Topology and its Applications 160 (2013) 50–55

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Lattices of uniformly continuous functions ☆

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ARTICLE INFO

Article history:
Received 10 November 2011
Received in revised form 17 September 2012
Accepted 18 September 2012

Keywords:
Lattices
Uniformly continuous functions
Isomorphism
Banach–Stone theorem

ABSTRACT

An explicit representation of the order isomorphisms between lattices of uniformly continuous functions on complete metric spaces is given. It is shown that every lattice isomorphism $T : U(Y) \to U(X)$ is given by the formula $(Tf)(x) = t(x, f(\tau(x)))$, where $\tau : X \to Y$ is a uniform homeomorphism and $t : X \times \mathbb{R} \to \mathbb{R}$ is defined by $t(x, c) = (Tc)(x)$. This provides a correct proof for a statement made by Shirota sixty years ago.

1. Introduction

The aim of this short note is to prove the following

Theorem. *Let X and Y be complete metric spaces and $T : U(Y) \to U(X)$ a lattice isomorphism. There is a uniform homeomorphism $\tau : X \to Y$ such that*

$$(Tf)(x) = t\big(x, f\big(\tau(x)\big)\big) \quad \big(f \in U(Y),\ x \in X\big), \tag{1}$$

where $t : X \times \mathbb{R} \to \mathbb{R}$ is given by $t(x, c) = (Tc)(x)$. Here c is first treated as a real number and then as a constant function on Y.

(We use $C(X)$, $U(X)$ and $U^*(X)$ for the lattices of continuous, uniformly continuous and bounded uniformly continuous real-valued functions on X, respectively.)

We emphasize that lattice isomorphisms are not assumed to be linear, they are just bijections that preserve the order in both directions. Of course the preceding theorem implies that each linear lattice isomorphism $T : U(Y) \to U(X)$ is a weighted composition operator of the form $(Tf)(x) = w(x)f(\tau(x))$, where $\tau : X \to Y$ is a uniform homeomorphism and $w = T(1)$. To the best of our knowledge, even this specialization is new.

Let us quickly review some earlier results closely related to the subject of this note.

Shirota proved in [10] that the lattice structure of $U^*(X)$ determines the uniform structure of X amongst the complete metric spaces. A similar result had been got earlier by Nagata [9] under various continuity assumptions.

From a modern perspective the result for bounded functions is as follows (see [2, Section 4]): there is a compactification σX of X such that $U^*(X) = C(\sigma X)$ in the sense that a bounded function $f : X \to \mathbb{R}$ extends to a continuous function on

☆ Research supported in part DGICYT projects MTM2007-6994-C02-02 and MTM2010-20190 and Junta de Extremadura program GR10113.
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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.09.010

σX if and only if it is uniformly continuous. (This construction is due to Smirnov and Samuel.) Therefore, each isomorphism between $U^*(Y)$ and $U^*(X)$ gives rise to an isomorphism $T : C(\sigma Y) \to C(\sigma X)$. By an old result of Kaplansky [8, Theorem 1], T induces a (necessarily uniform) homeomorphism $\tau : \sigma X \to \sigma Y$. But the only points in σX having countable neighborhood bases are those in X (and similarly for Y) and so τ restricts to a uniform homeomorphism between X and Y (see [5, Lemma 1]; this idea goes back to Čech). In a similar vein, it is proved in [5] that if there is a linear and unital isomorphism of lattices between $U(Y)$ and $U(X)$, then X and Y are uniformly homeomorphic, and the hypothesis about linearity was removed in [3]. For related results beyond the metric setting we refer the reader to the recent paper [6] as well as to [7], where a remarkable "internal" characterization of lattices of uniformly continuous functions appears.

To be true, it is claimed in [10] that the lattice structure of $U(X)$ determines the uniform structure of the complete metric space X. It seems, however, that the proof given there is not correct. See Section 3.4 below. In any case we believe that the result deserves a clean, correct proof.

2. Proof

The proof is organized as follows. First, we construct the homeomorphism τ on certain dense subsets of X and Y and we establish the functional representation (1) there.

After that we manage to prove that τ extends to a uniform homeomorphism between X and Y and we obtain (1) in general.

2.1. Regular open sets

This part makes heavy use of the ideas of [3]. An open set is said to be regular if it is the interior of its closure. The class of all regular open subsets of X is denoted by $R(X)$. We will consider the order given by inclusion in $R(X)$.

The support of a continuous $f : X \to \mathbb{R}$ is the closure of the (cozero) set $\{x \in X: f(x) \neq 0\}$ and we define U_f as the interior of the support of f. Note that U_f and U_{-f} are exactly the same set. Quite clearly, U_f is a regular open set and each regular open set arises in this way for some $f \in U(X)$. Indeed, if $U \in R(X)$, then $U = U_f$, where $f(x) = \operatorname{dist}(x, U^c)$.

Next, we remark that the condition $U_f \subseteq U_g$ can be expressed within the order structure of $U(X)^+$, the subset of nonnegative functions in $U(X)$. To see this, following Shirota [10], let us declare $f \subset g$ if, whenever $h \in U(X)^+$, $h \wedge g = 0$ implies $h \wedge f = 0$. It is easily seen that, given $f, g \in U(X)^+$, one has $f \subset g$ if and only if $U_f \subseteq U_g$. It follows that, given $f, g \in U(X)^+$ one has $U_f = U_g$ if and only if $f \subset g$ and $g \subset f$.

Now, assuming once and for all $T(0) = 0$ as we clearly may do, we see that the map $\mathfrak{T} : R(Y) \to R(X)$ given by $\mathfrak{T}(U_f) = U_{Tf}$ for $f \in U(Y)^+$ is correctly defined and it is an order isomorphism.

Although the definition of $\mathfrak{T}$ uses nonnegative functions only, the following result shows that, in some sense, $\mathfrak{T}$ governs the behavior of T on the whole of $U(Y)$. For the sake of completeness, we have included a proof which is perhaps simpler than the given in [3, Corollary 2].

Lemma 1. *Let $T : U(Y) \to U(X)$ be an isomorphism with $T(0) = 0$ and let $\mathfrak{T} : R(Y) \to R(X)$ be as before. Then, given $f, g \in U(Y)$ and $U \in R(Y)$, one has $f \leqslant g$ on U if and only if $Tf \leqslant Tg$ on $\mathfrak{T}(U)$. The same is true replacing "$\leqslant$" by "$\geqslant$" or by "$=$".*

Proof. First, we prove the lemma for $f, g \in U(Y)^+$. It suffices to check that $f \leqslant g$ on $U \in R(Y)$ if and only if $f \wedge h \leqslant g \wedge h$ for every $h \in U(Y)^+$ such that $U_h \subseteq U$. The "only if" part is nearly obvious from the definitions. The converse is as follows: if $f(y) > g(y)$ for some $y \in U$, then we may take $h \in U(Y)^+$ such that $h(y) = f(y)$ and $U_h \subseteq U$. We have $(f \wedge h)(y) = f(y) > g(y) = (g \wedge h)(y)$, which is enough.

Now, by symmetry (or applying the previous case to the map $f \mapsto -T(-f)$, which is a lattice isomorphism too), we also have the following: the map $\mathfrak{T}^- : R(Y) \to R(X)$ defined by $\mathfrak{T}^-(U_h) = U_{T(-h)}$ for $h \in U(Y)^+$ is correctly defined and preserves order in both directions. Moreover, given $f, g \in U(Y)^-$ and $U \in R(Y)$ one has $f = g$ on U if and only if $Tf = Tg$ on $\mathfrak{T}^-(U)$.

Next we prove that $\mathfrak{T}^- = \mathfrak{T}$. Let us verify that $\mathfrak{T}^-(U) \subseteq \mathfrak{T}(U)$ for every $U \in R(Y)$; the other containment is analogous, with the roles of $\mathfrak{T}^-(U)$ and $\mathfrak{T}(U)$ reversed. Suppose on the contrary that $\mathfrak{T}(U)$ does not contain $\mathfrak{T}^-(U)$. Then neither $\overline{\mathfrak{T}(U)}$ does and there is a (nonempty) regular open set $V \subseteq \mathfrak{T}^-(U)$ such that $d(V, \mathfrak{T}(U)) > 0$. Set $g' = T(1)$, $g'' = T(-1)$ and take a (nonempty) regular open $B'' \subset V$ where g'' is bounded. Take $A = (\mathfrak{T}^-)^{-1}(B'')$ and let B' be any (nonempty) regular open subset of $\mathfrak{T}(A)$ where g' is bounded. Then, since $d(B', B'') \geqslant d(V, \mathfrak{T}(U)) > 0$ and g' and g'' are bounded on B' and B'' respectively, there is $g \in U(X)$ which agrees with g' on B' and agrees with g'' on B''. Take $f \in U(Y)$ such that $g = Tf$ and put $f^+ = f \vee 0$ and $f^- = f \wedge 0$. Clearly, $T(f^+) = T(f \vee 0) = Tf \vee T0 = g^+$ and $T(f^-) = g^-$. Observe that $g^+ = g'$ on B' and $g^- = g''$ on B'', from where it follows that $f^+ = 1$ on $\mathfrak{T}^{-1}(B')$ and $f^- = -1$ on $(\mathfrak{T}^-)^{-1}(B'') = A$ and this is a contradiction since $\mathfrak{T}^{-1}(B')$ is a nonempty subset of A.

To complete the proof we observe that, given $f, g \in U(Y)$ and $U \in R(Y)$, one has $f \leqslant g$ on U if and only if $f^+ \leqslant g^+$ on U and $f^- \leqslant g^-$ on U. In this case one has $T(f^+) \leqslant T(g^+)$ on $\mathfrak{T}(U)$ and $T(f^-) \leqslant T(g^-)$ on $\mathfrak{T}(U)$ and since $T(f^\pm) = (Tf)^\pm$ and $T(g^\pm) = (Tg)^\pm$ we have $Tf \leqslant Tg$ on $\mathfrak{T}(U)$. $\square$

Now, we can use $\mathfrak{T}$ to construct a point map between certain dense subsets of X and Y. The following result is a particular case of [3, Lemma 6]. The proof is included to render this note self-contained.

Lemma 2. *If $\mathfrak{T} : R(Y) \to R(X)$ is a lattice isomorphism, then there exist dense subsets X' of X and Y' of Y and a homeomorphism $\tau : X' \to Y'$ such that given $x \in X'$ and $U \in R(Y)$ one has $x \in \mathfrak{T}(U)$ if and only if $\tau(x) \in U$.*

Proof. Given $(x, y) \in X \times Y$, let us write $x \sim y$ if

$$\bigcap_{y\in U} \mathfrak{T}(U) = \{x\} \quad \text{and} \quad \bigcap_{x\in V} \mathfrak{T}^{-1}(V) = \{y\},$$

where $U \in R(Y)$ and $V \in R(X)$. First of all notice that if $x \sim y$ and $x \sim y'$, then $y = y'$. Similarly, if $x \sim y$ and $x' \sim y$, then $x = x'$. Let X' be the set of those $x \in X$ for which there exists (a necessarily unique) $y \in Y$ such that $x \sim y$ and Y' the set of those $y \in Y$ such that $x \sim y$ for some $x \in X$. It is pretty obvious that the map $\tau : X' \to Y'$ sending each $x \in X'$ to the only $y \in Y'$ such that $x \sim y$ is a homeomorphism.

It remains to see that Y' is dense in Y. The corresponding statement for X' follows by symmetry.

Let U be a nonempty open subset of Y. We must show that U meets Y'. Take a nonempty $U_1 \in R(Y)$ such that $\overline{U}_1 \subseteq U$ and $\operatorname{diam} U_1 \leqslant 1$. Choose a nonempty $V_1 \subset \mathfrak{T}(U_1)$, with $\operatorname{diam} V_1 \leqslant 1$. Then choose a nonempty $U_2 \subset \mathfrak{T}^{-1}(V_1)$ with $\overline{U}_2 \subset U_1$ and $\operatorname{diam} U_2 \leqslant 1/2$. Next, take a nonempty $V_2 \subset \mathfrak{T}(U_2)$ such that $\overline{V}_2 \subset V_1$ and $\operatorname{diam} V_2 \leqslant 1/2$. In this way we get sequences (U_n) and (V_n) in $R(Y)$ and $R(X)$, respectively, such that, for each n:

- $\overline{U}_{n+1} \subset U_n$ and $\overline{V}_{n+1} \subset V_n$.
- U_n and V_n have diameter at most $1/n$.
- $\mathfrak{T}(U_{n+1}) \subset V_n \subset \mathfrak{T}(U_n)$.

Now, it is clear that there are $y \in Y$ and $x \in X$ such that

$$\{y\} = \bigcap_n U_n = \bigcap_n \overline{U}_n \quad \text{and} \quad \{x\} = \bigcap_n V_n = \bigcap_n \overline{V}_n.$$

From where it follows that $x \sim y$ and since $y \in U$ we see that Y' is dense in Y. □

2.2. Functional representation

The following result allows one to entwine a couple of functions near a point where they agree.

Lemma 3. *Suppose $f, g \in U(Y)$ agree at $y \in Y$. If $g \leqslant f$, then there is $h \in U(Y)$ such that every neighborhood of y contains a nonempty (regular) open set where h agrees with f and another nonempty open set where h agrees with g.*

Proof. If y is isolated, then there is nothing to prove. Otherwise we may take a sequence (y_n) converging to y, with $y_n \neq y_m$ for $n \neq m$. Both $f(y_n)$ and $g(y_n)$ converge to $c = f(y) = g(y)$ and so there is a sequence (c_n) converging to c such that $c_n > f(y_n)$ for even n and $c_n < g(y_n)$ for odd n. Take $\phi \in U(Y)$ such that $\phi(y_n) = c_n$ and put $h = (\phi \vee g) \wedge f$. □

Going back to T, we can now prove the formula in (1), at least for $x \in X'$ – which is the set defined during the proof of Lemma 2.

Corollary 1. *Given $f \in U(Y)$ and $x \in X'$, the value of Tf at x depends only on $f(\tau(x))$. Consequently, the formula (1) holds for every $x \in X'$ and every $f \in U(Y)$.*

Proof. Indeed suppose $f, g \in U(Y)$ agree at $y = \tau(x)$ and let us see that $(Tf)(x) = (Tg)(x)$. Replacing f and g by $f \vee g$ and $f \wedge g$ we may assume $g \leqslant f$. Take h as in the lemma and look at Th: every neighborhood of x contains an open set where Th agrees with Tf and another open set where Th agrees with Tg and so $(Tf)(x) = (Tg)(x) = (Th)(x)$.

To end, if $c = f(y)$, we have $(Tf)(x) = (Tc)(x) = t(x, c) = t(x, f(\tau(x)))$, as required. □

2.3. Uniform continuity

Next we prove that $\tau : X' \to Y'$ is a uniform homeomorphism. By symmetry, one only has to check that it is uniformly continuous.

If we assume the contrary we easily arrive at the following situation: there are sequences (x_n) and (x'_n) in X' and $\delta > 0$ such that:

- $0 < d(x_n, x'_n) \to 0$;
- $d(x_n, x_m) \geqslant \delta$ and $d(x'_n, x'_m) \geqslant \delta$ for $n \neq m$;
- $d(y_n, y'_m) \geqslant \delta$ for every n and m,

where $y_n = \tau(x_n)$ and $y'_n = \tau(x'_n)$. See [1, Lemma 3.4] or [3, Proof of Theorem 1, part I] for details. We owe to the referee the information that this was published long time ago by Efremovich in [4] and by Vilhelm and Vitner in [11].

Suppose there is a sequence (y''_n) in Y' such that $d(y'_n, y''_n) \to 0$, with $y''_n \neq y'_n$ for every n. Since neither (y_n) nor (y'_n) have convergent subsequences we may assume that $y''_n \neq y'_m$ for arbitrary n, m. Let $f \in U(Y)$ be a function vanishing at every y_n and taking the value 1 at every y'_n and put $g = Tf$. Then g vanishes at every (x_n) and $g(x'_n) \to 0$. Now look at the sequence (x''_n), where $\tau(x''_n) = y''_n$ and observe that $x''_n \neq x'_m$ for every n and m. Quite clearly, there is $g^* \in U(X)$ such that $g^*(x_n) = g^*(x''_n) = 0$ for every n, while $g^*(x'_n) = g(x'_n)$ for every n. Taking $f^* \in U(Y)$ such that $Tf^* = g^*$ we see that $f^*(y_n) = f^*(y''_n) = 0$, while $f^*(y'_n) = f(y'_n) = 1$ for every n, a contradiction with $d(y'_n, y''_n) \to 0$.

If there is no such a sequence, then passing to a subsequence we may assume the sequence (y'_n) uniformly isolated in Y, that is, there is $r > 0$ (independent on n) such that the only point of Y in the ball of radius r centred at y'_n is y'_n itself. This obviously implies that the lattice of restrictions

$$M = \left\{s \in \mathbb{R}^{\mathbb{N}}:\ s(n) = f\left(y'_n\right) \text{ for some } f \in U(Y) \text{ vanishing at every } y_k\right\}$$

is the whole of $\mathbb{R}^{\mathbb{N}}$. But, certainly,

$$L = \left\{s \in \mathbb{R}^{\mathbb{N}}:\ s(n) = g\left(x'_n\right) \text{ for some } g \in U(X) \text{ vanishing at every } x_k\right\}$$

is c_0, the lattice of null sequences. Clearly, T induces a lattice isomorphism between M and L (taking $s \in M$ to the sequence $(Tf)(x'_n)$, where f is any uniformly continuous function on Y such that $s(n) = f(y'_n)$ and $f(y_n) = 0$ for every n), which is impossible since $\mathbb{R}^{\mathbb{N}}$ and c_0 are not isomorphic. Indeed, let us consider the following property that a given lattice N may have or may not have:

(♡) If C is a countable subset of N and there is $h \in N$ such that $h = f \wedge g$ whenever f and g are different elements of C, then C has a supremum in N.

Then $\mathbb{R}^{\mathbb{N}}$ has (♡), while c_0 lacks it.

This shows that τ defines a uniform homeomorphism between X' and Y' which, by density, extends to a uniform homeomorphism between X and Y that we shall not relabel. It is easily seen that, with the notations of Section 2.1, one has $\mathfrak{T}(U) = \tau^{-1}(U)$ and this implies that $X' = X$ and $Y' = Y$. Now (1) follows from what we proved in Section 2.2, which completes the proof of the theorem.

3. Miscellaneous remarks and examples

3.1. Lattices of bounded uniformly continuous functions

As we already mentioned the main result is true, and well-known, replacing $U(\cdot)$ by $U^*(\cdot)$. Let us indicate the minor changes required in the proof. First, notice that every $U \in R(X)$ arises as U_f for some nonnegative $f \in U^*(X)$: just take $f = d(\cdot, U^c) \wedge 1$. The remainder of the proof goes undisturbed replacing $U(\cdot)$ by $U^*(\cdot)$ everywhere until the point where the lattices of restrictions appear. This time $M^* = \{s \in \mathbb{R}^{\mathbb{N}}:\ s(n) = f(y'_n) \text{ for some } f \in U^*(Y) \text{ vanishing at every } y_k\}$ equals ℓ_∞, the lattice of all bounded sequences, while $L^* = \{s \in \mathbb{R}^{\mathbb{N}}:\ s(n) = g(x'_n) \text{ for some } g \in U^*(X) \text{ vanishing at every } x_k\}$ is again c_0. The following observation ends the proof.

Lemma 4. *The lattices c_0 and ℓ_∞ are not isomorphic.*

Proof. Quite clearly, ℓ_∞ has a countable subset C such that for every $g \in \ell_\infty$ there is $f \in C$ so that $g \leqslant f$. Let us see that there is no such set in c_0. Let (f_i) be any sequence in c_0. Take an increasing sequence of integers (n_i) so that $f_i(n) < 1/i$ for $n \geqslant n_i$. Then set $g(n) = 1/i$ for $n \in [n_i, n_{i+1})$. Clearly, g belongs to c_0, but $g \leqslant f_i$ for no $i \in \mathbb{N}$. □

This argument provides a rather elementary proof of Shirota's theorem for bounded functions which, moreover, gives a very explicit functional representation for T.

3.2. Bounded functions and continuity of isomorphisms

One may wonder to what extend $U(X)$ "knows" which functions are bounded. Since for fixed $g \in U(X)$ the translation mapping $f \mapsto f + g$ is an automorphism the real question is whether an isomorphism $T : U(Y) \to U(X)$ (or an automorphism of $U(X)$) must send pairs having bounded differences into pairs of the same type. In general the answer is negative since automorphisms of $U(\mathbb{N})=\mathbb{R}^{\mathbb{N}}$ are as arbitrary as they can be. Indeed, given a strictly positive $g\in U(\mathbb{N})$, the multiplication map $f\mapsto g\cdot f$ is a linear automorphism of $U(\mathbb{N})$ sending 1 to g.

It turns out that this "pathological" behavior is possible only if $\mathbb{N}$ (with the discrete metric) appears as a direct summand in X in the "uniform category" – that is, there is $r>0$ and a sequence (x_n) in X such that the only point of X in the ball of radius r centered at x_n is x_n itself. Let us state this properly.

Corollary 2. *For a metric space X the following statements are equivalent*:

(a) *$\mathbb{N}$ is not a direct summand in X in the uniform category.*
(b) *Whenever T is an automorphism of $U(X)$ and $f,g\in U(X)$ are such that $g-f$ is bounded, $Tg-Tf$ is bounded.*
(c) *Every automorphism of $U(X)$ is continuous in the topology of uniform convergence.*

Proof. A metric space has the same uniformly continuous functions as its completion, and so we may assume X complete so that the main result applies.

It is clear that any of the conditions (b) or (c) implies (a).

Let us prove the implication (a) $\Rightarrow$ (b). Suppose there is an automorphism T of $U(X)$ and $f,g\in U(X)$ such that $g-f$ is bounded, but $Tg-Tf$ is not. Clearly, we may assume that the uniform homeomorphism associated to T is the identity on X so that $(Tf)(x)=t(x,f(x))$ for every $f\in U(X)$ and $x\in X$. Replacing f and g by $f\wedge g$ and $f\vee g$ we may and do assume $f\leqslant g$. Applying a translation if necessary we can assume that $f=a$ is constant and then that $g=b$ is also a constant. (Each translation is an order automorphism.) Consider the set $I=\{t\in[a,b]\colon\ T(t)-T(a)\text{ is bounded}\}$. Needless to say I is an interval and so either $I=[a,c)$ with $c\leqslant b$ or $I=[a,c]$, with $c<b$. We write the proof in the first case, the other is left to the reader. After subtracting $T(c)$ we arrive at the following situation: $T(c)=0$ and $T(c')$ is unbounded for $c'>c$. We fix a sequence (c_n) decreasing to c and we put $f_n=T(c_n)$. All these functions are unbounded (from above) and we can choose a sequence (x_n) such that $f_{n+1}(x_{n+1})\geqslant 1+f_1(x_n)$. We then have $f_1(x_{n+1})\geqslant 1+f_1(x_n)$, which guarantees that the terms of (x_n) are uniformly apart. We claim that there is no sequence (x_n') with $d(x_n,x_n')\to 0$ and $x_n'\neq x_n$ for every n. Assuming the contrary, there is $f\in U(X)$ such that $f(x_n)=c_n$ and $f(x_n')=c$. Therefore $g=Tf$ vanishes at every x_n' while $g(x_n)=f_n(x_n)\to\infty$, a contradiction.

Finally we prove that (a) implies (c). Let us begin with the observation that the formula appearing in the theorem already implies that each isomorphism $U(Y)\to U(X)$ is continuous (hence a homeomorphism) in the topology of pointwise convergence. This follows from the fact that, for each fixed $x\in X$, the function $c\in\mathbb{R}\to t(x,c)\in\mathbb{R}$ is continuous, since it is increasing and surjective.

Now suppose (c) fails. After a moment's reflection we realize that there is an automorphism T of $U(X)$ with $T(0)=0$, a sequence (c_n) decreasing to 0 and some $\varepsilon>0$ such that $\sup\{(Tc_n)(x)\colon\ x\in X\}>\varepsilon$. As before, we may assume that the spatial part of T is the identity.

For each n, pick x_n such that $(Tc_n)(x_n)>\varepsilon$. As $(Tc_n)(x)\to 0$ as $n\to\infty$ for each fixed x we see that (x_n) does not converge (and has no convergent subsequence). For if (x_n) converges, say to x, then we have $(Tc_n)(x)\to 0$, while if $m\geqslant n$, then $(Tc_n)(x_m)\geqslant(Tc_m)(x_m)>\varepsilon$, which is absurd. Hence there is some $r>0$ such that $d(x_n,x_m)>r$ for $n\neq m$. Let us check that there is no sequence (x_n') with $d(x_n,x_n')\to 0$ and $x_n'\neq x_n$ for every n. If such an (x_n') exists, then one finds $f\in U(X)$ such that $f(x_n)=c_n$ and $f(x_n')=0$ for every n and so $(Tf)(x_n)>\varepsilon$ and $(Tf)(x_n')=0$ for all n, a contradiction. □

3.3. *Lattice homomorphisms*

The main theorem does not extend to general lattice homomorphisms. The following example shows that, in general, homomorphisms $T:U(Y)\to U(X)$ do not correspond to point maps $\tau:X\to Y$, even in the linear case. Please consider $\mathbb{R}$ as the lattice of (uniformly continuous) functions on a single point.

Example 1.

(a) A linear surjective homomorphism $\phi:U(\mathbb{R})\to\mathbb{R}$ vanishing on every bounded function.
(b) A linear injective homomorphism $T:U(Y)\to U(X)$ which is not a weighted composition operator.

Proof. (a) Recall that every uniformly continuous function on the real line is Lipschitz for large distances and so, if $f\in U(\mathbb{R})$, the ratio $f(t)/t$ is bounded for large t. Let (s_n) be so that $|s_n|\to\infty$ and $\mathcal{U}$ a free ultrafilter on the integers. Then set $\phi(f)=\lim_{\mathcal{U}(n)} f(s_n)/s_n$.

(b) Take $Y=\mathbb{R}$ and let X be the disjoint union of $\mathbb{R}$ and a single (isolated) point, so that $U(X)=U(\mathbb{R})\times\mathbb{R}$. Put $Tf=(f,\phi(f))$, where ϕ is as in Part (a). □

3.4. *Was Shirota right?*

As we mentioned in the Introduction, Shirota claims in [10, Theorem 6, first part] that two complete metric spaces are uniformly homeomorphic provided they have isomorphic lattices of uniformly continuous functions.

We believe that this fact is not proved in [10] nor can even be deduced from the arguments given in that paper. (Please read the first paragraph of [5] and Problems 1 and 3 in [6]; by the way notice that our main result solves both problems in the affirmative sense.) Let us justify our opinion. Together with the order relation that we used in Section 2.1, Shirota considers the following stronger relation (Definition 4), where L can be either $U(X)^+$ or $U^*(X)^+$: Given $f, g \in L$ we write $f \Subset g$ if, whenever the family (h_α) has an upper bound in L and $h_\alpha \subset f$ for all α, there is an upper bound $h \in L$ such that $h \subset g$.

As far as we can understand, the proof of the part of [10, Theorem 6] concerning bounded functions is based on the fact that $d(U_f, U_g^c) > 0$ is equivalent to $f \Subset g$ when $L = U^*(X)^+$.

Allowing unbounded functions $d(U_f, U_g^c) > 0$ does not longer imply $f \Subset g$, as the following example, copied from [3, Section 5] and pasted here, shows. Consider $X = \mathbb{R}$ with the usual distance and the sets:

$$V = \bigcup_n (n - 1/8, n + 1/8) \quad \text{and} \quad W = \bigcup_n (n - 1/4, n + 1/4).$$

Clearly, $d(V, W^c) = 1/8$. Define f and g taking $f(x) = d(x, V^c)$ and $g(x) = d(x, W^c)$, so that $V = U_f$ and $W = U_g$. Let us see that the relation $f \Subset g$ does not hold in $U(\mathbb{R})$. Indeed, for $n \in \mathbb{N}$, let h_n be piecewise linear function defined by the conditions $h_n(n) = n$, $h_n(n \pm \frac{1}{8}) = 0$. Then $h_n \subset f$ for all n and the sequence (h_n) is bounded by $|\cdot|$. However no uniformly continuous function $h \subset g$ can be an upper bound for (h_n).

As the referee pointed out, the gap in [10] occurs in the (fifth line of the) proof of Lemma 1, where Shirota uses the product $g_1 h_1$ without realizing that h_1 may be unbounded and the product of a bounded uniformly continuous function with an unbounded one need not be uniformly continuous.

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