

Twisting non-commutative L_p spaces

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ABSTRACT. The paper makes the first steps into the study of extensions (“twisted sums”) of noncommutative L^p -spaces regarded as Banach modules over the underlying von Neumann algebra \mathcal{M} .

Our approach combines Kalton’s description of extensions by centralizers (these are certain maps which are, in general, neither linear nor bounded) with a general principle, due to Rochberg and Weiss saying that whenever one finds a given Banach space Y as an intermediate space in a (complex) interpolation scale, one automatically gets a self-extension $0 \rightarrow Y \rightarrow X \rightarrow Y \rightarrow 0$.

For semifinite algebras, considering $L^p = L^p(\mathcal{M}, \tau)$ as an interpolation space between \mathcal{M} and its predual \mathcal{M}_* one arrives at a certain self-extension of L^p that is a kind of noncommutative Kalton-Peck space and carries a natural bimodule structure. Some interesting properties of these spaces are presented.

For general algebras, including those of type III, the interpolation mechanism produces two (rather than one) extensions of one sided modules, one of left-modules and the other of right-modules. Whether or not one can find (nontrivial) self-extensions of bimodules in all cases is left open.

Introduction

In this paper we make the first steps into the study of extensions of noncommutative L^p -spaces. An extension (of Z by Y) is a short exact sequence of Banach spaces and (linear, continuous) operators

$$(1) \quad 0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0.$$

This essentially means that X contains Y as a closed subspace so that the corresponding quotient is (isomorphic to) Z .

We believe that the convenient setting in studying extensions of L^p -spaces is not that of Banach spaces, but that of Banach modules over the underlying von Neumann algebra \mathcal{M} . Accordingly, one should require the arrows in (1) to be homomorphisms.

In this regard it is remarkable and perhaps a little ironic that, while the study of the module structure of general L^p -spaces goes back to its inception, the only papers where one can find some relevant information about extensions, namely [14] and [15], deliberately neglected this point.

Let us summarize the main results and explain the organization of the paper.

Section 1 contains some preliminaries. Section 2 deals with the tracial (semifinite) case. It is shown that whenever one has a reasonably “symmetric” self-extension of the commutative $L^p(\mathbb{R}^+)$ (the usual Lebesgue space of p -integrable functions) one can get a similar self-extension

$$0 \rightarrow L^p(\mathcal{M}, \tau) \rightarrow X \rightarrow L^p(\mathcal{M}, \tau) \rightarrow 0$$

of bimodules over any semifinite von Neumann algebra \mathcal{M} , equipped with a trace τ .

Our approach combines Kalton’s description of extensions by centralizers (these are certain maps which are, in general, neither linear nor bounded) with a general principle, due to Rochberg

and Weiss that we can express by saying that whenever one finds a given Banach space Y as an intermediate space in a (complex) interpolation scale, one automatically gets a self-extension $0 \longrightarrow Y \longrightarrow X \longrightarrow Y \longrightarrow 0$.

Thus for instance, considering $L^p(\mathcal{M}, \tau)$ as an interpolation space between \mathcal{M} and its predual \mathcal{M}_* one arrives at a certain self-extension of $L^p(\mathcal{M}, \tau)$ that we regard as a kind of noncommutative Kalton-Peck space. Some interesting properties of these spaces are presented.

In Section 3 we leave the tracial setting and we consider L^p -spaces over general (but σ -finite) algebras, including those of type III. In this case the interpolation trick still works but produces two (rather than one) extensions of one sided modules, one of left-modules and the other of right-modules. Whether or not one can find (nontrivial) self-extensions of bimodules in all cases is left open.

1. Preliminaries

1.1. Extensions. Let A be a Banach algebra. A quasi-Banach (left) module over A is a quasi-Banach space X together with a jointly continuous outer multiplication $A \times X \rightarrow X$ satisfying the traditional algebraic requirements.

An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$(2) \quad 0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \longrightarrow 0.$$

The open mapping theorem guarantees that ι embeds Y as a closed submodule of X in such a way that the corresponding quotient is isomorphic to Z . Two extensions $0 \rightarrow Y \rightarrow X_i \rightarrow Z \rightarrow 0$ ($i = 1, 2$) are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \parallel & & \downarrow u & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

By the five-lemma [11, Lemma 1.1], and the open mapping theorem, u must be an isomorphism. We say that (2) splits if it is equivalent to the trivial sequence $0 \rightarrow Y \rightarrow Y \oplus Z \rightarrow Z \rightarrow 0$. This just means that Y is a complemented submodule of X , that is, there is a homomorphism $X \rightarrow Y$ which is a left inverse for the inclusion $Y \rightarrow X$; equivalently, there is a homomorphism $Z \rightarrow X$ which is a right inverse for the quotient $X \rightarrow Z$.

Operators and homomorphisms are assumed to be continuous. Otherwise we speak of linear maps and “morphisms”.

Taking $A = \mathbb{C}$ one recovers extensions in the Banach space setting.

Every extension of (quasi-) Banach modules is also an extension of (quasi-) Banach spaces. Clearly, if an extension of modules is trivial, then so is the underlying extension of (quasi-) Banach spaces. Simple examples show that the converse is not true in general. A Banach algebra A is amenable if every extension of Banach modules (2) in which Y is a dual module splits as long as it splits an an extension of Banach spaces. This is not the original definition, but an equivalent condition. The original definition reads as follows: A is amenable if every continuous derivation from A into a dual bimodule is inner. Here “derivation” means “operator satisfying Leibniz rule” and has nothing to do with the derivations appearing in Section 1.4.

Every Banach space is a quasi-Banach space and it is possible that the middle space X in (2) is only a quasi-Banach space even if both Z and Y are Banach spaces (see [16, Section 4]). This will never occur in this paper, among other things because X will invariably be a quotient of certain Banach space of holomorphic functions. Anyway, Kalton proved in [12] that if Z has nontrivial type $p > 1$ and Y is a Banach space, then X must be locally convex and so isomorphic to a Banach

space. In particular, any quasi-norm giving the topology of X must be equivalent to a norm – hence to the convex envelope norm. If Z is super-reflexive the proof is quite simple; see [3].

1.2. Centralizers and the extensions they induce. Let us introduce the main tool in our study of extensions.

DEFINITION 1. Let Z and Y be quasi-normed modules over the Banach algebra A and let \tilde{Y} be another module containing Y in the purely algebraic sense. A centralizer from Z to Y with ambient space \tilde{Y} is a homogeneous mapping $\Omega : Z \rightarrow \tilde{Y}$ having the following properties.

- (a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f + g) - \Omega(f) - \Omega(g) \in Y$ and $\|\Omega(f + g) - \Omega(f) - \Omega(g)\|_Y \leq Q(\|f\|_Z + \|g\|_Z)$.
- (b) There is a constant C so that if $a \in A$ and $f \in Z$, then $\Omega(af) - a\Omega(f) \in Y$ and $\|\Omega(af) - a\Omega(f)\|_Y \leq C\|a\|_A\|f\|_Z$.

We denote by $Q[\Omega]$ the least constant for which (a) holds and by $C[\Omega]$ the least constant for which (b) holds.

We now indicate the connection between centralizers and extensions. Let Z and Y be quasi-Banach modules and $\Omega : Z \rightarrow \tilde{Y}$ is a centralizer from Z to Y . Then

$$Y \oplus_{\Omega} Z = \{(g, f) \in \tilde{Y} \times Z : g - \Omega f \in Y\}$$

is a linear subspace of $\tilde{Y} \times Z$ and the functional $\|(g, f)\|_{\Omega} = \|g - \Omega f\|_Y + \|f\|_Z$ is a quasi-norm on it. Moreover, the map $\iota : Y \rightarrow Y \oplus_{\Omega} Z$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $\pi : Y \oplus_{\Omega} Z \rightarrow Z$ given as $\pi(g, f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces

$$(3) \quad 0 \longrightarrow Y \xrightarrow{\iota} Y \oplus_{\Omega} Z \xrightarrow{\pi} Z \longrightarrow 0$$

with relatively open maps. This already implies that $Y \oplus_{\Omega} Z$ is complete, i. e., a quasi-Banach space. Actually only quasi-linearity (a) is necessary here. The estimate in (b) implies that the multiplication $a(g, f) = (ag, af)$ makes $Y \oplus_{\Omega} Z$ into a quasi-Banach module over A in such a way that the arrows in (3) become homomorphisms. Indeed,

$$\|a(g, f)\|_{\Omega} = \|ag - \Omega(af)\|_Y + \|af\|_Z = \|ag - a\Omega f + a\Omega f - \Omega(af)\|_Y + \|af\|_Z \leq M\|a\|_A\|(g, f)\|_{\Omega}.$$

We will always refer to Diagram 3 as the extension (of Z by Y) induced by Ω .

It is easily seen that two centralizers Ω and Φ (acting between the same sets, say Z and \tilde{Y}) induce equivalent extensions if and only if there is a morphism $h : Z \rightarrow \tilde{Y}$ such that $\|\Omega(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_Z$. We write $\Omega \sim \Phi$ in this case and $\Omega \approx \Phi$ if the preceding inequality holds for $h = 0$ in which case we say that Ω and Φ are strongly equivalent. In particular Ω induces a trivial extension if and only if $\|\Omega(f) - h(f)\|_Y \leq K\|f\|_Z$ for some morphism $h : Z \rightarrow \tilde{Y}$. In this case we say that Ω is a trivial centralizer.

The corresponding definitions for right modules and bimodules are obvious. Thus, for instance, we define bicentralizers from Z to Y (which are now assumed to be Banach bimodules over the Banach algebra A) by requiring \tilde{Y} to be also a bimodule and replacing the estimate in Definition 1(b) by

$$\|\Omega(azb) - a\Omega(z)b\|_Y \leq C\|a\|_A\|f\|_Z\|b\|_A \quad (a, b \in A, z \in Z).$$

We insist that we are interested in the case of Banach spaces here, so one can assume Z and Y to be Banach spaces. However, the Ribe function $\|\cdot\|_{\Omega}$ will be only a quasi-norm on $Y \oplus_{\Omega} Z$, even if it is equivalent to a true norm. See the paragraph closing Section 1.1 and [5, Appendix 1.9].

1.3. Push-outs and extensions. The push-out construction appears naturally when one considers two operators defined on the same space. Given operators $\alpha : Y \rightarrow A$ and $\beta : Y \rightarrow B$, the associated push-out diagram is

$$(4) \quad \begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \beta' \\ B & \xrightarrow{\alpha'} & \text{PO} \end{array}$$

Here, the push-out space $\text{PO} = \text{PO}(\alpha, \beta)$ is the quotient of the direct sum $A \oplus B$ (with the sum norm, say) by S , the closure of the subspace $\{(\alpha y, -\beta y) : y \in Y\}$. The map α' is given by the inclusion of B into $A \oplus B$ followed by the natural quotient map $A \oplus B \rightarrow (A \oplus B)/S$, so that $\alpha'(b) = (0, b) + S$ and, analogously, $\beta'(a) = (a, 0) + S$.

The diagram (4) is commutative: $\beta'\alpha = \alpha'\beta$. Moreover, it is “minimal” in the sense of having the following universal property: if $\beta'' : A \rightarrow C$ and $\alpha'' : B \rightarrow C$ are operators such that $\beta''\alpha = \alpha''\beta$, then there is a unique operator $\gamma : \text{PO} \rightarrow C$ such that $\alpha'' = \gamma\alpha'$ and $\beta'' = \gamma\beta'$. Clearly, $\gamma((a, b) + S) = \beta''(a) + \alpha''(b)$ and one has $\|\gamma\| \leq \max\{\|\alpha''\|, \|\beta''\|\}$.

Suppose we are given an extension (2) and an operator $t : Y \rightarrow B$. Consider the push-out of the couple (ι, t) and draw the corresponding arrows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \longrightarrow & Z \longrightarrow 0 \\ & & t \downarrow & & \downarrow t' & & \\ & & B & \xrightarrow{\iota'} & \text{PO} & & \end{array}$$

Clearly, ι' is an isomorphic embedding. Now, the operator $\pi : X \rightarrow Z$ and the null operator $n : B \rightarrow Z$ satisfy the identity $\pi\iota = nt = 0$, and the universal property of push-outs gives a unique operator $\varpi : \text{PO} \rightarrow Z$ making the following diagram commutative:

$$(5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z \longrightarrow 0 \\ & & t \downarrow & & \downarrow t' & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{\iota'} & \text{PO} & \xrightarrow{\varpi} & Z \longrightarrow 0 \end{array}$$

Or else, just take $\varpi((x, b) + S) = \pi(x)$, check commutativity, and discard everything but the definition of PO . Elementary considerations show that the lower sequence in the preceding diagram is exact. That sequence will be referred to as the push-out sequence. The universal property of push-out diagrams yields:

LEMMA 1. *With the above notations, the push-out sequence splits if and only if t extends to X , that is, there is an operator $T : X \rightarrow B$ such that $T\iota = t$. \square*

1.4. Complex interpolation and twisted sums. These lines explain the main connection between interpolation and twisted sums we use throughout the paper. General references are [26, 7, 18, 15, 4]. Let (X_0, X_1) be a compatible couple of complex Banach spaces. This means that both X_0 and X_1 are embedded into a third topological vector space W and so it makes sense to consider its sum $\Sigma = X_0 + X_1 = \{w \in W : w = x_0 + x_1\}$ which we furnish with the norm $\|w\|_\Sigma = \inf\{\|x_0\|_0 + \|x_1\|_1 : w = x_0 + x_1\}$ as well as the intersection $\Delta = X_0 \cap X_1$ with the norm $\|x\|_\Delta = \max\{\|x\|_0, \|x\|_1\}$. We attach a certain space of analytic functions to (X_0, X_1) as follows.

Let \mathbb{S} denote the closed strip $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$, and \mathbb{S}° its interior. We denote by $\mathcal{G} = \mathcal{G}(X_0, X_1)$ the space of functions $g : \mathbb{S} \rightarrow \Sigma$ satisfying the following conditions:

- (1) g is $\|\cdot\|_\Sigma$ -bounded ;

- (2) g is $\|\cdot\|_\Sigma$ -continuous on \mathbb{S} and $\|\cdot\|_\Sigma$ -analytic on \mathbb{S}° ;
- (3) $g(it) \in X_0, g(it+1) \in X_1$ for each $t \in \mathbb{R}$;
- (4) the map $t \mapsto g(it)$ is $\|\cdot\|_0$ -bounded and $\|\cdot\|_0$ -continuous on \mathbb{R} ;
- (5) the map $t \mapsto g(it+1)$ is $\|\cdot\|_1$ -bounded and $\|\cdot\|_1$ -continuous on \mathbb{R} .

Then \mathcal{G} is a Banach space under the norm $\|g\|_{\mathcal{G}} = \sup\{\|g(j+it)\|_j : j = 0, 1; t \in \mathbb{R}\}$.

For $\theta \in [0, 1]$, define the interpolation space $X_\theta = [X_0, X_1]_\theta = \{x \in \Sigma : x = g(\theta) \text{ for some } g \in \mathcal{G}\}$ with the norm $\|x\|_\theta = \inf\{\|g\|_{\mathcal{G}} : x = g(\theta)\}$. We remark that $[X_0, X_1]_\theta$ is the quotient of \mathcal{G} by $\ker \delta_\theta$, the closed subspace of functions vanishing at θ , and so it is a Banach space.

Now, the basic result is the following.

LEMMA 2. *With the above notations, the derivative $\delta'_\theta : \mathcal{G} \rightarrow \Sigma$ is bounded from $\ker \delta_\theta$ onto X_θ for $0 < \theta < 1$.*

PROOF. For a fixed $\theta \in]0, 1[$, let φ be a conformal map of \mathbb{S}° onto the open unit disc sending θ to 0, for instance that given by

$$(6) \quad \varphi(z) = \frac{\exp(i\pi z) - \exp(i\pi\theta)}{\exp(i\pi z) - \exp(-i\pi\theta)} \quad \text{for } z \in \mathbb{S}.$$

If $g \in \mathcal{G}$ vanishes at θ , then one has $g = \varphi h$, with $h \in \mathcal{G}$ and $\|h\|_{\mathcal{G}} = \|g\|_{\mathcal{G}}$. Therefore, $g'(\theta) = \varphi'(\theta)h(\theta)$, so $g'(\theta) \in X_\theta$ and

$$\|g'(\theta)\|_{X_\theta} = |\varphi'(\theta)| \|h(\theta)\|_{X_\theta} \leq |\varphi'(\theta)| \|h\|_{\mathcal{G}} = |\varphi'(\theta)| \|g\|_{\mathcal{G}}.$$

Hence $\|\delta'_\theta : \ker \delta_\theta \rightarrow X_\theta\| \leq |\varphi'(\theta)|$. Notice that $|\varphi'(\theta)| = \pi/(2 \sin(\pi\theta))$ when φ is given by (6).

Let us see that δ'_θ maps $\ker \delta_\theta$ onto X_θ . Take $x \in X_\theta$, with $\|x\|_\theta = 1$ and choose $g \in \mathcal{G}$ so that $g(\theta) = x$, with $\|g\|_{\mathcal{G}} \leq 1 + \epsilon$. Then $h = \varphi g$ belongs to $\ker \delta_\theta$ and $h'(\theta) = x$. \square

In this way, for each $\theta \in]0, 1[$ we have a push-out diagram

$$(7) \quad \begin{array}{ccccc} \ker \delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & X_\theta \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ X_\theta & \longrightarrow & \text{PO} & \longrightarrow & X_\theta \end{array}$$

whose lower row is a self extension of X_θ . The derivation associated with the preceding diagram is the map $\Omega : X_\theta \rightarrow \Sigma$ obtained as follows: given $x \in X_\theta$ we choose $g = g_x \in \mathcal{G}$ (homogeneously) such that $x = g(\theta)$ and $\|g\|_{\mathcal{G}} \leq (1 + \epsilon)\|x\|_{X_\theta}$ for small $\epsilon > 0$ and we set $\Omega(x) = g'(\theta) \in \Sigma$. (Note that $\Omega(x)$ lies in X_θ at least for $x \in \Delta = X_0 \cap X_1$.) Homogeneously means that if g is the function attached to x and λ is a complex number, then the function attached to λx is λg – this makes $\Omega : X_\theta \rightarrow \Sigma$ homogeneous.

Needless to say, the map Ω depends on the choice of g . However, if $\tilde{\Omega}(x)$ is obtained as the derivative (at θ) of another $\tilde{g} \in \mathcal{G}$ such that $\tilde{g}(\theta) = x$ and $\|\tilde{g}\|_{\mathcal{G}} \leq M\|x\|$, then $\tilde{g} - g$ vanishes at θ , so (by Lemma 2)

$$\|\tilde{\Omega}(x) - \Omega(x)\|_{X_\theta} = \|\delta'_\theta(\tilde{g} - g)\|_{X_\theta} \leq \|\delta'_\theta : \ker \delta_\theta \rightarrow X_\theta\| (M + 1 + \epsilon)\|x\|_{X_\theta},$$

and thus $\tilde{\Omega} \approx \Omega$.

LEMMA 3. *The just defined map Ω is quasi-linear on X_θ . The extension induced by Ω is (equivalent to) the push-out sequence in (7).*

PROOF. That Ω is quasi-linear is straightforward from Lemma 2.

As for the second part, look at the basic Diagram 7. Consider the map $(\delta'_\theta, \delta_\theta) : \mathcal{G} \rightarrow X_\theta \oplus_\Omega X_\theta$ given by $(\delta'_\theta, \delta_\theta)(f) = (f'(\theta), f(\theta))$. Notice that $(f'(\theta), f(\theta))$ belongs to $X_\theta \oplus_\Omega X_\theta$ for every $f \in \mathcal{G}$. Indeed, letting $x = f(\theta)$ we have $f'(\theta) - \Omega(f(\theta)) = \delta'_\theta(f - g_x) \in X_\theta$. Moreover,

$$\|(f'(\theta), f(\theta))\|_\Omega = \|\delta'_\theta(f - g_x)\|_\theta + \|f(\theta)\|_\theta \leq M\|f\|_\mathcal{G}.$$

There is an obvious map $\iota : X_\theta \rightarrow X_\theta \oplus_\Omega X_\theta$ sending x to $(x, 0)$. If $f \in \ker \delta_\theta$ one has

$$(\delta'_\theta, \delta_\theta)(f) = (f'(\theta), 0) = \iota \delta'_\theta(f)$$

and the universal property of the push-out construction yields an operator u making commutative the following diagram

$$\begin{array}{ccccc} \ker \delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & X_\theta \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ X_\theta & \longrightarrow & \text{PO} & \longrightarrow & X_\theta \\ \parallel & & u \downarrow & & \parallel \\ X_\theta & \longrightarrow & X_\theta \oplus_\Omega X_\theta & \longrightarrow & X_\theta \end{array}$$

This completes the proof. \square

The preceding argument is closely related to the observation, due to Rochberg and Weiss [26], that $X_\theta \oplus_\Omega X_\theta = \mathcal{G}/(\ker \delta_\theta \cap \ker \delta'_\theta) = \{(f'(\theta), f(\theta)) : f \in \mathcal{G}\}$, where the third space carries the obvious (infimum) norm.

An important feature of the derivation process is that if we start with a couple (X_0, X_1) of Banach modules over an algebra A (this terminology should be self-explanatory by now), then the diagram (7) lives in the category of Banach modules and Ω is a centralizer over A .

2. The tracial (semifinite) case

2.1. Some special properties of centralizers on $L^p(\mathbb{R}^+)$. Let X be a Köthe space on $\mathbb{R}^+ = [0, \infty)$. It is clear from the definition that X is an $L^\infty(\mathbb{R}^+)$ -module under “pointwise” multiplication which turns out to be a submodule of $L^0(\mathbb{R}^+)$, the space of all measurable functions $f : \mathbb{R}^+ \rightarrow \mathbb{C}$. Here we apply the usual convention of identifying functions agreeing almost everywhere. Let $\Phi : X \rightarrow L^0(\mathbb{R}^+)$ be an $L^\infty(\mathbb{R}^+)$ -centralizer on X . Then Φ is said to be:

- Real if it takes real functions to real functions.
- Symmetric if (X is symmetric and) there is a constant S so that, whenever u is a measure-preserving automorphism of \mathbb{R}^+ one has $\|\Phi(u^\circ f) - u^\circ(\Phi f)\|_X \leq S\|f\|_X$.
- Lazy if, whenever $f = \sum_{k=1}^\infty t_k 1_{A_k}$ is a function taking only countably many values (σ -simple from now on), one has $\Phi(f) = \sum_{k=1}^\infty s_k 1_{A_k}$ for certain sequence of scalars (s_k) .

Important examples of centralizers are given as follows (see [13], Section 3 and specially Theorem 3.1). Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz function. Then the map $L^p(\mathbb{R}^+) \rightarrow L^0(\mathbb{R}^+)$ given by

$$(8) \quad f \longmapsto f\varphi \left(\log \frac{|f|}{\|f\|_p}, \log r_{|f|/\|f\|_p} \right).$$

is a (real, symmetric, lazy) centralizer on $L^p(\mathbb{R}^+)$. Here r_g is the so called rank-function of $g \in L^0(\mathbb{R}^+)$ defined by

$$r_g(t) = \lambda\{s \in \mathbb{R}^+ : |g(s)| > |g(t)| \text{ or } s \leq t \text{ and } |g(s)| = |g(t)|\},$$

which arises in real interpolation.

For what this paper is concerned, the crucial result on L^∞ -centralizers is the following.

THEOREM 1 (Kalton [15], Theorem 7.6). *There is a (finite) constant K so that whenever $1 < p \leq 2$ and X is a p -convex and q -concave Köthe function space with $p^{-1} + q^{-1} = 1$ and Φ is a real centralizer on X with $C[\Phi] < 200/q$ then there is a pair of Köthe function spaces (X_0, X_1) so that $X = [X_0, X_1]_{1/2}$ (with equivalent norms) and if $\Omega : X \rightarrow L^0$ is the corresponding derivation, then $\|\Phi(f) - \Omega(f)\| \leq K\|f\|$ for $f \in X$. In particular $\Phi \approx \Omega$.*

If Φ is symmetric, then X_0 and X_1 can be taken to be symmetric (that is, rearrangement invariant).

2.2. From commutative to noncommutative. Let \mathcal{M} be a semifinite von Neumann algebra with a faithful, normal, semifinite (fns) trace τ , acting on a Hilbert space \mathcal{H} . A closed densely defined operator on \mathcal{H} is affiliated with \mathcal{M} if its spectral projections belong to \mathcal{M} . A closed densely defined operator x affiliated with \mathcal{M} is called τ -measurable if, for any $\epsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $e\mathcal{H} \subset D(x)$ and $\tau(1 - e) \leq \epsilon$. We denote the set of all τ -measurable operators associated with a von Neumann algebra \mathcal{M} by $\widetilde{\mathcal{M}}$. The so called measure topology on $\widetilde{\mathcal{M}}$ is the least linear topology containing the sets

$$\{x \in \widetilde{\mathcal{M}} : \text{there exists a projection } e \in \mathcal{M} \text{ such that } \tau(1 - e) < \epsilon, xe \in \mathcal{M} \text{ and } \|xe\| < \epsilon\},$$

with $\epsilon > 0$. Endowed with measure topology, strong sum, strong product and adjoint operation as involution, $\widetilde{\mathcal{M}}$ becomes a topological *-algebra (see [21, 6] for basic information). The trace τ has a natural extension to $\widetilde{\mathcal{M}}_+$.

We define $L^p(\mathcal{M}, \tau)$ as the space of all τ -measurable operators x such that $\tau(|x|^p) < \infty$, with norm $\|x\|_p = (\tau(|x|^p))^{1/p}$.

More general spaces of operators can be introduced as follows [8, 22, 19]. Let x be a measurable operator, so that $\tau e_\lambda|x| < \infty$ for some $\lambda > 0$, where $e_\lambda|x|$ denotes the spectral resolution of $|x|$ corresponding to the indicator function of (λ, ∞) . The generalized singular numbers of x is the function $\mu(x) : (0, \infty) \rightarrow [0, \infty]$ given by

$$\mu(x)(t) = \mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda|x|) \leq t\}.$$

Now, if X is a symmetric (= rearrangement invariant Köthe) function space on $(0, \infty)$, the corresponding ‘symmetric operator space’ is

$$X(\tau) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in X\}, \quad \text{with} \quad \|x\| = \|\mu(x)\|_X.$$

An important feature of these spaces is that they are bimodules over \mathcal{M} with the obvious outer multiplications.

The following result and its proof are modeled on [15, Theorem 8.3].

THEOREM 2. (a) *Let Φ be a lazy, symmetric $L^\infty(\mathbb{R}^+)$ -centralizer on $L^p(\mathbb{R}^+)$, where $1 < p < \infty$. Then, for every semifinite von Neumann algebra (\mathcal{M}, τ) there is an \mathcal{M} -bicentralizer $\Phi_\tau : L^p(\tau) \rightarrow \widetilde{\mathcal{M}}$ whose action on σ -elementary operators can be obtained as follows. Given a σ -elementary operator $x = \sum_{i=1}^\infty t_i e_i$, choose $f \in L^p(\mathbb{R}^+)$ with the same distribution as x , that is, $f = \sum_{i=1}^\infty t_i 1_{A_i}$, where $m(A_i) = \tau(e_i)$. Then set $\Phi_\tau(x) = \sum_{i=1}^\infty s_i e_i$, where $\Phi(f) = \sum_{i=1}^\infty s_i e_i$.*

(b) *Two bicentralizers on $L^p(\tau)$ agreeing on σ -elementary operators are equivalent.*

(c) *Every symmetric centralizer on $L^p(\mathbb{R}^+)$ is equivalent to a (necessarily symmetric) lazy centralizer.*

PROOF. (a) Clearly, one can write $\Phi = \Phi_1 + i\Phi_2$ with Φ_1 and Φ_2 real centralizers and we can assume that Φ is real and arising as a derivation. Precisely, we are assuming there is a couple of (symmetric) Köthe spaces on \mathbb{R}^+ such that $L^p(\mathbb{R}^+) = [X_0, X_1]_{1/2}$, with equivalent norms in such a

way that, for each $f \in L^p(\mathbb{R}^+)$ one has $\Phi(f) = g'(\frac{1}{2})$, where $g \in \mathcal{G}(X_0, X_1)$ satisfies $g(\frac{1}{2}) = f$ and $\|g\|_{\mathcal{G}} \leq K\|f\|_p$.

Set $X = [X_0, X_1]_{1/2}$, with the natural norm. This is a symmetric Köthe space on \mathbb{R}^+ . The key point is that the formula

$$(9) \quad [X_0(\tau), X_1(\tau)]_{\frac{1}{2}} = X(\tau)$$

holds for all semifinite algebras (\mathcal{M}, τ) – see [8, Theorem 3.2] and [23]. Fixing τ we want to export Φ from $L^p(\mathbb{R}^+)$ to $L^p(\tau)$. In view of (9) one has $[X_0(\tau), X_1(\tau)]_{1/2} = L^p(\tau)$ (with equivalent norms) and we can consider the corresponding derivation. So we define a mapping $\Phi_\tau : L^p(\tau) \rightarrow \widetilde{\mathcal{M}}$ as follows. Given $x \in L^p(\tau) = X(\tau)$ we choose $g \in \mathcal{G}(X_0(\tau), X_1(\tau))$ such that $g(\frac{1}{2}) = x$ and $\|g\|_{\mathcal{G}} \leq (1 + \epsilon)\|x\|_{X(\tau)} \leq K\|x\|_p$ and then we put

$$\Phi_\tau(x) = g'(1/2) \in \widetilde{\mathcal{M}}.$$

Notice that if $x = \sum_i t_i e_i$ is a σ -elementary operator and $f = \sum_i t_i 1_{A_i}$ is as in the statement, then applying (9) to the restriction of τ to the subalgebra of \mathcal{M} spanned by the projections (e_i) and to the subalgebra of $L^\infty(\mathbb{R}^+)$ generated by the functions 1_{A_i} we see that there is an (almost) optimal representative of f having the form $z \mapsto \sum_i g_i(z) 1_{A_i}$ and also that $g(z) = \sum_i g_i(z) e_i$ is then an (almost) optimal representative of x . Thus we may assume $\Phi(f) = \sum_i s_i 1_{A_i}$ with $s_i = g'_i(\frac{1}{2})$ and so $\Phi_\tau(x) = \sum_i s_i e_i$, as desired.

Needless to say, the real content of Part (a) is that the map Φ_τ just defined is an \mathcal{M} -bicentralizer on $X(\tau) = L^p(\tau)$. This should be obvious by now, but let us record the proof for future reference. Take $x \in X(\tau)$ and $a, b \in \mathcal{M}$ (that we regard as constant functions on \mathbb{S}). We have $g_{axb} - ag_x b \in \ker \delta_{1/2}$ by the very definition. Moreover,

$$\|g_{axb} - ag_x b\|_{\mathcal{G}} \leq \|g_{axb}\|_{\mathcal{G}} + \|ag_x b\|_{\mathcal{G}} \leq 2(1 + \epsilon)\|a\|\|x\|_{X(\tau)}\|b\|,$$

so

$$\begin{aligned} \|\Phi_\tau(axb) - a\Phi_\tau(x)b\|_{X(\tau)} &= \|\delta'_{1/2}(g_{axb} - ag_x b)\|_{X(\tau)} \\ &\leq \|\delta'_{1/2} : \ker \delta_{1/2} \rightarrow X_{1/2}(\tau)\| 2(1 + \epsilon)\|a\|\|x\|_{X(\tau)}\|b\| \\ &\leq (1 + \epsilon)\pi\|a\|\|x\|_{X(\tau)}\|b\|. \end{aligned}$$

(b) It suffices to check that if $\Psi : L^p(\tau) \rightarrow \widetilde{\mathcal{M}}$ is a (say left) centralizer vanishing on every σ -elementary operator, then Ψ is bounded. First, let $x \in L^p(\tau)$ be self-adjoint. It is easy to see that $x = ay$, where y is a σ -elementary operator with $\|y\|_p \leq 2\|x\|_p$ and $\|a\|_{\mathcal{M}} \leq 1$. Hence $\|\Psi(x)\|_p = \|\Psi(x) - a\Psi(y)\|_p \leq 2C[\Psi]\|x\|_p$. Now, for arbitrary x , let $u|x|$ be the polar decomposition. We have $\|\Psi(x) - u\Psi(|x|)\|_p \leq C[\Psi]\|x\|_p$ and so $\|\Psi(x)\|_p \leq 3C[\Psi]\|x\|_p$.

(c) was proved during the proof of (a). □

Although we are unable to describe $\Phi_\tau(x)$ when x is not elementary for general Φ , the following result applies to many centralizers appearing in nature. In particular, it applies to the centralizers given by (8) when φ depends only on the first variable, by just taking $\theta(t) = t\varphi(\log t)$ for $t \in \mathbb{R}^+$.

COROLLARY 1. *Let $\theta : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a continuous function. Suppose the map $\Phi : L^p(\mathbb{R}^+) \rightarrow L^0(\mathbb{R}^+)$ given by $\Phi(f) = \|f\|_p u_f \theta_\circ(\|f\|/ \|f\|_p)$ is a centralizer $L^p(\mathbb{R}^+)$. Then, for every semifinite von Neumann algebra (\mathcal{M}, τ) the map $\Phi^\tau : L^p(\tau) \rightarrow \widetilde{\mathcal{M}}$ given by $\Phi^\tau(x) = \|x\|_p u_x \theta(|x|/\|x\|_p)$ is an \mathcal{M} -bicentralizer on $L^p(\tau)$ – strongly equivalent to Φ_τ .*

PROOF. Let us first assume \mathcal{M} abelian. By general representation results, \mathcal{M} is isomorphic to $L^\infty(\mu)$ for some (countably additive, nonnegative, strictly localizable) measure space (S, Σ, μ) . Moreover one can take the isomorphism $\chi : \mathcal{M} \rightarrow L^\infty(\mu)$ in such a way that if $a \in \mathcal{M}$ is nonnegative and $f = \chi(a)$, then $\tau(a) = \int_S f(s) d\mu(s)$. It is easily seen that χ extends to a continuous

homomorphism of $*$ -algebras $\widetilde{\mathcal{M}} \rightarrow L^0(\mu)$ that we still call χ – the topology we consider on $L^0(\mu)$ is that of convergence in measure on subsets of finite measure. Although χ is not in general surjective from $\widetilde{\mathcal{M}}$ to $L^0(\mu)$, it is an isometric isomorphism between $L^p(\tau)$ and $L^p(\mu)$. In fact, if $\theta : \mathbb{R}^+ \rightarrow \mathbb{C}$ is a Borel function, $x \in \widetilde{\mathcal{M}}$ is nonnegative and $f = \chi(x)$, then $\chi(\theta(x)) = \theta \circ f$, where $\theta(x)$ is given by functional calculus, that is,

$$\theta(x) = \int_0^\infty \theta(\lambda) d\epsilon(\lambda) \quad \text{with} \quad x = \int_0^\infty \lambda d\epsilon(\lambda).$$

The consequence of all this is that in order to prove Corollary 1 for abelian algebras it suffices to check that if θ and Φ are as in the statement of Corollary and μ is a (countably additive, nonnegative, strictly localizable) measure, the map $\Phi^\mu : L^p(\mu) \rightarrow L^0(\mu)$ given by $\Phi^\mu(f) = u\|f\|_p\theta_\circ(|f|/\|f\|_p)$ is an $L^\infty(\mu)$ -centralizer on $L^p(\mu)$.

Applying Theorem 2 to Φ with $\mathcal{M} = L^\infty(\mu)$ and embedding $\widetilde{\mathcal{M}}$ into $L^0(\mu)$ as before, we get a centralizer $\Phi_\mu : L^p(\mu) \rightarrow L^0(\mu)$ such that

$$\Phi_\mu(f) = \Phi^\mu(f) = u\|f\|_p\theta_\circ(|f|/\|f\|_p)$$

if f takes countably many values. If besides a takes only countably many values we have

$$\|\Phi^\mu(af) - a\Phi^\mu(f)\|_p = \|\Phi_\mu(af) - a\Phi_\mu(f)\|_p \leq C[\Phi_\mu]\|a\|_\infty\|f\|_p.$$

If $a \in L^\infty(\mu)$ and $f \in L^p(\mu)$ are arbitrary we may take sequences of σ -simple functions (a_n) and (f_n) such that $f_n \rightarrow f$ in $L^p(\mu)$, $a_n \rightarrow a$ in $L^\infty(\mu)$ and $\|f_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. This implies that $a_n\Phi^\mu(f_n) \rightarrow a\Phi^\mu(f)$ and $\Phi^\mu(a_nf_n) \rightarrow \Phi^\mu(af)$ in $L^0(\mu)$. As the unit ball of $L^p(\mu)$ is closed in $L^0(\mu)$ we get

$$\|\Phi^\mu(af) - a\Phi^\mu(f)\|_p \leq \lim_{n \rightarrow \infty} \|\Phi^\mu(a_nf_n) - a_n\Phi^\mu(f_n)\|_p \leq C[\Phi_\mu]\|a\|_\infty\|f\|_p,$$

and Φ^μ is a centralizer. By Theorem 2(b) we have $\Phi^\mu \approx \Phi_\mu$.

Going back to general (semifinite) algebras, we want to see that if $a \in \mathcal{M}$ and $x \in L^p(\tau)$ are positive and commute (that is, each spectral projection of a commutes with every spectral projection of x ; see [25, Chapter VIII, Section 5]), then

$$(10) \quad \|\Phi^\tau(ax) - a\Phi^\tau(x)\|_p \leq C\|a\|_\infty\|x\|_p,$$

for some C independent on a and x . And this is so because if \mathcal{A} is a maximal abelian subalgebra of \mathcal{M} containing the spectral projections of a and x (equipped with the restriction of τ), then $\widetilde{\mathcal{A}}$ embeds in $\widetilde{\mathcal{M}}$ in such a way that $L^p(\mathcal{A}, \tau)$ falls into $L^p(\mathcal{M}, \tau)$. In fact

$$L^p(\mathcal{A}, \tau) = \{y \in L^p(\mathcal{M}, \tau) : by = yb \text{ for every } b \in \mathcal{A}\}.$$

Obviously, the map $\Phi^\tau : L^p(\mathcal{M}, \tau) \rightarrow \widetilde{\mathcal{M}}$ sends $L^p(\mathcal{A}, \tau)$ to $\widetilde{\mathcal{A}}$ and have seen that this restriction acts as an \mathcal{A} -centralizer on $L^p(\mathcal{A}, \tau)$ and the centralizer constant depends only on θ through the centralizer constant of Φ , which proves (10).

We finally prove that $\Phi_\tau \approx \Phi^\tau$. Take a positive $x \in L^p(\tau)$ and write $x = ay$, where a and y are positive, commuting operators such that y is σ -elementary with $\|y\|_p \leq 2\|x\|_p$ and $\|a\|_\infty \leq 1$. One has $\|\Phi^\tau(x) - a\Phi^\tau(y)\|_p \leq C2\|x\|_p$, by (10) and $\|\Phi_\tau(x) - a\Phi_\tau(y)\|_p \leq C[\Phi_\tau]2\|x\|_p$. But $\Phi_\tau(y) = \Phi^\tau(y)$ and so $\|\Phi^\tau(x) - \Phi_\tau(x)\|_p \leq 2(C[\Phi_\tau] + C)\|x\|_p$, which completes the proof. \square

2.3. Noncommutative Kalton-Peck spaces. In this part we discuss the simplest case of self-extensions, namely that one obtains out from the identity $[\mathcal{M}, L^1(\mathcal{M}, \tau)]_\theta = L^p(\mathcal{M}, \tau)$ at $\theta = 1/p$. In order to simplify the computation of extremals we introduce a larger space of holomorphic functions as follows. We consider both \mathcal{M} and $L^1(\mathcal{M}, \tau)$ as subspaces of $\widetilde{\mathcal{M}}$ and we set $\Delta = \mathcal{M} \cap L^1(\mathcal{M}, \tau)$ and $\Sigma = \mathcal{M} + L^1(\mathcal{M}, \tau)$. Let $\mathcal{H} = \mathcal{H}(\mathcal{M}, \tau)$ be the space of functions $h : \mathbb{S} \rightarrow \Sigma$ satisfying the following conditions:

- (1) h is $\|\cdot\|_\Sigma$ -bounded.
- (2) For each $x \in \Delta$ the function $z \mapsto \tau(xh(z))$ is continuous on \mathbb{S} and analytic on \mathbb{S}° .
- (3) $h(it) \in \mathcal{M}, h(it+1) \in L^1(\tau)$ for each $t \in \mathbb{R}$;
- (4) the map $t \mapsto h(it)$ is $\|\cdot\|_\infty$ -bounded and $\sigma(\mathcal{M}, L^1(\tau))$ -continuous on \mathbb{R} ;
- (5) the map $t \mapsto h(it+1)$ is $\|\cdot\|_1$ -bounded and $\|\cdot\|_1$ -continuous on \mathbb{R} .

We equip \mathcal{H} with the norm $\|h\|_{\mathcal{H}} = \sup\{\|h(it)\|_{\mathcal{M}}, \|h(it+1)\|_1 : t \in \mathbb{R}\}$. Note that the elements of \mathcal{H} are in fact $\|\cdot\|_\Sigma$ -analytic on \mathbb{S}° .

Letting $\theta = 1/p \in (0, 1)$ we have that δ_θ maps \mathcal{H} onto $L^p(\tau)$ (without increasing the norm) and replacing \mathcal{G} by \mathcal{H} everywhere in the proof of Lemma 2 we see that the restriction of δ'_θ to $\ker \delta_\theta$ is a bounded operator onto $L^p(\tau)$ and we can form the push-out diagram

$$(11) \quad \begin{array}{ccccc} \ker \delta_\theta & \longrightarrow & \mathcal{H} & \xrightarrow{\delta_\theta} & L^p(\tau) \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ L^p(\tau) & \longrightarrow & \text{PO} & \longrightarrow & L^p(\tau) \end{array}$$

Please note that the above diagram lives in the category of bimodules over \mathcal{M} . Also, as \mathcal{H} contains the Calderón space \mathcal{G} it is really easy to see that this new push-out extension is in fact the same one gets by using \mathcal{G} .

Let us compute the extremals associated to the quotient $\delta_\theta : \mathcal{H} \rightarrow L^p(\tau)$. Suppose $f \in L^p(\tau)$ is a positive operator with $\|f\|_p = 1$. It is easily seen that the function $h(z) = f^{pz}$ belongs to \mathcal{H} (although it is not in \mathcal{G} in general) and also that $\|h\|_{\mathcal{H}} = 1$. Of course, $h'(\theta) = pf \log f$ and thus, the derivation associated to Diagram 11 is given by

$$(12) \quad \Omega_p(f) = pf \log(|f|/\|f\|_p) \quad (f \in L^p(\tau)).$$

Let us denote the corresponding twisted sum $L^p(\tau) \oplus_{\Omega_p} L^p(\tau)$ by $Z_p(\tau)$. Our immediate aim is to prove the following.

THEOREM 3. *$Z_p(\tau)$ is a nontrivial self extension of $L^p(\tau)$ as long as \mathcal{M} is infinite dimensional and $1 < p < \infty$.*

PROOF. Needless to say $Z_p(\tau)$ is a bimodule extension. We shall prove that it doesn't split even as an extension of Banach spaces. As \mathcal{M} is infinite dimensional there is a sequence (e_i) of mutually orthogonal projections having finite trace. Given $E \subset \widetilde{\mathcal{M}}$, we put $E_e = \{f \in E : f = \sum_i t_i e_i, \text{ with } t_i \in \mathbb{C}\}$.

Notice that Ω_p maps $L^p(\tau)_e$ to $\widetilde{\mathcal{M}}_e$ as an \mathcal{M}_e -centralizer and we have a commutative diagram

$$(13) \quad \begin{array}{ccccc} L^p(\tau)_e & \longrightarrow & L^p(\tau)_e \oplus_{\Omega_p} L^p(\tau)_e & \longrightarrow & L^p(\tau)_e \\ \downarrow & & \downarrow & & \downarrow \\ L^p(\tau) & \longrightarrow & Z_p(\tau) & \longrightarrow & L^p(\tau) \end{array}$$

On the other hand, the conditional expectation given by

$$\varepsilon(f) = \sum_i \tau(fe_i)e_i$$

is a contractive projection of $L^p(\tau)$ whose range is $L^p(\tau)_e$. The immediate consequence of all this is that if the lower extension in Diagram 13 splits, then so does the upper one.

Let us check that this is not the case. As \mathcal{M}_e is amenable (it is isometrically isomorphic to the algebra ℓ^∞) and $L^p(\tau)_e$ is a dual bimodule over $\mathcal{M}_e = \ell^\infty$ (it is isometrically isomorphic to ℓ^p) we have that the upper row in (13) splits as an extension of Banach spaces if and only if it splits as an extension of Banach \mathcal{M}_e -modules. And this happens if and only if there is a morphism $\phi : L^p(\tau)_e \rightarrow \widetilde{\mathcal{M}}_e$ approximating Ω_p in the sense that

$$(14) \quad \|\Omega_p(f) - \phi(f)\|_p \leq \delta \|f\|_p$$

for some constant δ and every $f \in L^p(\tau)_e$. It is clear that every morphism $\phi : L^p(\tau)_e \rightarrow \widetilde{\mathcal{M}}_e$ has the form $\phi(\sum_i t_i e_i) = \sum_i \phi_i t_i e_i$ for some sequence of complex numbers (ϕ_i) . Taking $f = e_i$ in (14) we see that $|\phi_i + \log \tau(e_i)| \leq \delta$. It follows that if (14) holds for some $\phi = (\phi_i)$ then it must hold for $\phi_i = -\log \tau(e_i)$, possibly doubling the value of δ .

Fix $n \in \mathbb{N}$ and take $f = \sum_{i=1}^n t_i e_i$ normalized in $L^p(\tau)$ in such a way that the nonzero summands in the norm of f agree:

$$f = \sum_{i=1}^n (n\tau(e_i))^{-1/p} e_i.$$

For this f and $\phi_i = -\log \tau(e_i)$ we have $\Omega_p(f) - \phi(f) = -\log(n)f$, so $\|\Omega_p(f) - \phi(f)\|_p = \log(n)$, which makes impossible the estimate in (14). \square

2.4. Duality. In this Section we extend Kalton-Peck duality results in [17] to all semifinite algebras by showing that for every trace τ the dual space of $Z_p(\tau)$ is isomorphic to $Z_q(\tau)$, where p and q are conjugate exponents. In order to achieve a sharp adjustment of the parameters, let us agree that, given $p \in (1, \infty)$ and a Lipschitz function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$, the associated Kalton-Peck centralizer $\Phi_p : L^p(\tau) \rightarrow \widetilde{\mathcal{M}}$ is defined by $\Phi_p(f) = f\varphi(p \log(|f|/\|f\|_p))$ and the corresponding Kalton-Peck space is $Z_p^\varphi(\tau) = L^p(\tau) \oplus_{\Phi_p} L^p(\tau)$. This is coherent with (12), where φ is the identity on \mathbb{R} .

THEOREM 4. *Let p and q be conjugate exponents, φ a Lipschitz function, and τ be a trace. Then $Z_q^\varphi(\tau)$ is isomorphic to the conjugate of $Z_p^\varphi(\tau)$ under the pairing*

$$(15) \quad \langle (x, y), (v, w) \rangle = \tau(xw - yv) \quad ((x, y) \in Z_p^\varphi(\tau), (v, w) \in Z_q^\varphi(\tau))$$

PROOF. The proof depends on the following elementary inequality: given $s, t \in \mathbb{C}$ one has

$$(16) \quad \left| ts \left(\log \frac{|t|^q}{|s|^p} \right) \right| \leq \frac{p}{e} (|t|^q + |s|^p).$$

This is (a rewording of) the case $n = 1$ of [17, Lemma 5.2] that Kalton and Peck use in the proof of [17, Theorem 5.1]. Let us see that the pairing is continuous. To this end write

$$xw - yv = (x - \Phi_q(y))w + \Phi_q(y)w - y(v - \Phi_p(w)) - y\Phi_p(w).$$

As $\|(x - \Phi_q(y))w\|_1 \leq \|x - \Phi_q(y)\|_q \|w\|_p$ and, similarly, $\|y(v - \Phi_p(w))\|_1 \leq \|y\|_q \|v - \Phi_p(w)\|_p$ it suffices to get an estimate of the form

$$(17) \quad |\tau(\Phi_q(y)w - y\Phi_p(w))| \leq M \|y\|_q \|w\|_p.$$

First, let us assume y and w are σ -elementary operators with $\|y\|_q = \|w\|_p = 1$ and representations $y = \sum t_i y_i$ and $w = \sum s_j w_j$ converging in $L^q(\tau)$ and $L^p(\tau)$, respectively. We may assume

with no loss of generality that $\sum_i y_i = \sum_j w_j = \mathbf{1}_{\mathcal{M}}$ (summation in the $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology). We have

$$\begin{aligned} \Phi_q(y)w - y\Phi_p(w) &= \left(\sum_i t_i \varphi(q \log |t_i|) y_i \right) \left(\sum_j s_j w_j \right) - \left(\sum_i t_i y_i \right) \left(\sum_j s_j \varphi(p \log |s_j|) w_j \right) \\ &= \sum_{i,j} t_i s_j (\varphi(\log |t_i|^q) - \varphi(\log |s_j|^p)) y_i w_j. \end{aligned}$$

Applying (16) and taking into account that if y and w are projections one has $\tau(yw) \geq 0$ we can estimate the left-hand of (17) as follows:

$$\begin{aligned} |\tau(\Phi_q(y)w - y\Phi_p(w))| &\leq \sum_{i,j} |t_i| |s_j| |\varphi(\log |t_i|^q) - \varphi(\log |s_j|^p)| \tau(y_i w_j) \\ &\leq L_\varphi \sum_{i,j} |t_i| |s_j| \left| \log \frac{|t_i|^q}{|s_j|^p} \right| \tau(y_i w_j) \\ &\leq L_\varphi \sum_{i,j} \frac{p}{e} (|t_i|^q + |s_j|^p) \tau(y_i w_j) \\ &= \frac{p}{e} L_\varphi \left(\sum_i |t_i|^q \left(\sum_j \tau(y_i w_j) \right) + \sum_j |s_j|^p \left(\sum_i \tau(y_i w_j) \right) \right) \\ &= \frac{p}{e} L_\varphi \left(\sum_i |t_i|^q \tau(y_i) + \sum_j |s_j|^p \tau(w_j) \right) \\ &= \frac{2pL_\varphi}{e}, \end{aligned}$$

where L_φ denotes the Lipschitz constant of φ . By symmetry one also has $|\tau(\Omega_q(y)w - y\Omega_p(w))| \leq 2e^{-1}qL_\varphi$ and so

$$(18) \quad |\tau(\Phi_q(y)w - y\Phi_p(w))| \leq 4e^{-1}L_\varphi \|y\|_q \|w\|_p,$$

whenever y and w are σ -elementary operators. Now, if y and w are arbitrary it is easy to find sequences of σ -elementary operators (y_n) and (w_n) for which the numerical sequences

$$\|y_n - y\|_q, \quad \|\Phi_q y_n - \Phi_q y\|_q, \quad \|w_n - w\|_p, \quad \|\Phi_p w_n - \Phi_p w\|_p$$

are all convergent to zero. This implies that

$$\|\Phi_q(y_n)w_n - y_n\Phi_p(w_n) - \Phi_q(y)w_n - y\Phi_p(w)\|_1 \rightarrow 0$$

and so (18) holds for every y and w .

Therefore, going back to (15) we have

$$\begin{aligned} |\tau(xw - yv)| &= |\tau((x - \Phi_q(y))w + \Phi_q(y)w - y(v - \Phi_p(w)) - y\Phi_p(w))| \\ &\leq \|(v - \Phi_p(w))\|_p \|y\|_q + 4e^{-1}L_\varphi \|w\|_p \|y\|_q + \|w\|_p \|(x - \Phi_q(y))\|_q \\ &\leq 2L_\varphi \|(x, y)\|_{\Phi_q} \|(v, w)\|_{\Phi_p}. \end{aligned}$$

The remainder of the proof is quite easy: we have just seen that the map $u : Z_q^\varphi(\tau) \rightarrow (Z_p^\varphi(\tau))^*$ given by $(u(x, y))(v, w) = \tau(xw - yv)$ is bounded. On the other hand, the following diagram is

commutative:

$$\begin{array}{ccccc} L^q(\tau) & \longrightarrow & Z_q^\varphi(\tau) & \longrightarrow & L^q(\tau) \\ \parallel & & \downarrow u & & \downarrow -1 \\ L^p(\tau)^* & \longrightarrow & (Z_p^\varphi(\tau))^* & \longrightarrow & L^p(\tau)^* \end{array}$$

Here, the lower row is the adjoint of the extension induced by Φ_p . It follows that u is onto, and open. \square

2.5. The role of the trace. Theorem 3 cannot be extended to arbitrary centralizers. Actually, the following example shows that the behavior of Φ_τ may depend strongly on the trace τ .

EXAMPLE 1. For each \pm and $p \in (1, \infty)$, consider the centralizer on $L^p(\mathbb{R}^+)$ given by $\Phi^\pm(f) = f(i^\pm(\log(|f|/\|f\|_p)))$, where $i^+(t) = \max\{0, t\}$ and $i^-(t) = \min\{0, t\}$. Then, with the notation of Theorem 2:

- (a) Φ^\pm is nontrivial on $L^p(\mathbb{R}^+)$.
- (b) If τ is bounded away from zero on the projections of \mathcal{M} , then Φ_τ^+ is trivial on $L^p(\mathcal{M}, \tau)$ – while Φ_τ^- is nontrivial as long as \mathcal{M} is infinite-dimensional.
- (c) If $\tau(\mathbf{1}_\mathcal{M}) < \infty$, then Φ_τ^- is trivial on $L^p(\mathcal{M}, \tau)$ – while Φ_τ^+ is nontrivial as long as \mathcal{M} is infinite-dimensional.

PROOF. (See the proof of Theorem 3.) Let $\Psi : L^p(\mu) \rightarrow L^0(\mu)$ be any centralizer. Let (S_i) be a sequence of disjoint measurable sets, with finite and positive measure. Given $E \subset L^0(\mu)$, we write E_S for the set of those functions f in E that can be written as $f = \sum_i t_i 1_{S_i}$, for some sequence of complex numbers (t_i) . Then, if Ψ maps $L^p(\mu)_S$ to $L^0(\mu)_S$, then it defines an $L^\infty(\mu)_S$ centralizer on $L^p(\mu)_S$. Moreover, if Ψ is trivial on $L^p(\mu)_S$ (as a quasi-linear map), then its restriction to $L^p(\mu)_S$ is trivial as an $L^\infty(\mu)_S$ centralizer.

(a) To check that Φ^+ is nontrivial on $L^p(\mathbb{R}^+)$ just take a sequence (A_i) with $|A_i| = 2^{-i}$. To check that Φ^- is nontrivial, take A_i with $|A_i| = 1$ for all $i \in \mathbb{N}$.

(b) We may assume $\tau(e) \geq 1$ for every projection $e \in \mathcal{M}$. Pick a positive, σ -elementary f normalized in $L^p(\tau)$ so that $f = \sum_{n=1}^\infty f_n e_n$, with $f_n \geq 0$ and e_n disjoint projections. Obviously $f_i \leq 1$ for every i and so $\Phi^+(f) = 0$. It follows that Φ^+ is bounded on $L^p(\tau)$.

As $\Phi_\tau^+ + \Phi_\tau^- = \Omega_p$ and Φ^+ is trivial we see that Φ^- must be nontrivial since Ω_p is nontrivial unless \mathcal{M} is finite-dimensional.

(c) We may assume $\tau(\mathbf{1}_\mathcal{M}) = 1$. Take a positive, normalized $f \in L^p(\tau)$ and let $f = \int_0^\infty \lambda de_\lambda$ be its spectral resolution. Set $g = \int_0^1 \lambda de_\lambda$ and $h = \int_{1+}^\infty \lambda de_\lambda$. One has

$$\|\Phi_\tau^-(f) - \Phi_\tau^-(g) - \Phi_\tau^-(h)\|_p \leq Q[\Phi_\tau^-](\|g\|_p + \|h\|_p) \leq 2Q[\Phi_\tau^-].$$

Obviously, $\Phi_\tau^-(h) = 0$, while $g \in \mathcal{M}$, with $\|g\|_\infty \leq 1$. Hence

$$\|\Phi_\tau^-(g) - g\Phi_\tau^-(\mathbf{1}_\mathcal{M})\|_p \leq C[\Phi_\tau^-]\|g\|_\infty\|\mathbf{1}\|_p \leq C[\Phi_\tau^-].$$

But $\Phi_\tau^-(\mathbf{1}_\mathcal{M}) = 0$ and so $\|\Phi_\tau^-(f)\|_p \leq 2Q[\Phi_\tau^-] + C[\Phi_\tau^-]$. \square

3. Type III algebras

In this Section we leave the comfortable tracial setting and we face the problem of twisting arbitrary L^p spaces, including those built over type III von Neumann algebras. There are several constructions of these L^p spaces, none of them elementary. All provide bimodule structures on the resulting spaces that turn out to be equivalent at the end.

It is natural to ask for (nontrivial) self-extensions of $L^p(\mathcal{M})$ is the category of Banach bimodules over \mathcal{M} . Unfortunately we have been unable to construct such objects; nevertheless we can still use the interpolation trick to obtain self extensions as (one-sided) modules. In this regard the most suited representation of L^p spaces is one due to Kosaki.

For the sake of clarity, we can restrict here to σ -finite algebras so that we can take weights in \mathcal{M}_* . So, let \mathcal{M} be a von Neumann algebra and $\phi \in \mathcal{M}_*$ a faithful positive functional. (We don't normalize ϕ because the restriction of a state to a direct summand is not a state; see Lemma 4(b) below.) We “include” \mathcal{M} into \mathcal{M}_* just taking $a \in \mathcal{M} \mapsto a\phi \in \mathcal{M}_*$ thus starting the interpolation procedure with $\Sigma = \mathcal{M}_*$ as “ambient” space and $\Delta = \mathcal{M}\phi$ – to which the norm and $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology are transferred without further mention. Then, the Kosaki (left) version of the space $L^p(\mathcal{M})$ is defined as

$$L^p(\phi) = L^p(\mathcal{M}, \phi) = [\mathcal{M}\phi, \mathcal{M}_*]_\theta, \quad (\theta = 1/p).$$

We emphasize we are referring to Kosaki's construction [20, 24, 23] and not to that of Terp [27, 28]. Recall that \mathcal{M}_* is an \mathcal{M} -bimodule with product given by

$$\langle a\psi b, x \rangle = \langle \psi, bxa \rangle \quad (\psi \in \mathcal{M}_*; a, b, x \in \mathcal{M}).$$

The inclusion $\cdot\phi : \mathcal{M} \rightarrow \mathcal{M}_*$ is, however, only a left-homomorphism: $(ba) \cdot \phi = b(a\phi)$. Asking for a two-sided homomorphism means that one should also have

$$(ab) \cdot \phi = ab\phi = a\phi b = (a\phi) \cdot b.$$

In particular (take $a = 1$) $b\phi = \phi b$ for all $b \in \mathcal{M}$, which happens if and only if ϕ is a trace.

Let $\mathcal{G} = \mathcal{G}(\mathcal{M}, \phi)$ denote the Calderón space associated to the couple $(\mathcal{M}\phi, \mathcal{M}_*)$ and put $\mathcal{G}_0 = \mathcal{G}(\mathcal{M}, \phi)_0 = \{g \in \mathcal{G} : g(\theta) = 0\}$, where $\theta = 1/p$ is fixed. These are left \mathcal{M} -modules in the obvious way and so is the quotients $L^p(\mathcal{M}, \phi) = \mathcal{G}/\mathcal{G}_0$. Plug and play to get the push-out diagram

$$(19) \quad \begin{array}{ccccc} \mathcal{G}_0 = \ker \delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & L^p(\mathcal{M}, \phi) \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ L^p(\mathcal{M}, \phi) & \longrightarrow & \text{PO} & \longrightarrow & L^p(\mathcal{M}, \phi) \end{array}$$

(where $\theta = 1/p$) and observe that every arrow here is a homomorphism of left \mathcal{M} -modules. Let us denote by $Z_p(\mathcal{M}, \phi)$ or $Z_p(\phi)$ the push-out space in the preceding diagram. This is coherent with the notation used in the tracial case.

We have mentioned that there is also a right action of \mathcal{M} on $L^p(\mathcal{M}, \phi)$ which is compatible with the given left action and makes $L^p(\mathcal{M}, \phi)$ into a bimodule. All known descriptions of that action are quite heavy and depend on Tomita-Takesaki theory. That action is in general incompatible with the arrows in the preceding diagram.

Now, we are confronted with the problem of deciding whether the lower extension in Diagram 19 is trivial or not. The pattern followed in the proof of Theorem 3 cannot be used now because we have only a left multiplication in $Z_p(\mathcal{M}, \phi)$.

Suppose we are given two von Neumann algebras \mathcal{M} and \mathcal{N} with distinguished faithful normal states ϕ and ψ . If $u_\infty : \mathcal{M} \rightarrow \mathcal{N}$ and $u_1 : \mathcal{M}_* \rightarrow \mathcal{N}_*$ are operators making the square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{u_\infty} & \mathcal{N} \\ \cdot\phi \downarrow & & \downarrow \cdot\psi \\ \mathcal{M}_* & \xrightarrow{u_1} & \mathcal{N}_* \end{array}$$

commutative, then interpolation yields operators $u_p : L^p(\mathcal{M}, \phi) \rightarrow L^p(\mathcal{N}, \psi)$ for each $p \in (1, \infty)$.

LEMMA 4. Let \mathcal{M} be a von Neumann algebra with a faithful positive normal functional ϕ . Let \mathcal{N} be a von Neumann subalgebra of \mathcal{M} equipped with the restriction of ϕ . Suppose either

- (a) There is a normal conditional expectation $\varepsilon : \mathcal{M} \rightarrow \mathcal{N}$ leaving ϕ invariant; or
- (b) \mathcal{N} is a direct summand (as a von Neumann algebra) in \mathcal{M} .

Then, for each $p \in [1, \infty]$, $L^p(\mathcal{N}, \phi|_{\mathcal{N}})$ embeds into $L^p(\mathcal{M}, \phi)$ and there is a \mathcal{N} -homomorphism on the latter space whose range is the former.

PROOF. (a) We have assembled the hypotheses in order to guarantee the commutativity of the diagram

$$(20) \quad \begin{array}{ccccc} \mathcal{N} & \xrightarrow{\iota} & \mathcal{M} & \xrightarrow{\varepsilon} & \mathcal{N} \\ \cdot\phi|_{\mathcal{N}} \downarrow & & \cdot\phi \downarrow & & \downarrow \cdot\phi|_{\mathcal{N}} \\ \mathcal{N}_* & \xrightarrow{\varepsilon_*} & \mathcal{M}_* & \xrightarrow{\iota_*} & \mathcal{N}_* \end{array}$$

Here, $\iota : \mathcal{N} \rightarrow \mathcal{M}$ the inclusion map and the subscript indicates preadjoint (in the Banach space sense), in particular ι_* is plain restriction.

Indeed, for $a \in \mathcal{N}$, one has $\varepsilon_*(a\phi) = a\varepsilon_*(\phi) = a\phi$, so the left square commutes. As for the right one, taking $a \in \mathcal{N}, b \in \mathcal{M}$ we have

$$\langle \varepsilon(b)\phi, a \rangle = \langle \phi, a\varepsilon(b) \rangle = \langle \phi, \varepsilon(ab) \rangle = \langle \phi, ab \rangle = \langle b\phi, a \rangle.$$

Notice, moreover, that $\varepsilon \circ \iota$ is the identity on \mathcal{N} , while $\iota_* \circ \varepsilon_*$ is the identity on \mathcal{N}_* . Therefore, interpolating (ι, ε_*) we get operators $\iota_p : L^p(\mathcal{N}, \phi|_{\mathcal{N}}) \rightarrow L^p(\mathcal{M}, \phi)$ for $1 \leq p \leq \infty$, while (ε, ι_*) gives operators $\varepsilon_p : L^p(\mathcal{M}, \phi) \rightarrow L^p(\mathcal{N}, \phi|_{\mathcal{N}})$. And since $\varepsilon_p \circ \iota_p$ is the identity on $L^p(\mathcal{N}, \phi|_{\mathcal{N}})$ we are done.

(b) In this case we can use the same diagram, just replacing ε by the projection $P : \mathcal{M} \rightarrow \mathcal{N}$ given by $P(a) = eae$, where e is the unit of \mathcal{N} . Then $P_* : \mathcal{N}_* \rightarrow \mathcal{M}_*$ is given by $\langle P_*(\psi), b \rangle = \langle \psi, ebe \rangle$. \square

The following step is the result we are looking for.

LEMMA 5. With the same hypotheses as in Lemma 4, $Z_p(\mathcal{N}, \phi|_{\mathcal{N}})$ is a complemented subspace of $Z_p(\mathcal{M}, \phi)$ for every $1 < p < \infty$.

PROOF. We write the proof assuming (a). The other case requires only minor modifications that are left to the reader. Let us begin with the embedding of $\text{PO}(\mathcal{N}) = Z_p(\mathcal{N}, \phi|_{\mathcal{N}})$ into $\text{PO}(\mathcal{M}) = Z_p(\mathcal{M}, \phi)$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}})_0 & \longrightarrow & \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}}) & \xrightarrow{\delta_\theta} & L^p(\mathcal{N}) \\ \downarrow \delta'_\theta & \searrow & \downarrow & \searrow (\varepsilon_*)_\circ & \downarrow \\ \mathcal{G}(\mathcal{M}, \phi)_0 & \longrightarrow & \mathcal{G}(\mathcal{M}, \phi) & \xrightarrow{\delta_\theta} & L^p(\mathcal{M}) \end{array}$$

Here, $(\varepsilon_*)_o$ sends a given function $f : \mathbb{S} \rightarrow \mathcal{N}_*$ to the composition $\varepsilon_* \circ f : \mathbb{S} \rightarrow \mathcal{N}_* \rightarrow \mathcal{M}_*$ and the mappings from $L^p(\mathcal{N})$ to $L^p(\mathcal{M})$ are all given by ι_p . It is not hard to check that this is a commutative diagram. Therefore, we can insert an operator $\kappa : \text{PO}(\mathcal{N}) \rightarrow \text{PO}(\mathcal{M})$ making the resulting diagram commutative because of the universal property of the push-out square

$$\begin{array}{ccc} \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}})_0 & \longrightarrow & \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}}) \\ \delta'_\theta \downarrow & & \downarrow \\ L^p(\mathcal{N}) & \longrightarrow & \text{PO}(\mathcal{N}) \end{array}$$

A similar argument shows the existence of an operator $\pi : \text{PO}(\mathcal{M}) \rightarrow \text{PO}(\mathcal{N})$ making commutative the diagram

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{M}, \phi)_0 & \longrightarrow & \mathcal{G}(\mathcal{M}, \phi) & \xrightarrow{\delta_\theta} & L^p(\mathcal{M}) \\ \downarrow \delta'_\theta & \searrow & \downarrow & \searrow (\iota_*)_o & \downarrow \\ & & \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}})_0 & \longrightarrow & \mathcal{G}(\mathcal{N}, \phi|_{\mathcal{N}}) & \xrightarrow{\delta_\theta} & L^p(\mathcal{N}) \\ & & \downarrow \delta'_\theta & & \downarrow & & \parallel \\ L^p(\mathcal{M}) & \xrightarrow{\delta'_\theta} & \text{PO}(\mathcal{M}) & \xrightarrow{\pi} & L^p(\mathcal{M}) & & \parallel \\ & \searrow & \downarrow & \searrow & \downarrow & & \parallel \\ & & L^p(\mathcal{N}) & \longrightarrow & \text{PO}(\mathcal{N}) & \longrightarrow & L^p(\mathcal{N}) \end{array}$$

The arrows from $L^p(\mathcal{M})$ to $L^p(\mathcal{N})$ are now given by ε_p . Putting together the two preceding diagrams it is easily seen that $\pi \circ \kappa$ is the identity on $\text{PO}(\mathcal{N})$. \square

Here is the main result about the twisting of Kosaki's L^p . As we shall see later (Section 3.2) $Z_p(\mathcal{M}, \phi)$ doesn't depend on ϕ and so the conclusion of the following Theorem holds for any ϕ .

THEOREM 5. *Let \mathcal{M} be an infinite dimensional von Neumann algebra. There is a faithful normal state ϕ for which the lower extension of the push-out diagram*

$$(21) \quad \begin{array}{ccccc} \ker \delta_\theta & \longrightarrow & \mathcal{G}(\mathcal{M}, \phi) & \xrightarrow{\delta_\theta} & L^p(\mathcal{M}, \phi) \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ L^p(\mathcal{M}, \phi) & \longrightarrow & \text{PO} & \longrightarrow & L^p(\mathcal{M}, \phi) \end{array}$$

is nontrivial.

PROOF. The idea of the proof is to choose ϕ in such a way that its “centralizer subalgebra”

$$\mathcal{M}^\phi = \{a \in \mathcal{M} : a\phi = \phi a\}$$

is infinite dimensional. After that we proceed as follows. By [2, Corollary III.4.7.9], there is a normal conditional expectation $\varepsilon : \mathcal{M} \rightarrow \mathcal{M}^\phi$ leaving ϕ invariant: $\phi = \phi|_{\mathcal{M}^\phi} \circ \varepsilon$. Actually ε is unique, by [2, Corollary II.6.10.8].

Apply now Lemma 5 to embed $\text{PO}(\mathcal{M}^\phi, \phi)$ as a complemented subspace (in fact as a “complemented subextension”) of $\text{PO}(\mathcal{M}, \phi)$ and please note that the restriction of ϕ to \mathcal{M}^ϕ is a (finite) trace by the very definition of \mathcal{M}^ϕ .

The nonsplitting of $\text{PO}(\mathcal{M}^\phi, \phi)$ is nothing but a particular case of Theorem 3 as for a finite trace τ one has $\mathcal{M} \subset L^1(\tau)$ and, after identifying $L^1(\tau)$ with \mathcal{M}_* , the inclusion agrees with Kosaki’s left method.

In order to find out the required ϕ , let us decompose $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$, with \mathcal{N} semifinite and \mathcal{L} without direct summands of type I (This can be done in several ways: for instance, taking \mathcal{N} as the semifinite part and \mathcal{L} as the type III part of \mathcal{M} , or taking \mathcal{N} as the discrete part and \mathcal{L} as the continuous part, see [2, Section III.1.4].)

By Lemma 5 we have an isomorphism $\text{PO}(\mathcal{M}, \phi) = \text{PO}(\mathcal{N}, \phi|_{\mathcal{N}}) \oplus \text{PO}(\mathcal{L}, \phi|_{\mathcal{L}})$ and we can consider the two cases separately.

★ First, assume \mathcal{M} has no direct summand of type I (so that it is either type II or III). Then, if ψ is any faithful normal state on \mathcal{M} , there is a faithful normal state ϕ (in the closure of the orbit of ψ under the inner automorphisms of \mathcal{M}) whose centralizer subalgebra \mathcal{M}^ϕ is of type II_1 ([10, Theorem 11.1]) and “we” are done.

★ Now, suppose \mathcal{M} semifinite and let us see that any ϕ works. Let τ be a (fns) trace on \mathcal{M} and let us identify \mathcal{M}_* with $L^1(\tau)$ so that we may consider ϕ as a τ -measurable operator on the ground Hilbert space. If ϕ is elementary, let us write it as $\phi = \sum_{i=1}^n t_i e_i$, where the e_i are mutually orthogonal projections in \mathcal{M} . Letting $\mathcal{M}_i = e_i \mathcal{M} e_i$ we see that $\bigoplus_i \mathcal{M}_i$ is an infinite dimensional subalgebra of \mathcal{M}^ϕ , which is enough. Otherwise ϕ has infinite spectrum and its spectral projections already generate an infinite dimensional subalgebra of \mathcal{M}^ϕ . \square

3.1. Duality again. We now give a description of the dual of $Z_p(\mathcal{M}, \phi)$ for general \mathcal{M} . To this end we consider the right embedding of \mathcal{M} into \mathcal{M}_* given by $a \mapsto \phi a$ which is a homomorphism of right modules and the new couple $(\phi \mathcal{M}, \mathcal{M}_*)$. The former couple using the left embedding is denoted by $(\mathcal{M} \phi, \mathcal{M}_*)$. The right version of Kosaki L^p is

$$L^p(\mathcal{M}, \phi)^r = [\phi \mathcal{M}, \mathcal{M}_*]_{1/p} = [\mathcal{M}_*, \phi \mathcal{M}]_{1-1/p}.$$

Let us define $Z_p(\mathcal{M}, \phi)^r$ as the push-out space (actually right module on \mathcal{M}) in the ubiquitous diagram

$$(22) \quad \begin{array}{ccccc} \ker \delta_\theta & \longrightarrow & \mathcal{G}(\mathcal{M}, \phi)^r & \xrightarrow{\delta_\theta} & L^p(\mathcal{M}, \phi)^r \\ \delta'_\theta \downarrow & & \downarrow & & \parallel \\ L^p(\mathcal{M}, \phi)^r & \longrightarrow & \text{PO} & \longrightarrow & L^p(\mathcal{M}, \phi)^r \end{array}$$

where $\theta = 1/p$ and $\mathcal{G}(\mathcal{M}, \phi)^r$ is the Calderón space associated to the couple $(\phi \mathcal{M}, \mathcal{M}_*)$.

We want to see that if $p, q \in (1, \infty)$ are conjugate exponents, then the conjugate of $Z_p(\mathcal{M}, \phi)^\ell$ (our former $Z_p(\mathcal{M}, \phi)$) is well isomorphic to $Z_q(\mathcal{M}, \phi)^r$.

Consider the couples $(\mathcal{M} \phi, \mathcal{M}_*)$ and $(\mathcal{M}_*, \phi \mathcal{M})$ (not $(\phi \mathcal{M}, \mathcal{M}_*)!$). Then

$$\Delta^\ell = \Delta(\mathcal{M} \phi, \mathcal{M}_*) = \mathcal{M} \phi \quad \text{and} \quad \Delta^r = \Delta(\mathcal{M}_*, \phi \mathcal{M}) = \phi \mathcal{M}.$$

Both $\mathcal{M} \phi$ and $\phi \mathcal{M}$ are dense in \mathcal{M}_* since ϕ is faithful. Define a bilinear form $\beta : \Delta^\ell \times \Delta^r \rightarrow \mathbb{C}$ by $\beta(a\phi, \phi b) = \phi(ba)$. The key point is that

$$\beta(a\phi, \phi b) = \langle a\phi, b \rangle = \langle a, \phi b \rangle,$$

where the brackets refer to the dual pairing between \mathcal{M}_* and \mathcal{M} . (Notice, moreover, that β is balanced in the sense that $\beta(cf, g) = \beta(f, gc)$ for $f \in \Delta^\ell, g \in \Delta^r$ and $c \in \mathcal{M}$.)

Then β is bounded both at $\theta = 0$ and $\theta = 1$. Indeed, for $\theta = 0$ one has

$$|\phi(ba)| \leq \|a\|_{\mathcal{M}} \|\phi b\|_{\mathcal{M}_*}.$$

Similarly, when $\theta = 1$,

$$|\phi(ba)| \leq \|a\phi\|_{\mathcal{M}_*} \|b\|_{\mathcal{M}}.$$

By bilinear interpolation [1, Theorem 4.4.1] β extends to a bounded bilinear form on $L^p(\mathcal{M}, \phi)^\ell \times L^q(\mathcal{M}, \phi)^r = [\mathcal{M}\phi, \mathcal{M}_*]_\theta^\ell \times [\mathcal{M}_*, \phi\mathcal{M}]_\theta^r$ which provides the dual pairing between $L^p(\mathcal{M}, \phi)^\ell$ and $L^q(\mathcal{M}, \phi)^r$ (see [20] or [24]). Let us call β to that extension.

For $1 < p < \infty$, let $\Omega_p^\ell : L^p(\mathcal{M}, \phi) \rightarrow \mathcal{M}_*$ be the derivation associated to the identity $[\mathcal{M}\phi, \mathcal{M}_*]_{1/p}^\ell = L^p(\mathcal{M}, \phi)$ and $\Omega_q^r : L^p(\mathcal{M}, \phi)^r \rightarrow \mathcal{M}_*$ that associated to $[\phi\mathcal{M}, \mathcal{M}_*]_{1/p}^r = L^p(\mathcal{M}, \phi)^r$. Note that if $\theta = 1/p$, then the derivation associated to $[\mathcal{M}_*, \phi\mathcal{M}]_\theta^r = L^q(\mathcal{M}, \phi)^r$ is just $-\Omega_q^r$. Proposition 1.3 in [9] yields

$$|\beta(\Omega_p^\ell(f), g) - \beta(f, \Omega_q^r(g))| \leq \frac{\pi}{\sin(\pi\theta)} \|f\|_p \|g\|_q$$

at least when $f \in \mathcal{M}\phi$ and $g \in \phi\mathcal{M}$. The following result is implicit in [26].

THEOREM 6. *Given all this paraphernalia, if $p, q \in (1, \infty)$ are conjugate exponents, then the dual of $Z_p(\mathcal{M}, \phi)^\ell = L^p(\mathcal{M}, \phi)^\ell \oplus_{\Omega_p^\ell} L^p(\mathcal{M}, \phi)^\ell$ is isomorphic to $-Z_q(\mathcal{M}, \phi)^r = L^q(\mathcal{M}, \phi)^r \oplus_{-\Omega_q^r} L^q(\mathcal{M}, \phi)^r$. More precisely, there is an isomorphism of right Banach modules over \mathcal{M} making commutative the following diagram*

$$(23) \quad \begin{array}{ccccc} (L^p(\mathcal{M}, \phi)^\ell)^* & \xrightarrow{\pi^*} & (Z_p(\mathcal{M}, \phi)^\ell)^* & \xrightarrow{i^*} & (L^p(\mathcal{M}, \phi)^\ell)^* \\ \parallel & & \uparrow u & & \parallel \\ L^q(\mathcal{M}, \phi)^r & \longrightarrow & L^q(\mathcal{M}, \phi)^r \oplus_{-\Omega_q^r} L^q(\mathcal{M}, \phi)^r & \longrightarrow & L^q(\mathcal{M}, \phi)^r \end{array}$$

PROOF. Put

$$(24) \quad (u(g', g))(f', f) = \beta(f, g') + \beta(f', g) \quad (g \in \Delta^r, f \in \Delta^\ell).$$

We have

$$\begin{aligned} |\beta(f, g') + \beta(f', g)| &= |\beta(f, g' + \Omega_q^r(g)) - \beta(f, \Omega_q^r(g)) + \beta(f' - \Omega_p^\ell(f), g) + \beta(\Omega_p^\ell(f), g)| \\ &\leq \|f\|_p \|g' + \Omega_q^r(g)\|_q + \|f' - \Omega_p^\ell(f)\|_p \|g\|_q + \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q \\ &\leq \frac{\pi}{\sin(\pi/p)} \|(g', g)\|_{-\Omega_q^r} \|(f', f)\|_{\Omega_p^\ell}. \end{aligned}$$

As Δ^ℓ is dense in $L^p(\mathcal{M}, \phi)^\ell$, we see that $L^p(\mathcal{M}, \phi)^\ell \oplus_{\Omega_p^\ell} \Delta^\ell$ is dense in $Z_p(\mathcal{M}, \phi)^\ell$ and so (24) shows that $u(g', g)$ acts, as a bounded linear functional on $Z_p(\mathcal{M}, \phi)^\ell$, with

$$\|u(g', g) : Z_p(\mathcal{M}, \phi)^\ell \rightarrow \mathbb{C}\| \leq M \|(g', g)\|_{-\Omega_q^r},$$

at least when g is in Δ^r . This defines an operator making the following diagram commute:

$$(25) \quad \begin{array}{ccccc} (L^p(\mathcal{M}, \phi)^\ell)^* & \xrightarrow{\pi^*} & (Z_p(\mathcal{M}, \phi)^\ell)^* & \xrightarrow{i^*} & (L^p(\mathcal{M}, \phi)^\ell)^* \\ \parallel & & \uparrow u & & \parallel \\ L^q(\mathcal{M}, \phi)^r & \longrightarrow & L^q(\mathcal{M}, \phi)^r \oplus_{-\Omega_q^r} \Delta^r & \longrightarrow & \Delta^r \end{array}$$

and where Δ^r is treated as a submodule of $L^q(\mathcal{M}, \phi)^r$. By density u extends to an operator that we still call u fitting in (23). The five-lemma and the open mapping theorem guarantee that u is a

linear homeomorphism. It remains to check it is also a homomorphism of right \mathcal{M} -modules. But for $g \in \Delta^r$ and $f \in \Delta^\ell$ one has

$$\begin{aligned} u((g', g)a)(f', f) &= (u(g'a, ga))(f', f) = \beta(f, g'a) + \beta(f', ga) \\ &= \beta(af, g') + \beta(af', g) = u(g', g)(af', af) = (u(g', g)a)(f', f). \end{aligned}$$

This completes the proof. \square

3.2. Change of state. In this Section we prove the extension $L^p(\mathcal{M}, \phi) \rightarrow Z_p(\mathcal{M}, \phi) \rightarrow L^p(\mathcal{M}, \phi)$ is essentially independent on the reference state ϕ in the following precise sense.

PROPOSITION 1. *Let ϕ_0 and ϕ_1 be faithful normal states on \mathcal{M} and $p \in (1, \infty)$. Then there is a commutative diagram*

$$\begin{array}{ccccc} L^p(\mathcal{M}, \phi_0) & \longrightarrow & Z_p(\mathcal{M}, \phi_0) & \longrightarrow & L^p(\mathcal{M}, \phi_0) \\ \alpha \downarrow & & \downarrow & & \downarrow \alpha \\ L^p(\mathcal{M}, \phi_1) & \longrightarrow & Z_p(\mathcal{M}, \phi_1) & \longrightarrow & L^p(\mathcal{M}, \phi_1) \end{array}$$

in which the vertical arrows are isomorphisms of left \mathcal{M} -modules.

PROOF. The proof is based on an idea explained and discarded by Kosaki in [20, p. 71]. We remark that our proof provides a very natural isometry between L^p spaces based on two different states.

It will be convenient to consider two more spaces of analytic functions. The first one is the obvious adaptation of the space \mathcal{H} appearing in Section 2.3 to the nonracial setting. So, given a faithful state $\phi \in \mathcal{M}_*$, we consider the couple $(\mathcal{M}\phi, \mathcal{M}_*)$, and the space $\mathcal{H} = \mathcal{H}(\mathcal{M}, \phi)$ of bounded functions $H : \mathbb{S} \rightarrow \mathcal{M}_*$ such such that:

- (1) H is continuous on \mathbb{S} and analytic on \mathbb{S}° with respect to $\sigma(\mathcal{M}_*, \mathcal{M})$.
- (2) $H(it) \in \mathcal{M}\phi$ for every $t \in \mathbb{R}$. The function $t \in \mathbb{R} \mapsto H(it) \in \phi\mathcal{M}$ is \mathcal{M} -bounded and $\sigma(\mathcal{M}, \mathcal{M}_*)$ -continuous.
- (3) The function $t \in \mathbb{R} \mapsto H(1 + it) \in \mathcal{M}_*$ is continuous in the norm of \mathcal{M}_* .

As one may expect we furnish \mathcal{H} with the norm $\|H\|_{\mathcal{H}} = \sup_t (\|H(it)\|_{\mathcal{M}}, \|H(1 + it)\|_{\mathcal{M}_*})$. Of course, \mathcal{H} is larger than \mathcal{G} . The second space we shall denote by $\mathcal{F} = \mathcal{F}(\mathcal{M}, \phi)$ is the space of those $f \in \mathcal{G}(\mathcal{M}, \phi)$ satisfying the additional condition that $f(it) \rightarrow 0$ in $\mathcal{M} = \mathcal{M}\phi$ as $|t| \rightarrow \infty$ and $f(1 + it) \rightarrow 0$ in \mathcal{M}_* as $|t| \rightarrow \infty$. Moreover the complex method of interpolation, applied to the couple $(\mathcal{M}, \mathcal{M}_*)$, leads to the same scale using \mathcal{F} , \mathcal{G} or \mathcal{H} :

$$[\mathcal{M}, \mathcal{M}_*]_{\theta}^{\mathcal{F}} = [\mathcal{M}, \mathcal{M}_*]_{\theta}^{\mathcal{G}} = [\mathcal{M}, \mathcal{M}_*]_{\theta}^{\mathcal{H}} = L^p(\mathcal{M}, \phi) \quad (\theta = 1/p)$$

with identical norms. This is very easy to check, once we know that $L^p(\mathcal{M}, \phi)$ is reflexive and agrees with the dual of the right space $L^q(\mathcal{M}, \phi)^r$, where q is the conjugate exponent of p . As Lemma 2 is true (with the same proof) replacing \mathcal{G} by \mathcal{F} or by \mathcal{H} we see that the lower extension in Diagram 19 does not vary after replacing \mathcal{G} by \mathcal{F} or by \mathcal{H} .

We shall use the following notations:

$$\begin{aligned} \mathcal{F}_0(\mathcal{M}, \phi) &= \{F \in \mathcal{F}(\mathcal{M}, \phi) : F(\theta) = 0\} \\ \mathcal{F}_1(\mathcal{M}, \phi) &= \{F \in \mathcal{F}(\mathcal{M}, \phi) : F(\theta) = F'(\theta) = 0\} \end{aligned}$$

and similarly for \mathcal{G} and \mathcal{H} . As we mentioned after Lemma 3 one has isomorphisms

$$Z_p(\mathcal{M}, \phi) = \frac{\mathcal{F}(\mathcal{M}, \phi)}{\mathcal{F}_1(\mathcal{M}, \phi)} = \frac{\mathcal{G}(\mathcal{M}, \phi)}{\mathcal{G}_1(\mathcal{M}, \phi)} = \frac{\mathcal{H}(\mathcal{M}, \phi)}{\mathcal{H}_1(\mathcal{M}, \phi)}.$$

It is important to realize how these quotient spaces arise as self-extensions of $L^p = L^p(\mathcal{M}, \phi)$. We describe the details for the smaller space \mathcal{F} ; replacing it by \mathcal{G} or \mathcal{H} makes no difference.

Recall that we have $\mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}$ and therefore an exact sequence

$$0 \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \xrightarrow{j} \mathcal{F}/\mathcal{F}_1 \xrightarrow{\varpi} \mathcal{F}/\mathcal{F}_0 \longrightarrow 0$$

where j and ϖ are the obvious maps. This becomes a self-extension of L^p after identifying $\mathcal{F}/\mathcal{F}_0$ with L^p through the (factorization) of the evaluation map $\delta_\theta : \mathcal{F} \rightarrow L^p$ at $\theta = 1/p$, while the identification of $\mathcal{F}_0/\mathcal{F}_1$ with L^p is provided by the (factorization) of the derivative $\delta'_\theta : \mathcal{F}_0 \rightarrow L^p$ (at $\theta = 1/p$) which is an isomorphism of left modules over \mathcal{M} .

We conclude these prolegomena with the following observation. Let $\mathcal{E}(\mathcal{M}, \phi)$ denote the subspace of those $F \in \mathcal{F}(\mathcal{M}, \phi)$ having the form $F(z) = f(z)\phi$, where $f : \mathbb{S} \rightarrow \mathcal{M}$ is continuous and analytic on the interior. It turns out that $\mathcal{E}(\mathcal{M}, \phi)$ is dense in $\mathcal{F}(\mathcal{M}, \phi)$. Indeed, the set of functions having the form $F(z) = f(z)\phi$, with

$$f(z) = \exp(\lambda z^2) \sum_{i=1}^n \exp(\lambda_i z) a_i \quad (\lambda, \lambda_i \in \mathbb{R}, a_i \in \mathcal{M})$$

is already a dense subspace of $\mathcal{F}(\mathcal{M}, \phi)$. See [1, Lemma 4.2.3].

Let $\varphi : \mathbb{S} \rightarrow \mathbb{D}$ be the function given by (6). Replacing \mathcal{G} by \mathcal{F} everywhere in the proof of Lemma 2 we see that $\mathcal{F}_0 = \varphi\mathcal{F}$ in the sense that the multiplication operator $f \mapsto \varphi f$ is an isomorphism between \mathcal{F} and \mathcal{F}_0 . Similarly, $f \mapsto \varphi^2 f$ is an isomorphism between \mathcal{F} and \mathcal{F}_1 . It follows that $\mathcal{E} \cap \mathcal{F}_0$ and $\mathcal{E} \cap \mathcal{F}_1$ are dense in \mathcal{F}_0 and in \mathcal{F}_1 , respectively.

Now we need a bit of (relative) modular theory for which we refer the reader to [20] or [24]. We fix two faithful states $\phi_0, \phi_1 \in \mathcal{M}_*$ and we consider the Connes-Radon-Nikodým cocycle of ϕ_0 relative to ϕ_1 :

$$(D\phi_0; D\phi_1)_t = \Delta_{\phi_0\phi_1}^{it} \Delta_{\phi_0}^{-it} \quad (t \in \mathbb{R}).$$

As it happens, $t \mapsto (D\phi_0; D\phi_1)_t$ is a strongly continuous path of unitaries in \mathcal{M} and so

$$(26) \quad t \mapsto (D\phi_0; D\phi_1)_t \phi_1$$

defines a continuous function from \mathbb{R} to \mathcal{M}_* . Now the point is that (26) extends to a function from the horizontal strip $-i\mathbb{S} = \{z \in \mathbb{C} : -1 \leq \Im(z) \leq 0\}$ to \mathcal{M}_* we may denote by $\overline{(D\phi_0; D\phi_1)}_{(\cdot)\phi_1}$ having the following properties:

- (a) For each $x \in \mathcal{M}$, the function $z \mapsto \overline{\langle (D\phi_0; D\phi_1)_{(z)} \phi_1, x \rangle}$ is continuous on $-i\mathbb{S}$ and analytic on $i\mathbb{S}^\circ$.
- (b) $\overline{(D\phi_0; D\phi_1)}_{(-i+t)\phi_1} = \phi_0(D\phi_0; D\phi_1)_t$ for every real t .

We are going to define an isometric embedding of modules $I : \mathcal{F}(\mathcal{M}, \phi_0) \rightarrow \mathcal{H}(\mathcal{M}, \phi_1)$. First, for $F \in \mathcal{E}(\mathcal{M}, \phi_0)$, we put

$$(27) \quad (IF)(z) = f(z) \overline{(D\phi_0; D\phi_1)_{(-iz)} \phi_1} \quad (F(z) = f(z)\phi_0, z \in \mathbb{S}).$$

We observe that for such an F one has $\|F\|_{\mathcal{F}} = \max\{\|f(it)\|_{\mathcal{M}}, \|f(1+it)\phi_0\|_{\mathcal{M}_*} : t \in \mathbb{R}\}$.

Let us check that $IF \in \mathcal{H}(\mathcal{M}, \phi_1)$. That IF satisfies (1) is obvious from (a). Regarding the values of IF on the boundary of \mathbb{S} we have for real t :

$$(28) \quad (IF)(it) = f(it)(D\phi_0; D\phi_1)_t \phi_1$$

which certainly falls in $\mathcal{M}\phi_1$ since $(D\phi_0; D\phi_1)_t$ is unitary and, besides, $\|f(it)(D\phi_0; D\phi_1)_t\|_{\mathcal{M}} = \|f(it)\|_{\mathcal{M}}$. Moreover the function $t \in \mathbb{R} \mapsto f(it)(D\phi_0; D\phi_1)_t \in \mathcal{M}$ is $\sigma(\mathcal{M}, \mathcal{M}_*)$ continuous since $t \in \mathbb{R} \mapsto f(it) \in \mathcal{M}$ is continuous for the norm and $t \in \mathbb{R} \mapsto (D\phi_0; D\phi_1)_t \in \mathcal{M}$ is strongly (hence $\sigma(\mathcal{M}, \mathcal{M}_*)$) continuous. So (2) holds as well.

On the other hand,

$$(IF)(1+it) = f(1+it)\overline{(D\phi_0; D\phi_1)_{(-i+t)}\phi_1} = f(1+it)\phi_0(D\phi_0; D\phi_1)_t,$$

so $\|(IF)(1+it)\|_{\mathcal{M}_*} = \|f(1+it)\phi_0(D\phi_0; D\phi_1)_t\|_{\mathcal{M}_*} = \|f(1+it)\phi_0\|_{\mathcal{M}_*}$ and $(IF)(1+it)$ is continuous in t for the norm topology of \mathcal{M}_* . Finally, that IF is \mathcal{M}_* bounded on the whole \mathbb{S} now follows by interpolation, using (28). Hence IF belongs to $\mathcal{H}(\mathcal{M}, \phi_1)$ and, moreover, the norm of IF in $\mathcal{H}(\mathcal{M}, \phi_1)$ and the norm of F in $\mathcal{F}(\mathcal{M}, \phi_0)$ coincide.

By density I extends to an isometric homomorphism of left \mathcal{M} -modules from $\mathcal{F}(\mathcal{M}, \phi_0)$ into $\mathcal{H}(\mathcal{M}, \phi_1)$ we call again I .

Now we observe that I maps $\mathcal{F}_0(\mathcal{M}, \phi_0)$ into $\mathcal{H}_0(\mathcal{M}, \phi_1)$. Indeed, it is obvious from (27) that IF vanishes at θ if $F \in \mathcal{E}$ vanishes at θ and for arbitrary F the result follows by a density argument, taking into account that $\mathcal{H}_0(\mathcal{M}, \phi_1)$ is closed in $\mathcal{H}(\mathcal{M}, \phi_1)$. In particular, I induces a contractive homomorphism from $L^p(\mathcal{M}, \phi_0)$ to $L^p(\mathcal{M}, \phi_1)$.

Similarly, I maps $\mathcal{F}_1(\mathcal{M}, \phi_0)$ into $\mathcal{H}_1(\mathcal{M}, \phi_1)$. Indeed, for $F \in \mathcal{E}(\mathcal{M}, \phi_0)$ one has

$$(IF)'(z) = f'(z)\overline{(D\phi_0; D\phi_1)_{(-iz)}\phi_1} + f(z)\frac{d}{dz}\overline{(D\phi_0; D\phi_1)_{(-iz)}\phi_1} \quad (F(z) = f(z)\phi_0).$$

Thus, if $F(\theta) = F'(\theta) = 0$, then $f(\theta) = f'(\theta) = 0$ and therefore $(IF)(\theta) = (IF)'(\theta) = 0$ and we proceed as before for general F .

Therefore we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}_0/\mathcal{F}_1 & \longrightarrow & \mathcal{F}(\phi_0)/\mathcal{F}_1 & \longrightarrow & \mathcal{F}/\mathcal{F}_0 \\
\downarrow \delta'_\theta & \searrow & \downarrow & \searrow & \downarrow \\
\mathcal{H}_0/\mathcal{H}_1 & \xrightarrow{(\delta'_\theta, \delta_\theta)} & \mathcal{H}(\phi_1)/\mathcal{H}_1 & \xrightarrow{\delta_\theta} & \mathcal{H}/\mathcal{H}_0 \\
\downarrow \delta'_\theta & \searrow & \downarrow & \searrow & \downarrow \\
L^p(\mathcal{M}, \phi_0) & \xrightarrow{\beta} & Z_p(\mathcal{M}, \phi_0) & \xrightarrow{(\delta'_\theta, \delta_\theta)} & L^p(\mathcal{M}, \phi_0) \\
\downarrow \beta & \searrow & \downarrow \gamma & \searrow & \downarrow \alpha \\
L^p(\mathcal{M}, \phi_1) & \xrightarrow{\gamma} & Z_p(\mathcal{M}, \phi_1) & \xrightarrow{\alpha} & L^p(\mathcal{M}, \phi_1)
\end{array}$$

The arrows in the preceding Diagram can be described as follows. First, all arrows in the upper face going from left to right are the obvious ones. All arrows in the upper face going from spaces based on ϕ_0 to spaces based on ϕ_1 are induced by I .

All vertical arrows are given by (factorization of) evaluations at $\theta = 1/p$, as indicated in the diagram. They are all isomorphisms of left modules over \mathcal{M} . Thus, for instance $(\delta'_\theta, \delta_\theta) : \mathcal{H}(\phi_1)/\mathcal{H}_1 \rightarrow Z_p(\mathcal{M}, \phi_1)$ takes (the class of) $H \in \mathcal{H}(\mathcal{M}, \phi_1)$ into the pair $(H'(\theta), H(\theta)) \in Z_p(\mathcal{M}, \phi_1)$ and so on.

The arrows lying in the bottom face are mere “shadows” of the corresponding arrows in the top face. It is really easy to see that all arrows in the bottom face going from left to right act as expected. Let us identify the arrows of the bottom face going from objects based on ϕ_0 to objects based on ϕ_1 . We begin with α . Suppose $x \in L^p(\phi_0)$ has the form $x = a\phi_0$, with $a \in \mathcal{M}$. Let

$\varepsilon : \mathbb{S} \rightarrow \mathbb{C}$ be an analytic function such that $\varepsilon(\theta) = 1$ and $\varepsilon(\infty) = 0$. Letting $F(z) = \varepsilon(z)a\phi_0$ we have $(IF)(z) = \varepsilon(z)a\overline{(D\phi_0; D\phi_1)_{(-iz)}\phi_1}$ and so $\alpha(a\phi_0) = (IF)(\theta) = a\overline{(D\phi_0; D\phi_1)_{(-i\theta)}\phi_1}$.

In order to identify β we take again $x = a\phi_0$ and we “jump” to \mathcal{F}_0 taking $F(z) = (\varphi'(\theta))^{-1}\varepsilon(z)\varphi(z)a\phi_0$, where φ is the function defined by (6). Note that

$$\varphi'(\theta) = \frac{\pi \exp(i\pi\theta)}{2 \sin(\pi\theta)} \neq 0.$$

One has

$$F'(\theta) = \frac{(\varepsilon\varphi)'(\theta)}{\varphi'(\theta)}a\phi_0 = a\phi_0.$$

Hence $\beta(a\phi_0) = (IF)'(\theta)$, where $(IF)(z) = (\varphi'(\theta))^{-1}\varepsilon(z)\varphi(z)a\overline{(D\phi_0; D\phi_1)_{(-iz)}\phi_1}$ and so by Leibniz rule

$$\beta(a\phi_0) = (IF)'(\theta) = a\overline{(D\phi_0; D\phi_1)_{(-i\theta)}\phi_1} = \alpha(a\phi_0).$$

Finally, the same argument shows that

$$\gamma(b\phi_0, a\phi_0) = (b\overline{(D\phi_0; D\phi_1)_{(-i\theta)}\phi_1}, a\overline{(D\phi_0; D\phi_1)_{(-i\theta)}\phi_1}).$$

To complete the proof we have to prove that α is an isomorphism – that γ is an isomorphism then follows from the five-lemma. This is not automatic because $I : \mathcal{F}(\phi_0) \rightarrow \mathcal{H}(\phi_1)$ is not surjective.

Anyway, reversing the rôles of ϕ_0 and ϕ_1 we know that there is a homomorphism of left \mathcal{M} modules $\omega : L^p(\mathcal{M}, \phi_1) \rightarrow L^p(\mathcal{M}, \phi_0)$ such that

$$\omega(x) = a\overline{((D\phi_1; D\phi_0)_{(-i\theta)}\phi_0)} \quad (x = a\phi_1 \in L^p(\phi_1)).$$

We will prove that α and ω are inverse of each other.

To this end, let us say that $a \in \mathcal{M}$ is “analytic” if the map

$$t \in \mathbb{R} \mapsto a(D\phi_0; D\phi_1)_t \in \mathcal{M}$$

extends to an entire function we shall denote by $\overline{a(D\phi_0; D\phi_1)_{(\cdot)}}$. This is a “left” version of the usual definition; see, e. g., [20, p. 73]. Let \mathcal{A} denote the set of “analytic” operators in \mathcal{M} . It is not hard to see that the \mathcal{A} is σ -weak ($= \sigma(\mathcal{M}, \mathcal{M}_*)$) dense in \mathcal{M} and so the set $\mathcal{A}\phi_0 = \{a\phi_0 : a \in \mathcal{A}\}$ is dense in $L^p(\phi_0)$. Thus the proof will be complete if we show that $\omega(\alpha(a\phi_0)) = a\phi_0$ for $a \in \mathcal{A}$. But for such an a we have

$$\alpha(a\phi_0) = \overline{(a(D\phi_0; D\phi_1)_{(-i\theta)})} \cdot \phi_1 = a \cdot \overline{((D\phi_0; D\phi_1)_{(-i\theta)}\phi_1)}$$

by the uniqueness of analytic continuation. Therefore, as $\overline{a(D\phi_0; D\phi_1)_{(-i\theta)}}$ belongs to \mathcal{M} ,

$$\omega(\alpha(a\phi_0)) = \omega\left(\overline{(a(D\phi_0; D\phi_1)_{(-i\theta)})} \cdot \phi_1\right) = \overline{(a(D\phi_0; D\phi_1)_{(-i\theta)})} \cdot \overline{((D\phi_1; D\phi_0)_{(-i\theta)}\phi_0)} = a\phi_0,$$

again by the uniqueness of analytic continuation, taking into account that $(D\phi_0; D\phi_1)_t = ((D\phi_1; D\phi_0)_t)^* = ((D\phi_1; D\phi_0)_t)^{-1}$ for $t \in \mathbb{R}$. \square

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