$. Let $U : \ell_q(I) \rightarrow Y \oplus_F X$ be any operator lifting the quotient $Y \oplus_F X \rightarrow X$. Clearly u , the restriction of U to K , takes values in Y and we have a commutative diagram

$$(7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \ell_q(I) & \xrightarrow{\varpi} & X \longrightarrow 0 \\ & & u \downarrow & & U \downarrow & & \parallel \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_F X & \xrightarrow{\pi} & X \longrightarrow 0 \end{array}$$

which shows that $\eta([u]) = [F]$. Should you need some more explanations, please identify $\ell_q(I) = K \oplus_\Phi X$ as indicated before. After that, U has the form $U(k, x) = (u(k) + \ell(x), x)$, where $u \in L(K, Y)$ and $\ell : X \rightarrow Y$ is a linear (but maybe discontinuous) map. Anyway U is bounded, so

$$\|u(k) + \ell(x) - F(x)\|_Y + \|x\|_X \leq \|U\|(\|k - \Phi(x)\|_K + \|x\|_X).$$

Taking $k = \Phi(x)$ we have

$$\|u(\Phi(x)) + \ell(x) - F(x)\|_Y + \|x\|_X \leq \|U\|\|x\|_X,$$

hence $\|F - (u \circ \Phi + \ell)\| \leq \|U\| - 1$ and, in particular, $[F] = [u \circ \Phi] = \eta[u]$. We have thus seen that η is a continuous linear bijection. The preceding argument shows a bit more: given a quasilinear $F : X \rightarrow Y$, $\eta^{-1}([F])$ is (the class in $\text{Ext}^\varpi(X, Y)$ of) the restriction to K of any operator $U : \ell_q(I) \rightarrow Y \oplus_F X$ lifting the quotient $Y \oplus_F X \rightarrow X$. As $\eta^{-1} : \mathcal{Q}(X, Y) \rightarrow \text{Ext}^\varpi(X, Y)$ is linear (η is), continuity will follow if we show that there is a constant C such that, given $F : X \rightarrow Y$ with $Q(F) \leq 1$, there is a lifting $U : \ell_q(I) \rightarrow Y \oplus_F X$ with $\|U\| \leq C$. We will prove that the obvious lifting does the work — see (8) below.

To this end let us remark that if (z_i) converges to z in the quasi-normed space Z , then $\|z\| \leq \Delta_Z \limsup_i \|z_i\|$, where Δ_Z is the modulus of concavity of Z .

Next, if $F : X \rightarrow Y$ is quasilinear, then the modulus of concavity of $Y \oplus_F X$ is at most $\max\{\Delta_Y^2, \Delta_Y Q(F) + \Delta_X\}$. Indeed, one has

$$\begin{aligned} \|(y + y', x + x')\|_F &= \|y + y' - F(x + x')\| + \|x + x'\| \\ &\leq \Delta_Y (\|y + y' - F(x) - F(x')\| \\ &\quad + \|F(x) + F(x') - F(x + x')\|) + \|x + x'\| \\ &\leq \Delta_Y (\Delta_Y (\|y - F(x)\| + \|y' - F(x')\|) \\ &\quad + Q(F)(\|x\| + \|x'\|) + \Delta_X (\|x\| + \|x'\|)) \\ &\leq \max\{\Delta_Y^2, \Delta_Y Q(F) + \Delta_X\} (\|(y, x)\|_F + \|(y', x')\|_F). \end{aligned}$$

We have mentioned that extensions of p -normable spaces are q -normable for $0 < q < p$. On the other hand, a quasilinear map F gives rise to a q -normable extension if and only if it satisfies the estimate

$$\left\| F\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n F(x_i) \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q}$$

for some M and all n , with x_i in the domain of F . An obvious ‘‘amalgamation’’ argument shows the existence of a constant $M = M(p, q)$ such that, if F is a quasilinear map acting between p -normed spaces, $0 < q < p$, and x_1, \dots, x_n belong to the domain of F , then

$$\left\| F\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n F(x_i) \right\| \leq M Q(F) \left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q}.$$

After that, suppose we are given a quasilinear map $F : X \rightarrow Y$. We define a lifting $U : \ell_q(I) \rightarrow Y \oplus_F X$ of the quotient map $\varpi : \ell_q(I) \rightarrow X$ through the formula

$$(8) \quad U\left(\sum_{i \in I} \lambda_i e_i\right) = \sum_{i \in I} \lambda_i (F \varpi e_i, \varpi e_i).$$

The summation of the right-hand side of (8) is performed in the quasi-norm topology of $Y \oplus_F Z$. The series converges because $Y \oplus_F Z$ is q -normable and complete and

$$\sum_i \|\lambda_i (F \varpi e_i, \varpi e_i)\|_F^q = \sum_i |\lambda_i|^q < \infty.$$

Let us estimate $\|U\|$ assuming Z and Y are p -normed and $Q(F) \leq 1$.

$$\begin{aligned} & \left\| U \left(\sum_i \lambda_i e_i \right) \right\|_F \\ &= \left\| \sum_i \lambda_i (F \varpi e_i, \varpi e_i) \right\|_F \\ &\leq \Delta \cdot \sup \left\{ \left\| \sum_{i \in J} \lambda_i (F \varpi e_i, \varpi e_i) \right\|_F : J \text{ is a finite subset of } I \right\} \\ &= \Delta \cdot \sup_J \left\{ \left\| \sum_{i \in J} \lambda_i (F \varpi e_i) - F \left(\sum_{i \in J} \lambda_i \varpi e_i \right) \right\|_Y + \left\| \sum_{i \in J} \lambda_i \varpi e_i \right\|_X \right\} \\ &\leq \Delta \cdot \sup_J \left\{ M \left(\sum_{i \in J} |\lambda_i|^q \right)^{1/q} + \left(\sum_{i \in J} |\lambda_i|^q \right)^{1/q} \right\} \\ &\leq \Delta(M + 1) \left\| \sum_i \lambda_i e_i \right\|_q, \end{aligned}$$

where $\Delta \leq \max(\Delta_Y^2, \Delta_Y + \Delta_X)$ is the modulus of concavity of $Y \oplus_F X$. To sum up, we have seen that if $Q(F) \leq 1$, then $\|U\| \leq (M(p, q) + 1) \max(\Delta_Y^2, \Delta_Y + \Delta_X)$ and the same bound holds for $u = U|_K$. As $\eta([u]) = [F]$ and taking into account that η is linear, in particular homogeneous, we have

$$\eta^{-1}([F]) \leq CQ([F]),$$

where $C = (M(p, q) + 1) \max(\Delta_Y^2, \Delta_Y + \Delta_X)$. This completes the proof. ■

It is perhaps worth noting that the correspondences $F \mapsto U$ and $F \mapsto u$ described in the preceding proof are not linear (or even homogeneous): $F \mapsto u$ becomes linear only when one passes to the quotient structures.

5. Approximation constants and the Hausdorff character of Ext

THEOREM 2: *Let X and Y be quasi-Banach spaces. Then $\text{Ext}(X, Y)$ is Hausdorff if and only if $K_0(X, Y)$ is finite.*

Proof. An operator $t : A \rightarrow B$ acting between quasi-Banach spaces has closed range if and only if it is relatively open, that is, there is a constant C such that whenever $b = t(a)$ for some $a \in A$ there is $a' \in A$ such that $b = t(a')$ and $\|a'\| \leq C\|b\|$.

We will work with the space $\text{Ext}^\omega(X, Y)$ arising from a relatively projective presentation of X , as in (5). As we have already shown, this space and $\text{Ext}(X, Y)$ (or $\mathcal{Q}(X, Y)$) are linearly isomorphic. Looking at the exact sequence

$$0 \longrightarrow \mathfrak{L}(X, Y) \xrightarrow{\varpi^*} \mathfrak{L}(\ell_q(I), Y) \xrightarrow{\varkappa^*} \mathfrak{L}(K, Y) \xrightarrow{\omega} \text{Ext}^\omega(X, Y) \longrightarrow 0$$

we see that $\text{Ext}^\omega(Z, Y)$ is Hausdorff if and only if the (restriction) map \varkappa^* has closed range. By the preceding remarks this is equivalent to \varkappa^* being relatively open, namely:

- There is a constant C such that whenever $u : K \rightarrow Y$ can be extended to $\ell_q(I)$ there is an extension $\tilde{u} \in \mathfrak{L}(\ell_q(I), Y)$ satisfying $\|\tilde{u}\| \leq C\|u\|$.

We will show that this happens if and only if $K_0(Z, Y)$ is finite. To this end let us make explicit the following fact, obtained during the proof of Theorem 1: there is a constant $N = N(X, Y, \Phi)$ such that, for each quasilinear map $F : X \rightarrow Y$, there exist $u \in \mathfrak{L}(K, Y)$, with $\|u\| \leq NQ(F)$, and a linear map $\ell : X \rightarrow Y$ satisfying

$$\|F - (u \circ \Phi + \ell)\| \leq NQ(F).$$

On the other hand, identifying $\ell_q(I)$ with $K \oplus_\Phi X$, we see that given $u \in \mathfrak{L}(K, Y)$, any (not necessarily continuous) linear extension of u to $K \oplus_\Phi X$ has the form $v(k, x) = u(k) - \ell'(x)$, where $\ell' : X \rightarrow Y$ is a linear map. Since

$$\|v\| = \sup \frac{\|u(k) - \ell'(x)\|_Y}{\|(k, x)\|_\Phi} = \sup \frac{\|u(k) - u(\Phi(x)) + u(\Phi(x)) - \ell'(x)\|_Y}{\|(k - \Phi(x))\|_K + \|x\|_X},$$

it follows that

$$\|u \circ \Phi - \ell'\| \leq \|v\| \leq \Delta_Y(\|u\| + \|u \circ \Phi - \ell'\|).$$

Now, suppose $K_0(X, Y)$ is finite and let K_0^+ be any number greater than $K_0(X, Y)$.

Suppose that $u \in \mathfrak{L}(K, Y)$ admits an extension to $\ell_q(I)$. There is no loss of generality in assuming $\|u\| \leq 1$, so that $Q(u \circ \Phi) \leq Q(\Phi)$. Let $\ell : Z \rightarrow Y$ be a linear map at finite distance from $u \circ \Phi$, so that $u \circ \Phi - \ell$ is bounded. As $Q(u \circ \Phi - \ell) = Q(u \circ \Phi) \leq Q(\Phi)$ there is $\ell' \in \mathfrak{L}(X, Y)$ such that $\|u \circ \Phi - \ell - \ell'\| = \|u \circ \Phi - (\ell + \ell')\| \leq Q(\Phi)K_0^+$. The operator $v(k, x) = u(k) - (\ell + \ell')(x)$ is then an extension of u to $K \oplus_\Phi X$, with

$$\|v\| \leq \Delta_Y(\|u\| + \|u \circ \Phi - (\ell + \ell')\|) \leq \Delta_Y(1 + Q(\Phi)K_0^+).$$

Hence \varkappa^* is relatively open, which proves the “if” part.

As for the converse, suppose (\bullet) holds and let $B : X \rightarrow Y$ be bounded, with $Q(B) \leq 1$. Choose $u \in \mathfrak{L}(K, Y)$ and a linear $\ell : X \rightarrow Y$ such that $\|u\| \leq N$ and

$$\|B - (u \circ \Phi + \ell)\| \leq N.$$

The operator u obviously extends to $K \oplus_{\Phi} X$ and the hypothesis yields an extension $v \in \mathfrak{L}(K \oplus_{\Phi} X, Y)$ with $\|v\| \leq C\|u\| \leq CN$. As mentioned before such v has the form $v(k, x) = u(k) - \ell'(x)$, where $\ell' : X \rightarrow Y$ is a linear map and

$$\|u \circ \Phi - \ell'\| \leq \|v\| \leq CN.$$

Hence

$$\|B - (\ell + \ell')\| \leq \Delta_Y(\|B - (u \circ \Phi + \ell)\| + \|u \circ \Phi + \ell - (\ell' + \ell)\|) \leq \Delta_Y N(1 + C).$$

This completes the proof. ■

6. Counterexamples

Let us recall that a Banach space is said to be weakly compactly generated (WCG) if it contains a weakly compact subset whose linear span is dense in the whole space. Obviously each separable Banach space is WCG and so are all reflexive spaces, in particular $\ell_p(I)$ for $1 < p < \infty$ and every index set I . Also, $c_0(I)$ is WCG for every I , while $\ell_1(I)$ is WCG if and only if I is countable. A good general reference is [19], in particular Section 3.

A quasi-Banach space X is called a \mathcal{K} -space if $K(X, \mathbb{K}) < \infty$; equivalently, if every quasilinear function $X \rightarrow \mathbb{K}$ is trivial; or else, if every short exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow X \longrightarrow 0$$

splits. Banach spaces having nontrivial type $p > 1$ are \mathcal{K} -spaces (cf. [6, Theorem 2.6]) as well as all \mathcal{L}_{∞} -spaces and their quotients [10, Theorem 6.5]. As for specific examples, if I is any infinite set, then $c_0(I)$ is a \mathcal{K} -space, and $\ell_p(I)$ is a \mathcal{K} -space if and only if $p \neq 1$, where $p \in (0, \infty]$ (cf. [6, Theorem 3.5]).

Example 1: Let X be a WCG Banach space.

- (a) If X is nonseparable, then $K(X, c_0) = \infty$.
- (b) If X is a \mathcal{K} -space, then $K_0(X, c_0) < \infty$.

Therefore, if X is a nonseparable WCG Banach \mathcal{K} -space, then $K_0(X, c_0)$ is finite but $K(X, c_0)$ is not.

Proof. (a) It is shown in [4, Theorem 3.4] that if X is a nonseparable WCG Banach space, then there is a nontrivial twisted sum of c_0 and X . Hence (see Section 2.3) nontrivial quasilinear maps $F : X \rightarrow c_0$ exist and so $K(X, c_0) = \infty$ — when X is either $c_0(I)$ or $\ell_2(I)$ for uncountable I the result goes back to [5]. For a remarkably simple proof based on Parovichenko’s theorem see the recent paper [18]. Different constructions can also be found in [13].

(b) We prove now that $K_0(X, c_0)$ is finite when X is a WCG Banach \mathcal{K} -space. Let $F : X \rightarrow c_0$ be a homogeneous bounded map, with $Q(F) \leq 1$. The idea is to form the twisted sum $c_0 \oplus_F X$ and obtain a projection p onto c_0 having small norm. (Recall from Section 2.3 that in this case there is a linear map $L : X \rightarrow c_0$ such that $\|F - L\| \leq \|p\|$.) It is a classical result in Banach space theory that c_0 is complemented by a projection of norm at most 2 in any WCG Banach space. This is Rosenthal’s improvement of Veech’s proof of Sobczyk’s theorem [17]. Of course $c_0 \oplus_F X$ is WCG since F is trivial and so it is isomorphic to $c_0 \oplus X$. The problem here is that even if $c_0 \oplus_F X$ is isomorphic to a Banach space, the functional $\|\cdot\|_F$ is only a quasi-norm and we need a true norm to control the norm of the projection. This is the point where the hypothesis that X is a \mathcal{K} -space enters.

The details are as follows. Let $K = K(X, \mathbb{K})$ be the \mathcal{K} -space constant of X . Then, for every homogeneous $f : X \rightarrow \mathbb{K}$ with $Q(f) \leq 1$ there is a linear map $\ell : X \rightarrow \mathbb{K}$ such that $|f(x) - \ell(x)| \leq K\|x\|$ for all $x \in X$. So, for finitely many points of X we have,

$$\begin{aligned} \left| f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) \right| &= \left| f\left(\sum_{i=1}^n x_i\right) - \ell\left(\sum_{i=1}^n x_i\right) + \sum_{i=1}^n \ell(x_i) - \sum_{i=1}^n f(x_i) \right| \\ &\leq K \left(\left\| \sum_{i=1}^n x_i \right\| + \sum_{i=1}^n \|x_i\| \right) \\ &\leq 2K \left(\sum_{i=1}^n \|x_i\| \right). \end{aligned}$$

It follows that if $F : X \rightarrow c_0$ is quasilinear, with $Q(F) \leq 1$, then

$$\left\| F\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n F(x_i) \right\| \leq 2K \sum_{i=1}^n \|x_i\|.$$

Hence, for every $n \in \mathbb{N}$ and $x_i \in X$,

$$\begin{aligned} \left\| \sum_{i=1}^n (y_i, x_i) \right\|_F &\leq \left\| \sum_{i=1}^n y_i - \sum_{i=1}^n F(x_i) \right\|_Y \\ &\quad + \left\| \sum_{i=1}^n F(x_i) - F\left(\sum_{i=1}^n x_i\right) \right\|_Y + \sum_{i=1}^n \|x_i\| \\ &\leq (2K + 1) \sum_{i=1}^n \|(y_i, x_i)\|_F. \end{aligned}$$

We define a norm on $c_0 \oplus_F X$ by letting

$$|(y, x)|_{c_0} = \inf \left\{ \sum_{i=1}^n \|(y_i, x_i)\|_F : (y, x) = \sum_{i=1}^n (y_i, x_i) \right\}.$$

By the preceding inequality,

$$|(y, x)|_{c_0} \leq \|(y, x)\|_F \leq (2K + 1)|(y, x)|_{c_0}.$$

To conclude, consider the natural inclusion map $\iota : c_0 \rightarrow (c_0 \oplus_F X, |\cdot|_{c_0})$ given by $\iota(y) = (y, 0)$. We have that $\iota(c_0)$ is a separable subspace of $(c_0 \oplus_F X, |\cdot|_{c_0})$, a WCG Banach space. Since

$$\|y\|_\infty = \|(y, 0)\|_F \leq (2K + 1)|(y, 0)|_{c_0}$$

the map $(y, 0) \mapsto y$ has norm at most $(2K + 1)$ from $(\iota(c_0), |\cdot|_{c_0})$ to c_0 and by Veech’s result it extends to a “projection” p of $(c_0 \oplus_F X, |(\cdot, \cdot)|_{c_0})$ onto c_0 having norm at most $2(2K + 1)$. It is clear that the norm of p as an operator from the quasi-Banach space $c_0 \oplus_F X$ with its original norm $\|(\cdot, \cdot)\|_F$ to c_0 is at most $2(2K + 1)$ too. Writing $p(y, x) = y - L(x)$ as in Section 2.3 we conclude that $\|F - L\| \leq 2(2K + 1)$. ■

7. “Positive” results

In most cases, however, $K_0(X, Y)$ is finite (if and) only if $K(X, Y)$ is. Let us recall a few basic facts about ultrapowers of quasi-Banach space spaces. Given a quasi-Banach space Y , a set of indices I and an ultrafilter \mathcal{U} on I one considers the space of bounded families $\ell_\infty(I, Y)$ with the sup quasi-norm and the subspace $N_{\mathcal{U}} = \{(y_i) : \lim_i \|y_i\| = 0 \text{ along } \mathcal{U}\}$. The ultrapower of Y with respect to \mathcal{U} , denoted $Y_{\mathcal{U}}$, is defined as the quotient space $\ell_\infty(I, Y)/N_{\mathcal{U}}$ with the quotient quasi-norm. Let $q_{\mathcal{U}}$ be the quotient map. As long as the quasi-norm

of Y is continuous the quasi-norm in the ultrapower can be computed with the formula

$$\|[(y_i)]_{\mathcal{U}}\| = \liminf_{\mathcal{U}} \|y_i\|,$$

where we set $[(y_i)]_{\mathcal{U}} = q_{\mathcal{U}}((y_i))$. In this case the diagonal map δ sending $y \in Y$ to the class of the constant family (y) in $Y_{\mathcal{U}}$ is an isometric embedding.

An ultrasummand is a quasi-Banach space which appears complemented in all its ultrapowers through the diagonal embedding. For Banach spaces this is equivalent to “it is complemented in its bidual (or in any other dual Banach space)”. Typical nonlocally convex ultrasummands are $\ell_p(I)$ and the Hardy classes H_p for $p \in (0, 1)$.

A quasi-Banach space X has the Bounded Approximation Property (BAP) if there is a uniformly bounded net of finite rank operators converging pointwise to the identity on X . Obviously such a X has separating dual (meaning that for every nonzero $x \in X$ there is $x^* \in X^*$ such that $x^*(x) \neq 0$). The BAP is equivalent to the following statement: there is a constant λ such that, whenever E is a finite-dimensional subspace of X , there is a finite rank operator u_E such that $u_E(x) = x$ for every $x \in E$ and $\|u\| \leq \lambda$. Every quasi-Banach space with a Schauder basis has the BAP.

- THEOREM 3:** (a) *Let X be a quasi-Banach space with the BAP and Y an ultrasummand. If $K_0(X, Y)$ is finite, then so is $K(X, Y)$.*
 (b) *Suppose X is a \mathcal{K} -space and Y an ultrasummand with the BAP. If $K_0(X, Y)$ is finite, then so is $K(X, Y)$.*

Proof. By the Aoki–Rolewicz theorem we may assume the quasi-norm of Y is continuous.

To prove (a), let \mathcal{F} denote the family of all finite-dimensional subspaces of X ordered by inclusion. Let \mathcal{U} be any ultrafilter refining the Fréchet (order) filter on \mathcal{F} and let us consider the corresponding ultrapower $Y_{\mathcal{U}}$. Since Y is an ultrasummand, there is a bounded linear projection $p : Y_{\mathcal{U}} \rightarrow Y$. Since X has the BAP, for each $E \in \mathcal{F}$ we may fix a finite rank $u_E \in \mathfrak{L}(X)$ fixing E with $\|u_E\| \leq \lambda$, where λ is the “BAP constant” of X . To conclude these prolegomena, let $1_E : X \rightarrow X$ denote the characteristic function of E ; i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise.

Now, let $F : X \rightarrow Y$ be quasilinear, with $Q(F) \leq 1$. We define a mapping $G : X \rightarrow \ell_{\infty}(\mathcal{F}, Y)$ as

$$G(x) = (1_E(x)F(u_E(x)))_{E \in \mathcal{F}}.$$

(Observe that the family is bounded.) When $x \in E$ the E -th coordinate of $G(x)$ is $F(x)$. Since for fixed $x \in X$ the set $\{E \in \mathcal{F} : x \in E\}$ belongs to \mathcal{U} , it follows that $q_{\mathcal{U}} \circ G = \delta \circ F$. Consequently, $F = p \circ q_{\mathcal{U}} \circ G$. The map $F \circ u_E : X \rightarrow Y$ is quasilinear and bounded because quasilinear maps are bounded on finite-dimensional spaces. Thus there is $\ell_E : X \rightarrow Y$ such that

$$\|\ell_E - F \circ u_E\| \leq K_0^+ Q(F \circ u_E) \leq K_0^+ Q(F) \|u_E\| \leq K_0^+ \lambda,$$

provided $K_0^+ > K_0(X, Y)$. This allows us to define a (probably nonlinear) map $\ell : X \rightarrow \ell_\infty(\mathcal{F}, Y)$ by

$$\ell(x) = (1_E(x)\ell_E(x))_{E \in \mathcal{F}}.$$

The definition makes sense because the family $(1_E(x)\ell_E(x))_{E \in \mathcal{F}}$ is bounded for each $x \in X$. In fact, one has

$$\|G(x) - \ell(x)\|_\infty = \sup_{E \in \mathcal{F}} \|1_E(x)(F(u_E(x)) - \ell_E(x))\|_Y \leq K_0^+ \lambda \|x\|.$$

Hence if we put $L = q_{\mathcal{U}} \circ \ell$, then $\|q_{\mathcal{U}} \circ G - L\| \leq \|G - \ell\| \leq K_0^+ \lambda$.

Now, the point is that L is linear: it is obviously homogeneous and moreover, given $x, y \in X$, the set $\{E \in \mathcal{F} : x, y \in E\}$ belongs to \mathcal{U} . For these E one has $1_E(x + y)\ell_E(x + y) = 1_E(x)\ell_E(x) + 1_E(y)\ell_E(y)$ and so

$$L(x + y) = [(1_E(x)\ell_E(x))_E]_{\mathcal{U}} + [(1_E(y)\ell_E(y))_E]_{\mathcal{U}} = L(x) + L(y).$$

Therefore $p \circ L : X \rightarrow Y$ is a linear map and

$$\|F - p \circ L\| = \|p \circ q_{\mathcal{U}} \circ G - p \circ L\| \leq \|p\| K_0^+ \lambda.$$

Thus, every quasilinear map from X to Y is trivial and $K(X, Y)$ is finite, by [6, Proposition 3.3].

The proof of (b) is analogous. This time the index set is the family of all finite-dimensional subspaces of Y (instead of X) which we denote again by \mathcal{F} . For each $E \in \mathcal{F}$ we take a finite rank operator $u_E \in \mathfrak{L}(Y)$ such that $u_E(y) = y$ for $y \in E$, with $\|u_E\| \leq \lambda$.

Suppose $F : X \rightarrow Y$ is quasilinear, with $Q(F) \leq 1$. As before, we fix $K_0^+ > K_0(X, Y)$. For each $E \in \mathcal{F}$, consider the composition $u_E \circ F$ as a map from X to $u_E(Y)$, a finite-dimensional space. As X is a \mathcal{K} -space there is a linear map $\ell_E : X \rightarrow u_E(Y)$ at finite distance from $u_E \circ F$. Considering now the difference $u_E \circ F - \ell_E$ as a bounded homogeneous map from X to Y we

have $Q(u_E \circ F - \ell_E) = Q(u_E \circ F) \leq \lambda$, and so there is a linear map $\ell'_E : X \rightarrow Y$ such that

$$\|(u_E \circ F - \ell_E) - \ell'_E\| = \|u_E \circ F - (\ell_E + \ell'_E)\| \leq \lambda K_0^+.$$

Taking $L_E = \ell_E + \ell'_E$ we define a mapping $L : X \rightarrow \ell_\infty(\mathcal{F}, Y)$ as

$$L(x) = (1_E(F(x))L_E(x)).$$

The map is well defined (i.e., the family is bounded) since when $F(x) \in E$, then $u_E \circ F(x) = F(x)$ and $\|L_E(x) - F(x)\| \leq K_0^+ \lambda \|x\|$; hence

$$\sup_{E \in \mathcal{F}} \|1_E(F(x))L_E(x)\| \leq \Delta_Y (\|F(x)\| + K_0^+ \lambda \|x\|).$$

Let \mathcal{U} be an ultrafilter refining the Fréchet filter on \mathcal{F} . Let us check that $q_{\mathcal{U}} \circ L : X \rightarrow Y_{\mathcal{U}}$ is linear although L was not. It is obviously homogeneous; and it is also additive: indeed, if $x, y \in X$, then as long as E contains $F(x), F(y)$ and $F(x + y)$ one has

$$1_E(F(x + y))L_E(x + y) = 1_E(F(x))L_E(x) + 1_E(F(y))L_E(y)$$

which yields $q_{\mathcal{U}}(L(x + y)) = q_{\mathcal{U}}(L(x)) + q_{\mathcal{U}}(L(y))$. Finally, if $p : Y_{\mathcal{U}} \rightarrow Y$ is a bounded projection then $p \circ q_{\mathcal{U}} \circ L : X \rightarrow Y$ is a linear map at finite distance from F : observe that $u_E(F(x)) = F(x)$ as long as $F(x) \in E$, so $\delta(F(x)) = [(1_E(F(x))u_E(F(x)))]_{\mathcal{U}}$. One then has $\|\delta \circ F - L\| \leq K_0^+ \lambda$, and thus

$$\|F - p \circ L\| = \|p \circ \delta \circ F - p \circ L\| \leq \|p\| \leq K_0^+ \lambda.$$

This shows that F is trivial, and since F is arbitrary we conclude that $K(X, Y)$ is finite. ■

★ Theorems 2 and 3 together indicate that for a wide class of Banach and quasi-Banach spaces, $\text{Ext}(X, Y)$ is Hausdorff only if it is zero. Thus, for instance, $\text{Ext}(X, Y)$ fails to be Hausdorff if X and Y are super reflexive and one of them has a basis.

★ The spaces appearing in Example 1 are rather specific. It should be interesting to have other examples of Banach spaces for which $\text{Ext}(X, Y)$ is nonzero and Hausdorff. Having separable counterexamples should be interesting, even allowing Banach space extensions only. To be more precise, if X and Y are Banach spaces, let us denote by $\text{Ext}_{\mathbf{B}}(X, Y)$ the space of extensions $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ in which Z is a Banach space, where we identify equivalent extensions.

The simplest way to describe the “natural” topology in $\text{Ext}_{\mathbf{B}}(X, Y)$ (which in general does not agree with the restriction of the topology in $\text{Ext}(X, Y)$) is to take a projective presentation of X in the category \mathbf{B} of Banach spaces and operators, say

$$0 \longrightarrow K \longrightarrow \ell_1(I) \xrightarrow{\varpi} X \longrightarrow 0,$$

with $K = \ker \varpi$, and to consider

$$\text{Ext}_{\mathbf{B}}^{\varpi}(X, Y) = \frac{\mathfrak{L}(K, Y)}{\varpi^*(\mathfrak{L}(\ell_1(I), Y))}$$

with the obvious seminorm.

★ Let $\ell_1 \rightarrow L_1$ be a quotient map and K its kernel. The space $\text{Ext}_{\mathbf{B}}^{\varpi}(L_1, K)$ is obviously nonzero. Is it Hausdorff in its natural topology? Needless to say the space $\text{Ext}(L_1, K)$ (of extensions of quasi-Banach spaces) is not Hausdorff as it contains a complemented copy of $\text{Ext}(\ell_1, \mathbb{K})$, which is nonzero [6, 14] and fails to be Hausdorff in view of Proposition 2 and Theorem 3(a).

★ Kalton and Ostrovskii asked (see the comments preceding Theorem 3.7 in [7]) if $K_0(X, \mathbb{K}) < \infty$ implies $K(X, \mathbb{K}) < \infty$ for all Banach spaces X . They state without proof that the answer is affirmative when X has the BAP. Theorem 3(a) is of course stronger.

Theorem 2 shows that in the quasi-Banach setting the answer to the Kalton–Ostrovskii question is no. Indeed, it is shown in [2, Corollary 4] that if Z is a quasi-Banach \mathcal{K} -space with trivial dual and Y is a subspace of Z , then one has $Y^* = \text{Ext}(Z/Y, \mathbb{K})$, up to linear homeomorphism. In particular, for $0 < p < 1$ one has that $\text{Ext}(L_p/\mathbb{K}, \mathbb{K}) = \mathbb{K}$ is nonzero, hence $K(L_p/\mathbb{K}, \mathbb{K}) = \infty$, and Hausdorff, hence $K_0(L_p/\mathbb{K}, \mathbb{K}) < \infty$.

★ Concerning Theorem 3 (b) it is worth noting that if X fails to be a \mathcal{K} -space, then $\text{Ext}(X, Y) \neq 0$ for all quasi-Banach spaces Y having nontrivial dual (a property implied by the BAP).

★ Whenever one has a couple of spaces X, Y for which $\text{Ext}(X, Y)$ is not Hausdorff — say, $X = Y = \ell_2$ — one has a non-trivial quasi-linear map $F : X \rightarrow Y$ such that $Q([F]) = 0$; i.e., for each given $\varepsilon > 0$ there is some B bounded for which $Q(F + B) \leq \varepsilon$. We do not know if this can happen to all elements; i.e., if the condition “ $Q([F]) = 0$ for all quasilinear maps $F : X \rightarrow Y$ ” implies $\text{Ext}(X, Y) = 0$.

8. Stability constants

In this final Section we explain why Example 1 solves the main problem raised in [11]. Let D be a convex set in a linear space X and Y a Banach space. A function $f : D \rightarrow Y$ is said to be δ -affine if

$$\|f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)\|_Y \leq \delta$$

for every $x, y \in D$ and every $t \in [0, 1]$. If the preceding inequality holds merely for $t = \frac{1}{2}$ we say that f is δ -Jensen. Of course 0-affine functions are just the popular affine functions and 0-Jensen functions are the so-called Jensen (or midpoint affine) functions.

Following [11] we define $A_0(D, Y)$ as the infimum of those constants C for which the following statement holds: if $f : D \rightarrow Y$ is a uniformly bounded δ -affine function, then there is an affine function $a : D \rightarrow Y$ such that $\|f(x) - a(x)\|_Y \leq C\delta$. We define $A(D, Y)$ as before, but omitting the words “uniformly bounded”. Also, we define $J_0(D, Y)$ and $J(D, Y)$ by replacing “affine” by “Jensen” everywhere in the definition of $A_0(D, Y)$ and $A(D, Y)$, respectively.

Laczkovich and Paulin proved in [11, Theorem 2.1] that

$$A(D, Y) \leq J(D, Y) \leq 2A(D, Y).$$

On the other hand, it trivially follows from [11, Lemma 2.2(iii)] that every uniformly bounded δ -Jensen function is 2δ -affine and, therefore,

$$A_0(D, Y) \leq J_0(D, Y) \leq 2A_0(D, Y).$$

The “second Whitney constant for bounded functions” $w_2^Y(D)$ also appears in [11]. We will not use it in this paper since it is clear from the discussion preceding Theorem 3.1 in [11] that $w_2^Y(D) = \frac{1}{2}J_0(D, Y)$. We refer the reader to [1] for a comprehensive study of the Whitney constants of the balls of various Banach and quasi-Banach spaces.

PROPOSITION 2: *Let X and Y be Banach spaces. If B is the unit ball of X , then:*

- (a) $K(X, Y) \leq A(B, Y) \leq 3 + 6K(X, Y)$.
- (b) $K_0(X, Y) \leq A_0(B, Y) \leq 3 + 6K_0(X, Y)$.

Proof. We write the proof of (a). The proof of (b) is the same, adding the word “bounded” from time to time to the text.

Let us check the first inequality. Suppose $F : X \rightarrow Y$ is quasilinear, with $Q(F) \leq 1$. Then the restriction of F to B is 1-affine. Indeed, taking $x, y \in B$ and $t \in [0, 1]$ we have

$$\begin{aligned} \|F(tx + (1 - t)y) - tF(x) - (1 - t)F(y)\| \\ &= \|F(tx + (1 - t)y) - F(tx) - F((1 - t)y)\| \\ &\leq \|tx\| + \|(1 - t)y\| \leq 1. \end{aligned}$$

If A^+ is any number greater than $A(B, Y)$, then there is an affine function $a : B \rightarrow Y$ such that $\|F(x) - a(x)\| \leq A^+$ for every $x \in B$. It is obvious that there is a (unique) linear map $L : X \rightarrow Y$ such that $L(x) = \frac{1}{2}(a(x) - a(-x))$ for $x \in B$. But F is odd and so $\|F(x) - L(x)\| \leq A^+$ for every $x \in B$. Now, by homogeneity, we have

$$\|F(x) - L(x)\| \leq A^+ \|x\| \quad (x \in X)$$

and, therefore, $K(X, Y) \leq A(B, Y)$.

As for the second inequality in (a) let $f : B \rightarrow Y$ be 1-affine. With no loss of generality we may assume $f(0) = 0$. Put

$$f_1(x) = \frac{f(x) - f(-x)}{2} \quad \text{and} \quad f_2(x) = \frac{f(x) + f(-x)}{2}.$$

Clearly, $f = f_1 + f_2$. Moreover, f_1 is 1-affine on B and $\|f_2(x)\| \leq 1$ for every $x \in B$. Let Λ be the set of lines of X passing through the origin. Given $\lambda \in \Lambda$, let f_λ denote the restriction of f_1 to $\lambda \cap B$, a set which is affinely equivalent to $[0, 1]$. As f_λ is 1-affine and $A([0, 1], Y) \leq 2$ by [11, Lemma 2.2(iii)] there is a linear map $f_\lambda^* : \lambda \rightarrow Y$ such that $\|f_1(x) - f_\lambda^*(x)\|_Y \leq 2$ for $x \in \lambda \cap B$.

Define $f^* : X \rightarrow Y$ taking $f^*(x) = f_\lambda^*(x)$ for $x \in \lambda$. Then f^* is homogeneous on X and 3-affine on B . Thus, for $x, y \in B$ we have

$$\|f^*(x + y) - f^*(x) - f^*(y)\| = 2 \left\| f^* \left(\frac{x + y}{2} \right) - \frac{f^*(x)}{2} - \frac{f^*(y)}{2} \right\| \leq 6.$$

And, by homogeneity, for arbitrary $x, y \in X$,

$$\|f^*(x + y) - f^*(x) - f^*(y)\| \leq 6 \max(\|x\|, \|y\|) \leq 6(\|x\| + \|y\|),$$

that is, $Q(f^*) \leq 6$.

Now, if K^+ is a number greater than $K(X, Y)$, there is a linear map $L : X \rightarrow Y$ such that $\|f^*(x) - L(x)\| \leq 6K^+ \|x\|$ for all $x \in X$. Hence for

$x \in B$ we have

$$\|f(x) - L(x)\| \leq \|f(x) - f_1(x)\| + \|f_1(x) - f^*(x)\| + \|f^*(x) - L(x)\| \leq 1 + 2 + 6K^+$$

and we are done. ■

COROLLARY 1: *Let B be the unit ball of a nonseparable WCG Banach \mathcal{K} -space. Then $A_0(B, c_0)$ is finite while $A(B, c_0)$ is not.* ■

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Is this correct?

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