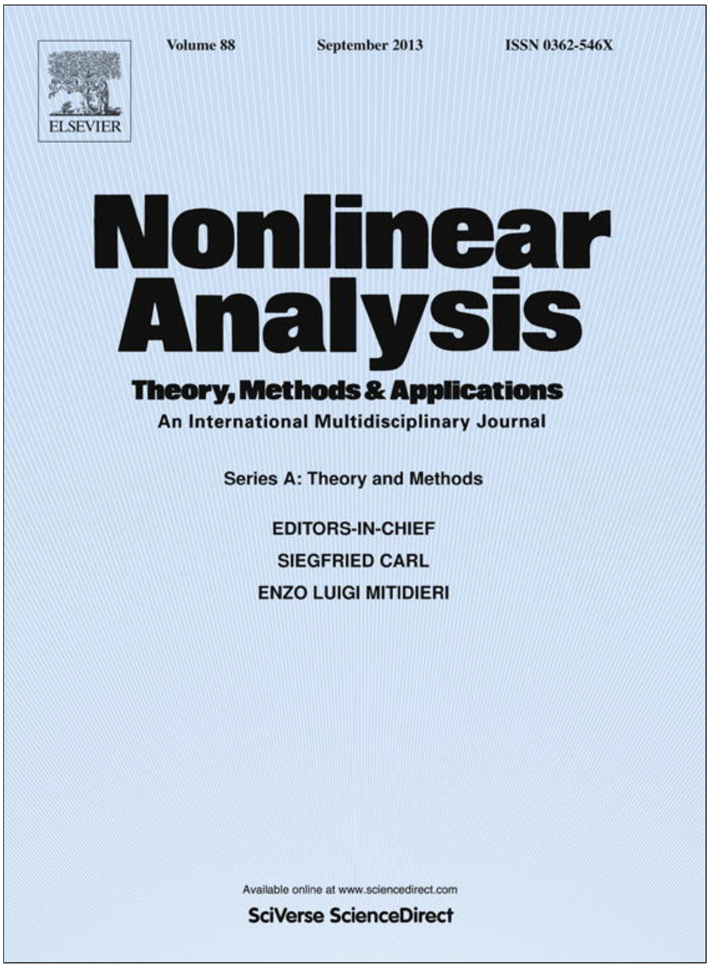
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Nonlinear centralizers with values in L_0 ☆

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ABSTRACT

It is shown that every centralizer from any "metric function space" X to L_0 is continuous at the origin of X. As a consequence, every short exact sequence of L_∞-modules $0 \to L_0 \to Z \to X \to 0$ splits if X is a "minimal" function space, and in particular if $X = L_0$. There are pairs of Orlicz function spaces U, V such that $\mathrm{Hom}(U, V) = 0$, but $\mathrm{Ext}(U, V) \neq 0$.

1. Introduction

1.1. Background

This paper deals with centralizers on function spaces. These are (in general neither linear nor continuous) maps that "almost commute" with multiplication operators.

The notion of a centralizer is an invention of Kalton, who introduced it in the memoir [1], isolating a property shared by most "derivations" appearing in interpolation theory. Later on, Kalton himself proved that every centralizer (on a super-reflexive Köthe space) arises as the derivation induced by interpolation of three Köthe function spaces [2, Theorems 7.6 and 7.9].

Our interest in centralizers stems from the fact that they are automatically quasilinear maps and therefore they give rise to "twisted sums". Many quasilinear maps appearing in nature are centralizers (actually the quasilinear maps considered in [3] are all centralizers). Indeed, if $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function (vanishing at the origin) and X is a Banach function space on an interval I, then the map $X \to L_0$ given by

$$f \longmapsto f\varphi\left(\log \frac{|f|}{\|f\|_X}, \log \frac{r_f}{\|f\|_X}\right) \tag{1}$$

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is a centralizer on X. Here r_f is the so-called rank-function of f defined by

$$r_f(t) = \lambda\{s \in I : |f(s)| > |f(t)| \text{ or } s \le t \text{ and } |f(s)| = |f(t)|\}$$

(λ is Lebesgue measure on I), which arises in real interpolation; see [1, Section 3] and especially Theorem 3.1.

Let us indicate the connection between centralizers and interpolation theory. Typically, an interpolation method takes a suitable couple of Banach or quasi-Banach spaces (X_0, X_1) and produces a parametric family of spaces that lie between X_0 and X_1 in a certain sense that depends on the method.

In practice, whenever one encounters a given space X inside an interpolation scale one finds a certain (neither linear nor bounded) mapping, defined on X, which "almost commutes" with those linear operators which are simultaneously bounded on both spaces of the starting couple. Such maps often arise as derivatives and are called "derivations". Good sources on these matters and many related things are [4–6].

Thus, for instance, for the complex method of Calderón, starting with the couple (L_∞, L_1) the intermediate spaces are the Lebesgue spaces L_p for $p \in [1, \infty]$ and the "derivation" is $f \mapsto f \log(|f|/\|f\|_p)$ for $f \in L_p$, which corresponds to the choice $\varphi(x, y) = x$ in (1).

If we use the real method instead, the intermediate spaces are the family of Lorentz spaces $L_{p,q}$ for $p, q \in [1, \infty]$ (in particular $L_{p,p} = L_p$) and the corresponding map is given by $f \mapsto f \log(r_f/\|f\|_{p,q})$, which corresponds to the choice $\varphi(x, y) = y$ in (1).

All this can be seen in [4] where it is remarked that, while these "derivations" are unbounded on L_p for $0 < p < \infty$, they become bounded both on L_∞ and on L_0 and it is asked whether this is an instance of a more general fact [4, p. 201].

In this note we give a partial answer by showing that all centralizers are continuous at the origin both on L_∞ and on L_0.

We hasten to remark that, although our initial motivation (together with the idea of a range space) comes from [4], our approach is more akin to those of the Kalton studies [1,2], and the main applications of the paper concern twisted sums.

1.2. The plan of the paper

Let us summarize the results of the paper. We consider centralizers in a very general sense. Indeed, if X and Y are (metric) function spaces, then a centralizer from X to Y will be a mapping $\Phi : X \to L_0$ (not Y) such that $\Phi(af) - a\Phi(f) \to 0$ in Y as $(a, f) \to (0, 0)$ in $L_\infty \times X$.

Section 2 deals with centralizers as pure maps. It is shown that every centralizer from any function space X to L_0 is continuous at the origin of X—nothing more can be expected since any mapping continuous at the origin is a centralizer. Also, we prove that if the map $f \mapsto f \log(|f|/\|f\|_X)$ is trivial as a centralizer on X and X is locally bounded and symmetric, then $X = L_\infty$, up to an equivalent quasi-norm.

We exhibit examples showing that this result cannot be extended beyond the locally bounded setting.

In Section 3 we exploit the connections between centralizers and twisted sums and we show that every short exact sequence of L_∞-modules $0 \to L_0 \to Z \to X \to 0$ splits if X is a "minimal" function space, and in particular if $X = L_0$.

We also show that the so-called domain of Φ, which is defined as $\mathrm{Dom}(\Phi) = \{f \in X : \Phi(f) \in Y\}$, is always a complete function space and we derive the curious fact that every exact sequence of L_∞-modules $0 \to Y \to Z \to X \to 0$ which splits in the purely algebraic sense also splits "topologically".

In Section 4 we use the idea of a range space (which is in some way dual to that of a domain space) and we show that every centralizer Ω from X to Y gives rise to a centralizer $\mho$ from $\mathrm{Ran}\,\Omega$ to $\mathrm{Dom}\,\Omega$. From this we derive that there are couples of Orlicz function spaces U, V such that $\mathrm{Hom}(U, V) = 0$ but $\mathrm{Ext}(U, V) \neq 0$.

1.3. Function spaces

Let L_0 denote the space of all real-valued measurable functions on the unit interval $\mathbb{I} = [0, 1]$, where we identify two functions if they agree almost everywhere. As usual, we consider the topology of convergence in measure on L_0. This topology can be described with the F-norm

$$|f|_0 = \int_0^1 \frac{|f|}{1 + |f|}.$$

By a "function space" we mean a linear subspace X of L_0 equipped with an F-norm $|\cdot|_X$ such that if $|f| \le g$ and $g \in X$, then $f \in X$ and $|f|_X \le |g|_X$. Besides, we assume that $1 \in X$ and that the inclusion $X \subset L_0$ is continuous. We say that X is minimal if L_∞ is dense in X.

When dealing with locally bounded spaces we will use a quasi-norm to describe the topology. Accordingly, a quasi-normed function space is a linear subspace X of L_0 equipped with a quasi-norm $\|\cdot\|_X$ in such a way that if $|f| \le g$ and $g \in X$, then $f \in X$ and $\|f\|_X \le \|g\|_X$. As before, we assume that $1 \in X$ and that the inclusion $X \subset L_0$ is continuous. Notice that quasi-norms need not be F-norms. However, in view of the Aoki–Rolewicz theorem, every quasi-normed function space has an equivalent p-norm $\|\cdot\|_X$ for some $p \in (0, 1]$ and so the function $|f|_X = \|f\|_X^p$ is a "function space" F-norm on X which gives the same topology as the original quasi-norm.

A quasi-Banach function space X is said to be symmetric if for every Borel, measure-preserving map σ on $\mathbb{I}$ and every $f \in X$, one has $f \circ \sigma \in X$ and $\|f \circ \sigma\|_X = \|f\|_X$.

2. Centralizers

Let X and Y be function spaces. A centralizer from X to Y is a mapping $\Phi : X \to L_0$ (not Y) such that $\Phi(0) = 0$ and, for every $f \in X$ and every $a \in L_\infty$, the difference $\Phi(af) - a\Phi(f)$ belongs to Y and converges to zero in Y as (a, f) goes to zero in $L_\infty \times X$.

The set of all centralizers from X to Y will be denoted by $\mathscr{C}(X, Y)$, or just $\mathscr{C}(X)$ when $Y = X$ in which case we speak of self-centralizers.

There are two kinds of mappings which are trivially centralizers. First, every mapping $\Phi : X \to Y$ (not L_0) which is continuous at the origin, with $\Phi(0) = 0$, is a centralizer. Second, any mapping $\Phi : X \to L_0$ satisfying that $\Phi(af) = a\Phi(f)$ for every $a \in L_\infty$ and every $f \in X$ is a centralizer. Such maps are necessarily implemented by multiplication by a fixed function, as we now prove.

Lemma 1. *Suppose that $\Phi : X \to L_0$ is a mapping satisfying that $\Phi(af) = a\Phi(f)$ for every $a \in L_\infty$ and every $f \in X$. Then there is $\phi \in L_0$ such that $\Phi(f) = \phi f$ for every $f \in X$.*

Proof. Set $\phi = \Phi(1)$. We want to see that $\Phi(f) = \phi f$ for every $f \in X$. Pick $f \in X$ and put $g = 1 + |f|$. Then $1 = (1/g)g$ and since $1/g \in L_\infty$ we have $\phi = (1/g)\Phi(g)$ and so $\Phi(g) = g\phi$. But $f = (f/g)g$, with $f/g \in L_\infty$. Hence

$$\Phi(f) = \Phi\left((f/g)g\right) = (f/g)\Phi(g) = (f/g)\phi g = \phi f,$$

as required. $\square$

Accordingly, we will say that Φ is a trivial centralizer (from X to Y) if it is the sum of two mappings of the preceding types, or, what comes to the same, if there is $\phi \in L_0$ such that $f \mapsto \Phi(f) - \phi f$ takes values in Y and is continuous at zero.

It is obvious that every centralizer from L_∞ is trivial. Indeed, if Y is any function space and $\Phi \in \mathscr{C}(L_\infty, Y)$, then $\Phi(f) - \phi f \to 0$ in Y as $f \to 0$ in X, where $\phi = \Phi(1)$.

Our immediate aim is to show that centralizers from any function space to L_0 are trivial (in fact continuous at zero). First, we need the following result, whose proof closely follows that of [1, Lemma 4.2].

Lemma 2. *Every centralizer is quasi-additive. More precisely, if $\Phi : X \to L_0$ is a centralizer from X to Y, then for every $f, g \in X$ one has $\Phi(f + g) - \Phi(f) - \Phi(g) \in Y$ and $|\Phi(f + g) - \Phi(f) - \Phi(g)|_Y \to 0$ as $|f|_X + |g|_X \to 0$.*

Proof. Let $\varepsilon > 0$. Fix $\gamma > 0$ such that $|\Phi(af) - a\Phi(f)|_Y \le \varepsilon$ for $\|a\|_\infty, |f|_X \le \gamma$. Now, choose $\delta > 0$ such that $|\gamma^{-1}h|_X < \gamma$ provided $h = |f| + |g|$, and $|f|_X, |g|_X < \delta$.

Suppose that $f, g \in X$ are such that $|f|_X, |g|_X \le \delta$. Take $h = |f| + |g|$ and write $f = uh$, $g = vh$ and $f + g = (u + v)h$, where $\|u\|_\infty, \|v\|_\infty, \|u + v\|_\infty \le 1$. One then has

$$\begin{aligned} |\Phi(f+g) - \Phi f - \Phi g|_Y &= |\Phi(\gamma(u+v)\gamma^{-1}h) - \Phi(\gamma u\gamma^{-1}h) - \Phi(\gamma v\gamma^{-1}z)|_Y \\ &\le \left|\Phi\left(\gamma(u+v)\frac{h}{\gamma}\right) - \gamma(u+v)\Phi\left(\frac{h}{\gamma}\right)\right|_Y + \left|\Phi\left(\frac{\gamma uh}{\gamma}\right) - \gamma u\Phi\left(\frac{h}{\gamma}\right)\right|_Y \\ &\quad + \left|\Phi\left(\frac{\gamma vh}{\gamma}\right) - \gamma v\Phi\left(\frac{h}{\gamma}\right)\right|_Y \\ &\le \varepsilon + \varepsilon + \varepsilon. \end{aligned}$$

Finally, that $\Phi(f + g) - \Phi(f) - \Phi(g)$ belongs to Y for every $f, g \in X$ obviously follows from the definition of a centralizer, since $\Phi(\lambda f) - \lambda\Phi(f) \in Y$ for every scalar λ and every $f \in X$. $\square$

Theorem 1. *Let X be a (not necessarily complete) function space and $\Phi \in \mathscr{C}(X, L_0)$. Then Φ is continuous at the origin of X.*

Proof. First suppose that $X = L_\infty$ and $|\cdot|_X = \|\cdot\|_\infty$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|\Phi(cu) - c\Phi(u)|_0 \le \varepsilon$ for $|c|, \|u\|_\infty \le \delta$. Now, take $\gamma > 0$ such that $|u\delta^{-1}\Phi(\delta 1)|_0 \le \varepsilon$ for all $\|u\|_\infty < \gamma$. Suppose that $\|u\|_\infty \le \min\{\delta^2, \gamma\}$. Then,

$$|\Phi(u)|_0 = |\Phi(\delta^{-1}u\delta 1)|_0 \le |\Phi(\delta^{-1}u\delta 1) - \delta^{-1}u\Phi(\delta 1)|_0 + |\delta^{-1}u\Phi(\delta 1)|_0 \le \varepsilon + \varepsilon,$$

as required.

Returning to the general case, let us fix $\varepsilon > 0$ again. Choose $\gamma > 0$ in such a way that $|f|_X, |g|_X, \|a\|_\infty \le \gamma$ implies:

(a) $|\Phi(f + g) - \Phi(f) - \Phi(g)|_0 \le \varepsilon$;
(b) $|\Phi(a)|_0 \le \varepsilon$;
(c) $|\Phi(af) - a\Phi(f)|_0 \le \varepsilon$.

As the set $V = \{f \in L_0 : \lambda(\{t \in \mathbb{I} : |f(t)| > \gamma\}) < \varepsilon\}$ is open in L_0, we can take $0 < \delta_1 \le \gamma$ such that every $f \in X$ with $|f|_X \le \delta_1$ belongs to V.

On the other hand multiplication by γ^{-1} is a continuous automorphism of X and we may take $\delta_2 > 0$ such that if $|f|_X \le \delta_2$, then $|\gamma^{-1}f|_X \le \gamma$.

Put $\delta = \min(\delta_1, \delta_2)$.

Suppose now that $|f|_X \le \delta$ and let us estimate $|\Phi(f)|_0$. Put $A = \{t \in \mathbb{I} : |f(t)| > \gamma\}$ and write $f = s + b$, where $b = 1_A f$ and $s = (1 - 1_A)f$. Then $|s|_X, |b|_X \le |f|_X < \delta \le \gamma$ and

$$|\Phi f|_0 \le |\Phi(s+b) - \Phi s - \Phi b|_0 + |\Phi s|_0 + |\Phi b|_0 \le \varepsilon + |\Phi s|_0 + |\Phi b|_0,$$

by (a). On the other hand, $|\Phi(s)|_0 \le \varepsilon$, by (b), since $\|s\|_\infty \le \gamma$. Finally,

$$\begin{aligned}|\Phi(b)|_0 &= |\Phi(\gamma 1_A \gamma^{-1} f)|_0 \\ &\le |\Phi(\gamma 1_A \gamma^{-1} f) - \gamma 1_A \Phi(\gamma^{-1} f)|_0 + |\gamma 1_A \Phi(\gamma^{-1} f)|_0 \le \varepsilon + \lambda(A) \le 2\varepsilon.\end{aligned}$$

Hence $|\Phi(f)|_0 \le 4\varepsilon$ and we are done. □

Now recall from the introduction that if X is any quasi-Banach function space, then the Kalton–Peck map $\Omega : X \to L_0$ given by $\Omega(f) = f \log(|f|/\|f\|_X)$ is a self-centralizer on X. It turns out that this map has a strong tendency to not be trivial when it acts on a locally bounded space. To be precise, we have the following "continuous" counterpart of [7, Corollary 2].

Example 1. Let X be a symmetric quasi-Banach function space. Then the centralizer $\Omega : X \to L_0$ given by $\Omega(f) = f \log(|f|/\|f\|_X)$ is trivial on X if and only if $X = L_\infty$, up to an equivalent norm.

Proof. If Ω is trivial, then there are $\phi \in L_0$ and M such that $\|\Omega f - \phi f\|_X \le M\|f\|_X$ for every $f \in X$. Take $A \subset \mathbb{I}$ with $\lambda(A) \ge \frac{1}{2}$ such that $1_A\phi \in L_\infty$ and set $X(A) = \{f \in X : \operatorname{supp} f \subset A\}$. Then, for some constant M,

$$\|\Omega(f)\|_X \le M\|f\|_X \quad (f \in X(A)).$$

Now, if $B \subset A$ we have $\Omega(1_B) = -\log(\|1_B\|_X)1_B$ and taking $f = 1_B$ in the preceding inequality and dividing by $\|1_B\|_X$ we have $|\log \|1_B\|_X| \le M$. It follows that there is $\delta_0 > 0$ such that

$$\|1_B\| \ge \delta_0 \quad (B \subset A, \lambda(B) > 0).$$

Pick now $f \in X(A)$. For each $k < \|f\|_\infty$ there is a set $B \subset A$ with $\lambda(B) > 0$ such that $|f| \ge k1_B$. Hence $\|f\|_X \ge k\delta_0$. It follows that every function $f \in X(A)$ is essentially bounded, with $\|f\|_X \ge \delta_0\|f\|_\infty$, and so

$$\delta_0\|f\|_\infty \le \|f\|_X \le \|1_A\|_X\|f\|_\infty \quad (f \in X(A)).$$

Thus, $X(A) = L_\infty(A)$ and, by symmetry, $X = L_\infty$, up to an equivalent quasi-norm. □

The result cannot be extended beyond the locally bounded setting, as the next example shows. Following [8], for $p \in (0, \infty)$, we denote by L_{p-} the space $\bigcap_{q<p} L_q$ with the projective topology. That topology is induced by the F-norm

$$|f|_{p-} = \sum_{n=1}^{\infty} \frac{\|f\|_{p_n}^{p_n}}{2^n(1 + \|f\|_{p_n}^{p_n})},$$

where (p_n) is any increasing sequence converging to p. Let us say that a centralizer Φ is strictly symmetric if $\Phi(f \circ \sigma) = (\Phi f) \circ \sigma$ for every Borel, measure-preserving automorphism $\sigma : \mathbb{I} \to \mathbb{I}$.

Example 2. Every strictly symmetric centralizer on L_{p-} is continuous at zero.

Proof. Let Φ be a strictly symmetric centralizer on L_{p-}. Observe that $\Phi(f) + \Phi(-f) \to 0$ in L_{p-} as $f \to 0$ in L_{p-} and so we may assume that Φ is an odd map, that is, $\Phi(-f) = -\Phi(f)$ for every $f \in L_{p-}$.

Claim. *For every $q < p$ there exist $r \in (q, p)$ and $\delta > 0$ such that* $\sup\{\|\Phi(f)\|_q : \|f\|_r = \delta, f \in L_{p-}\} < \infty$.

Proof of the Claim. Fix $q < p$. By the very definition of a centralizer there are $r \in (q, p)$ and $\delta > 0$ such that $\|\Phi(af) - a\Phi f\|_q \le 1$ whenever $\|a\|_\infty, \|f\|_r \le \delta$.

We define then $\Phi_{(q,r)}$ by letting

$$\Phi_{(q,r)}(f) = \frac{\|f\|_r}{\delta}\Phi\left(\frac{\delta f}{\|f\|_r}\right).$$

Let us check that $\Phi_{(q,r)}$ is a homogeneous centralizer from $(L_{p-}, \|\cdot\|_r)$ to L_q. Suppose that $\|a\|_\infty \le 1$ and $\|f\|_r \le \delta$. Then

$$\begin{aligned}\|\Phi_{(q,r)}(af) - a\Phi_{(q,r)}(f)\|_q &= \frac{1}{\delta}\left\| \|af\|_r\, \Phi\left(\frac{\delta af}{\|af\|_r}\right) - a\|f\|_r\, \Phi\left(\frac{\delta f}{\|f\|_r}\right)\right\|_q \\ &\le \frac{\Delta_q}{\delta}\left(\left\| \|af\|_r\, \Phi\left(\frac{\delta af}{\|af\|_r}\right) - \Phi\left(\frac{\delta\|af\|_r af}{\|af\|_r}\right)\right\|_q \right.\\ &\quad \left. + \left\|\Phi\left(\frac{\delta\|f\|_r af}{\|f\|_r}\right) - a\|f\|_r\, \Phi\left(\frac{\delta f}{\|f\|_r}\right)\right\|_q\right) \\ &\le \frac{2\Delta_q}{\delta},\end{aligned}$$

where Δ_q denotes the "modulus of concavity" of the quasi-norm $\|\cdot\|_q$, that is, $\Delta_q = 2^{1/q-1}$ for $q \in (0, 1]$ and $\Delta_q = 1$ for $q \ge 1$. Now, by homogeneity,

$$\|\Phi_{(q,r)}(af) - a\Phi_{(q,r)}(f)\|_q \le \frac{2\Delta_q}{\delta^2}\|a\|_\infty\|f\|_r \quad (a \in L_\infty, f \in L_{p-}).$$

By the main result of [9], $\Phi_{(q,r)}$ is trivial (as a centralizer from L_r to L_q) and so there are $\phi \in L_0$ and a constant M such that $\|\Phi_{(q,r)}(f) - \phi f\|_q \le M\|f\|_r$ for every $f \in L_{p-}$.

Take $A \subset \mathbb{I}$ such that $\lambda(A) = \frac{1}{2}$ and $1_A\phi \in L_\infty$. Then, if $f \in L_{p-}(A)$, one has $\|\Phi_{(q,r)}(f)\|_q \le M'\|f\|_r$ and, by symmetry, the same holds for $f \in L_{p-}(B)$, where $B = \mathbb{I} \setminus A$. For arbitrary $f \in L_{p-}$ we have

$$\|\Phi_{(q,r)}f\|_q \le C(\|\Phi_{(q,r)}(1_Af)\|_q + \|\Phi_{(q,r)}(1_Bf)\|_q) \le M'C(\|1_Af\|_r + \|1_Bf\|_r) \le M''\|f\|_r.$$

Therefore,

$$\left\|\Phi\left(\frac{\delta f}{\|f\|_r}\right)\right\|_q \le M''\delta \quad (f \in L_{p-}),$$

which proves the claim. □

To complete the proof, we fix $\varepsilon > 0$ and $q < p$ and we take r and δ as before, so

$$M = \sup_{\|f\|_r=\delta} \|\Phi(f)\|_q < \infty.$$

Also, there exist $r' > r$ and $0 < \delta' < \min(\delta, \varepsilon/M)$ such that

$$\|\Phi(\lambda f) - \lambda\Phi(f)\|_q \le \varepsilon \quad (|\lambda|, \|f\|_{r'} \le \delta', f \in L_{p-}).$$

Thus, for $|\lambda| < \delta'$ and $\|f\|_{r'} < \delta'$ we have

$$\|\Phi(\lambda f)\|_q \le \Delta_q(\varepsilon + |\lambda|\, \|\Phi(f)\|_q) \le \Delta_q(\varepsilon + \delta' M) \le 2\Delta_q\varepsilon.$$

We conclude that $\|\Phi(f)\|_q \le 2\Delta_q\varepsilon$ if $\|f\|_{r'} \le \delta'^2$ and $f \in L_{p-}$, which is enough. □

3. Extensions of modules

We now give some applications to the homology of L_∞-modules. A metric linear space X is said to be a module over L_∞ if there is a jointly continuous outer multiplication $L_\infty \times X \to X$ satisfying the traditional algebraical requirements. Evidently, every function space is a module over L_∞ under the "pointwise" product. From now on, our "default" category will be that of (metric) L_∞-modules and continuous homomorphisms. If X and Y are (metric) L_∞-modules, and in particular if they are function spaces, then a (continuous) homomorphism $h : X \to Y$ is a linear operator such that $h(ax) = ah(x)$ for every $a \in L_\infty$ and every $x \in X$. The space of homomorphisms from X to Y is denoted by $\mathrm{Hom}(X, Y)$.

Let X and Y be complete metric modules. An extension of X by Y is a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 0 \tag{2}$$

in which Z is a complete metric module and the arrows represent continuous homomorphisms. (Less technically, we may regard Y as a closed submodule of Z in such a way that the quotient Z/Y is isomorphic to X.) The extension (2) is said to be trivial (or to split) if there is a continuous homomorphism $\varpi : Z \to Y$ such that $\varpi \circ \imath = \mathbf{I}_Y$ (less technically, Y is complemented inside Z by a continuous projection of modules). Equivalently, there is a continuous homomorphism $\jmath : X \to Z$ such that $\pi \circ \jmath = \mathbf{I}_X$.

When properly classified and organized, the set of all extensions of X by Y becomes a linear space, denoted by $\mathrm{Ext}(X, Y)$, in such a way that the class of trivial sequences corresponds to zero. Thus, $\mathrm{Ext}(X, Y) = 0$ means "every short exact sequence of modules of the form (2) splits".

Corollary 1. *Let X be a complete, minimal, function space. Every extension (of modules) of X by L_0 splits.*

Proof. Suppose that $0 \longrightarrow L_0 \longrightarrow Z \xrightarrow{\pi} X \longrightarrow 0$ is an extension of modules. We may and do assume that $L_0 = \ker\pi$. As $\pi : Z \to X$ is open, there is a mapping $s : X \to Z$ such that $\pi \circ s = \mathbf{I}_X$, with $s(0) = 0$, and which is continuous at the origin (see [10, Lemma 2.2(a)]). We define Φ on $X_0 = X \cap L_\infty$ by $\Phi f = s(f) - fs(1)$. Then $\pi(\Phi(f)) = 0$, so Φ takes values in $L_0 = \ker\pi$. On the other hand,

$$\Phi(af) - a\Phi f = s(af) - as(f) \to 0$$

in L_0 as $(a, f) \to 0$ in $L_\infty \times X_0$, that is, $\Phi \in \mathscr{C}(X_0, L_0)$. Theorem 1 implies that Φ is continuous at zero and therefore $f \longmapsto fs(1)$ defines a continuous homomorphism $X_0 \to Z$ which is a right-inverse for π on X_0. Extending it to X completes the proof. □

We hasten to remark that L_0 is not injective amongst L_∞-modules. Indeed, let $L_0(\mathbb{I}^2)$ be the space of all measurable functions on the square and consider the (linear, topological) embedding $\imath : L_0 \to L_0(\mathbb{I}^2)$ given by $\imath(f)(s, t) = f(s)$. This is a continuous homomorphism of L_∞-modules, where the product $L_\infty \times L_0(\mathbb{I}^2) \to L_0(\mathbb{I}^2)$ is given by $(af)(s, t) = a(s)f(s, t)$. The extension

$$0 \longrightarrow L_0 \xrightarrow{\imath} L_0(\mathbb{I}^2) \xrightarrow{\pi} L_0(\mathbb{I}^2)/\imath L_0 \longrightarrow 0$$

does not split in the category of F-spaces, let alone in that of L_∞-modules. See [11].

There is another connection between centralizers and extensions which is based on the idea of a twisted sum. Let $\Phi \in \mathscr{C}(X, Y)$, where X and Y are complete function spaces. Consider

$$Y \oplus_\Phi X = \{(g, f) \in L_0 \times X : g - \Phi f \in Y\}.$$

(We are following the usual notation for twisted sums, with the subspace Y on the left. Note that in interpolation theory this order is reversed.) Using Lemma 2 it is very easy to see that $Y \oplus_\Phi X$ is a linear subspace of $L_0 \times X$. There is a (metrizable) linear topology on $Y \oplus_\Phi X$ for which the sets

$$U_\varepsilon = \{(g, f) : |g - \Phi f|_Y + |f|_X < \varepsilon\}$$

form a neighborhood-base at zero (see [10, Proposition 3.1]). Besides, the map $L_\infty \times (Y \oplus_\Phi X) \to Y \oplus_\Phi X$ given by $(a, (g, f)) \mapsto (ag, af)$ is jointly continuous and so it makes $Y \oplus_\Phi X$ into an L_∞-module. We define maps $\imath : Y \to Y \oplus_\Phi X$ and $\pi : Y \oplus_\Phi X \to X$ by $\imath(g) = (g, 0)$ and $\pi(g, f) = f$, respectively. Both maps are easily seen to be relatively open continuous homomorphisms of L_∞-modules and, moreover, $\imath(Y) = \ker\pi$, so $Y \oplus_\Phi X/\imath(Y)$ is isomorphic to X. As completeness is a "3-space" property (cf. [12, Theorem 12.1]), this implies that $Y \oplus_\Phi X$ is complete and

$$0 \longrightarrow Y \xrightarrow{\imath} Y \oplus_\Phi X \xrightarrow{\pi} X \longrightarrow 0 \tag{3}$$

is an extension of X by Y, which for good reasons is called the extension induced by Φ. It is not hard to see that (3) is trivial if and only if Φ is trivial.

Set $D = \mathrm{Dom}(\Phi) = \{f \in X : \Phi f \in Y\}$, which is a submodule of X in the purely algebraic sense. The map $\jmath : D \to Y \oplus_\Phi X$ given by $\jmath(f) = (0, f)$ is correctly defined and so we can put in D the induced topology. Thus a typical neighborhood of zero in D is

$$\{f \in X : |\Phi f|_Y + |f|_X < \varepsilon\} = \jmath^{-1}(U_\varepsilon). \tag{4}$$

Proposition 1. *If X and Y are complete function spaces and $\Phi \in \mathscr{C}(X, Y)$, then $\mathrm{Dom}(\Phi)$ is complete and the inclusion into L_0 is continuous.*

Proof. To prove that $D = \mathrm{Dom}(\Phi)$ is complete, it suffices to check that it is closed in $Y \oplus_\Phi X$, which is complete. Suppose that $(0, f_n)$ converges to (g, f) in $Y \oplus_\Phi X$, that is,

$$|g - \Phi(f - f_n)|_Y + |f - f_n|_X \to 0 \quad (n \to \infty).$$

Then $f_n \to f$ in X and $\Phi(f - f_n) \to g$ in Y. But Theorem 1 implies that $\Phi(f - f_n) \to 0$ in L_0 and since the inclusion of Y into L_0 is continuous, we have $g = 0$.

Finally, it is obvious from Theorem 1 that the inclusion of D in L_0 is continuous. □

The following result explains why centralizers take values in L_0 and not in the "target space" Y.

Corollary 2. *Let X and Y be complete function spaces and $\Phi \in \mathscr{C}(X, Y)$. If $\mathrm{Dom}(\Phi) = X$, then Φ is continuous at zero. Consequently, every extension of L_∞-modules $0 \to Y \to Z \to X \to 0$ which splits in the pure algebraic sense also splits topologically.*

Proof. The composition $\pi \circ \jmath : \mathrm{Dom}(\Phi) \to Y \oplus_\Phi X \to X$ is injective and continuous. If $\mathrm{Dom}(\Phi) = X$ it is also open, the sets appearing in (4) are neighborhoods of the origin for the topology of X and so $|\Phi f|_Y \to 0$ as $|f|_X \to 0$.

As for the second part, suppose that $0 \longrightarrow Y \longrightarrow Z \xrightarrow{\pi} X \longrightarrow 0$ splits algebraically, that is, there is a linear section m for the quotient map such that $m(af) = am(f)$ for every $a \in L_\infty$ and every $f \in X$. If s is a section for π, with $s(0) = 0$, and continuous at zero, then the difference $\Phi = s - m$ takes values in $Y = \ker\pi$ and is a centralizer. Thus $X = \mathrm{Dom}(\Phi)$ and Φ is continuous at zero and so is m. □

4. Hom versus Ext

Many results in homology revolve around the vague idea that if the space $\mathrm{Hom}(X, Y)$ is "small", then $\mathrm{Ext}(X, Y)$ "must" vanish. For instance, it can be proved that if X and Y are (quasi-)Banach function spaces and $\mathrm{Hom}(X, Y)$ is nonzero and q-concave for some finite q, then $\mathrm{Ext}(X, Y) = 0$. Details will appear elsewhere.

Recall that if X and Y are function spaces, then every homomorphism of modules $m : X \to Y$ has the form $m(f) = \phi f$ for some fixed $\phi \in L_0$; see Lemma 1.

Here we will adapt some ideas given in [4] to present examples of (quasi-)Banach function spaces X, Y where $\mathrm{Hom}(X, Y)$ vanishes and $\mathrm{Ext}(X, Y)$ does not. It is perhaps a little ironic that this is in fact a ubiquitous phenomenon (see Corollary 3).

From now on we consider quasi-Banach function spaces and homogeneous centralizers only and we use quasi-norms (instead of F-norms) to describe the topology. Note that if X and Y are quasi-Banach function spaces, then a homogeneous map $\Omega : X \to L_0$ is a centralizer if and only if there is a constant C such that

$$\|\Omega(af) - a\Omega f\|_Y \le C\|a\|_\infty \|f\|_X$$

for every $a \in L_\infty$ and every $f \in X$. So a homogeneous centralizer also obeys an estimate of the form $\|\Omega(f+g) - \Omega f - \Omega g\|_Y \le Q(\|f\|_X + \|g\|_X)$ for some constant Q and every $f, g \in X$. The topology of the twisted sum $Z = Y \oplus_\Omega X$ comes from the quasi-norm

$$\|(g, f)\|_\Omega = \|g - \Omega f\|_Y + \|f\|_X.$$

This quasi-norm satisfies the inequality $\|(ag, af)\|_\Omega \le M\|a\|_\infty \|(g, f)\|_\Omega$ for some constant M, but not for $M = 1$. This can be amended by replacing it by the equivalent quasi-norm given by

$$(g, f) \mapsto \sup_{\|a\|_\infty \le 1} \|(ag, af)\|_\Omega. \tag{5}$$

Now, let $D = \mathrm{Dom}(\Omega)$ with the quasi-norm $\|f\|_D = \|(0, f)\|_\Omega = \|\Omega f\|_Y + \|f\|_X$. This is a quasi-Banach function space (actually we may put the quasi-norm (5) on $Y \oplus_\Omega X$ to get a lattice quasi-norm on D by restriction).

Considering the isometric inclusion $\jmath : D \to Z$ sending f to $(0, f)$ and the corresponding quotient $R = Z/\jmath D$, we have a diagram

$$\begin{tikzcd} & D \arrow[d, "\jmath"] & \\ Y \arrow[r, "\imath"] & Z \arrow[r, "\pi"] \arrow[d, "\varpi"] & X \\ & R & \end{tikzcd} \tag{6}$$

where the vertical arrows in (6) represent an extension of R by D. The compositions $\pi \circ \jmath : D \to X$ and $\varpi \circ \imath : Y \to R$ are homomorphisms. Both are injective, since $\jmath D \cap \imath Y = 0$.

We may identify R with the set of those $g \in L_0$ such that $(g, f) \in Z$ for some $f \in X$ with the quasi-norm

$$\|g\|_R = \inf_{f \in X} \|(g, f)\|_\Omega.$$

This space is defined in a slightly different way, called the range of Ω, and denoted by $\mathrm{Ran}(\Omega)$ in [4, Section IV, part A]. It is easily seen that R is a quasi-Banach function space. That R embeds continuously in L_0 is seen as follows: if $\|g_n\|_R \to 0$, then we can find a sequence (f_n) in X such that $\|g_n\|_R \le \|g_n - \Omega f_n\|_Y + \|f_n\|_X \to 0$. We have $\|f_n\|_X \to 0$, so $\Omega f_n \to 0$ in L_0 (by Theorem 1) and since the inclusion of Y into L_0 is continuous, we have $g_n \to 0$ in L_0.

There is a centralizer $\mho \in \mathscr{C}(R, D)$ inducing the vertical extension in (6) whose description is an amusing exercise: given $g \in R$ we select (homogeneously) $f \in X$ (almost) minimizing $\|(g, f)\|_\Omega$ and we put $\mho g = f$. We observe that if f and f' are two allowable choices for $\mho g$, then

$$\begin{aligned} \|f' - f\|_D &= \|\Omega(f' - f)\|_Y + \|f' - f\|_X \le C(\|\Omega f' - \Omega f\|_Y + \|f'\|_X + \|f\|_X) \\ &\le C'(\|\Omega f' - g\|_Y + \|g - \Omega f\|_Y + \|f'\|_X + \|f\|_X) \le C''\|g\|_R. \end{aligned}$$

So any two "versions" of $\mho$ will differ in a bounded map, only. The rôles of Ω and $\mho$ are perfectly symmetric. As pure maps, Ω goes from X to R, while $\mho$ goes from R to X. Also, $\mathrm{Dom}(\mho) = \{f \in R : \mho f \in D\} = Y$ and $\mathrm{Ran}(\mho) = \{g \in L_0 : g - \mho f \in D \text{ for some } f \in R\} = Z$.

To sum up, we have:

Proposition 2. *Let X and Y be quasi-Banach function spaces and $\Omega : X \to L_0$ a homogeneous centralizer from X to Y. Set $D = \mathrm{Dom}(\Omega) = \{f \in X : \Omega f \in Y\}$, with quasi-norm $\|f\|_D = \|\Omega f\|_Y + \|f\|_X$ and*

$$R = \mathrm{Ran}(\Omega) = \{g \in L_0 : g - \Omega f \in Y \text{ for some } f \in X\},$$

with quasi-norm $\|g\|_D = \inf_{f\in X} \|g - \Omega f\|_Y + \|f\|_X$. *Then D and R are quasi-Banach function spaces. We define a mapping* $\mho : R \to L_0$ *as follows. For each* $g \in R$ *we choose (homogeneously)* $f \in X$ *such that* $\|g - \Omega f\|_Y + \|f\|_X \le (1+\varepsilon)\|g\|_R$ *and we put* $f = \mho(g)$. *Then* $\mho$ *is a centralizer from R to D. Besides* $X = \mathrm{Ran}(\mho)$, $Y = \mathrm{Dom}(\mho)$ *and the twisted sums* $Y \oplus_\Omega X$ *and* $D \oplus_\mho R$ *are isomorphic quasi-Banach modules over* L_∞.

If X and Y are symmetric quasi-Banach function spaces, then a homogeneous centralizer Φ from X to Y is said to be symmetric if there is a constant C such that for every Borel, measure-preserving map σ on $\mathbb{I}$ and every $f \in X$ one has

$$\|\Phi(f\circ\sigma) - (\Phi f)\circ\sigma\|_Y \le C\|f\|_X.$$

Corollary 3. *Let X and Y be symmetric quasi-Banach function spaces and* Ω *a nontrivial, homogeneous, symmetric centralizer from X to Y. If* $\mathrm{Hom}(X, Y) \ne 0$, *then* $\mathrm{Hom}(R, D) = 0$ *and* $\mho$ *is nontrivial, and hence* $\mathrm{Ext}(R, D) \ne 0$.

Proof. We first observe that the quasi-norm of $Z = Y \oplus_\Omega X$ is equivalent to $(g, f) \mapsto \sup_{a,\sigma} \|(a(g\circ\sigma), a(f\circ\sigma))\|_\Omega$, where $\|a\|_\infty \le 1$ and σ runs over all Borel, measure-preserving automorphisms of $\mathbb{I}$. This shows that we may amend the quasi-norms of D and R to make them into symmetric, quasi-Banach function spaces.

Next, we observe that if U and V are symmetric quasi-Banach function spaces, then either $\mathrm{Hom}(U, V) = 0$ or $U \subset V$ and the inclusion is bounded. Indeed, let us assume that a nonzero homomorphism $m : U \to V$ exists. By Lemma 1 we know that $m(f) = \phi f$, where $\phi = m(1)$ is not zero. Clearly, there are $c > 0$ and an integer n such that $\lambda\{t \in \mathbb{I} : |\phi(t)| \ge c\} \ge 1/n$ and so there is a Borel $A_1 \subset \mathbb{I}$ of measure $1/n$ where $|\phi(t)| \ge c$. Now, with the notation of the proof of Example 1, if $f \in U(A_1)$, then

$$\|cf\|_V \le \|\phi f\|_V = \|m(f)\|_V \le \|m : U \to V\|\, \|f\|_U.$$

It follows that $U(A_1) \subset V(A_1)$ and one moreover has $\|f\|_V \le c^{-1}\|m\|\,\|f\|_U$ for $f \in U(A_1)$.

Now, assuming that $n \ge 2$, let us take a partition $\mathbb{I} = A_1 \oplus \cdots \oplus A_n$, where A_i are Borel sets of measure $1/n$. Observe that, given $1 < i \le n$, there is a Borel, measure-preserving automorphism σ of $\mathbb{I}$ such that $\sigma(A_i) = A_1$. It follows from the very definition of symmetric quasi-Banach function space that for each $1 \le i \le n$, one has $U(A_i) \subset V(A_i)$ and $\|f\|_V \le c^{-1}\|m\|\,\|f\|_U$ for $f \in U(A_i)$.

Pick $f \in U$ and write $f = f_1 + \cdots + f_n$, with $f_i \in U(A_i)$. Then $f_i \in V$ and $\|f_i\| \le c^{-1}\|m\|\,\|f\|_U$. In particular $f \in V$. Moreover it is clear that there is a constant C—depending only on n and on the moduli of concavity of U and V—such that $\|f\|_V \le Cc^{-1}\|m\|\,\|f\|_U$ for $f \in U$.

Going back to the proof of the corollary, and referring to (6), we always have the continuous inclusions $Y \subset R$ and $D \subset X$. Hence if $\mathrm{Hom}(X, Y) \ne 0$ and $\mathrm{Hom}(R, D) \ne 0$, we have continuous inclusions $X \subset Y \subset R \subset D \subset X$, so $X = D$ with equivalent quasi-norms and Ω is continuous at zero, and hence trivial.

Thus, if Ω is nontrivial and $\mathrm{Hom}(X, Y) \ne 0$ then $\mathrm{Hom}(R, D) = 0$. Let us see that in this case $\mho$ cannot be trivial. Suppose on the contrary that $\mho$ is trivial, and so there is a unique $u \in \mathrm{Hom}(Z, D)$ such that $u \circ \jmath = \mathbf{I}_D$—uniqueness follows from $\mathrm{Hom}(R, D) = 0$. Please note that u must be of the form $u(g, f) = f + \phi g$ for some fixed $\phi \in L_0$ and $f \in D$, $g \in Y$. Now, since the "conjugation" $(g, f) \in Z = Y \oplus_\Omega X \mapsto (u(g\circ\sigma, f\circ\sigma))\circ\sigma^{-1}$ is a projection of Z onto D as well, it has to agree with u. It follows that ϕ is a constant. If $\phi = 0$, then u vanishes on Y and so it factors through a homomorphism $X \to D$: this implies that $X = D$ and Ω is trivial, a contradiction. Otherwise, the composition

$$Y \xrightarrow{\imath} Z \xrightarrow{u/c} D \xrightarrow{\jmath} Z \xrightarrow{\pi} X \xrightarrow{I} Y,$$

where I is the formal inclusion of X into Y, turns out to be the identity on Y and Ω is again trivial, which cannot be. □

Finally, let us consider a concrete example. Let $\Omega \in \mathscr{C}(L_2)$ be the popular centralizer given by $\Omega f = f\log(|f|/\|f\|_2)$. It is not hard to check that $D = \mathrm{Dom}\,\Omega$ agrees with the Orlicz space L_N, where $N(t) = t^2\log^2 t$. In order to identify the quotient $R = (L_2 \oplus_\Omega L_2)/D$, we observe that $L_2 \oplus_\Omega L_2$ is isomorphic to its own dual under the pairing

$$\langle (g', f'), (f, g)\rangle = \int_0^1 (g'g - f'f)\,dt.$$

This can be seen as in [3, Theorem 5.1] (or by interpolation, using [13, Proposition 2.11]), but notice that the pairing is slightly different here. It follows that R is isomorphic to $D^* = L_N'$ and so it agrees with the Orlicz space associated with the complementary function of N. (Actually it is true that the spaces $\mathrm{Dom}\,\Omega$ and $\mathrm{Ran}\,\Omega$ are duals of each other for every $\Omega \in \mathscr{C}(L_2)$; see [9, Corollary 4].) So we have the diagram

$$\begin{tikzcd} & L_N \arrow[d, "\jmath"] & \\ L_2 \arrow[r, "\imath"] & L_2 \oplus_\Omega L_2 \arrow[r, "\pi"] \arrow[d, "\varpi"] & L_2 \\ & L_N' & \end{tikzcd}$$

which shows that $\mathrm{Hom}(L_N', L_N) = 0$, while $\mathrm{Ext}(L_N', L_N) \ne 0$.

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