



## On separably injective Banach spaces

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### Abstract

We deal with two weak forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. We show several structural and stability properties of these classes of Banach spaces. We provide natural examples of (universally) separably injective spaces, including  $\mathcal{L}_\infty$  ultraproducts built over countably incomplete ultrafilters, in spite of the fact that these ultraproducts are never injective. We obtain two fundamental characterizations of universally separably injective spaces. (a) A Banach space  $E$  is universally separably injective if and only if every separable subspace is contained in a copy of  $\ell_\infty$  inside  $E$ . (b) A Banach space  $E$  is universally separably injective if and only if for every separable space  $S$  one has  $\text{Ext}(\ell_\infty/S, E) = 0$ . Section 6 focuses on special properties of 1-separably injective spaces. Lindenstrauss proved in the middle sixties that, under CH, 1-separably injective spaces are 1-universally separably injective and left open the question in ZFC. We construct a consistent example of a Banach space of type  $C(K)$  which is 1-separably injective but not universally 1-separably injective.

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## 1. Introduction

A Banach space  $E$  is said to be injective if for every Banach space  $X$  and every subspace  $Y$  of  $X$ , each operator  $t: Y \rightarrow E$  admits an extension  $T: X \rightarrow E$ . The space  $E$  is said to be  $\lambda$ -injective if, moreover,  $T$  can be chosen so that  $\|T\| \leq \lambda\|t\|$ . The space  $\ell_\infty$  is the best known example of 1-injective space. The two basic facts about injective spaces are that 1-injective spaces are isometric to  $C(K)$ -spaces with  $K$  extremely disconnected and that is not known if all injective spaces are isomorphic to 1-injective spaces.

In this paper we deal with two weak forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. A Banach space  $E$  is said to be separably injective if it satisfies the extension property in the definition of injective spaces under the restriction that  $X$  is separable; and it is said to be universally separably injective if it satisfies the extension property when  $Y$  is separable. Obviously, injective spaces are universally separably injective and these, in turn, are separably injective; the converse implications fail. We will mainly be concerned with nonseparable Banach spaces since  $c_0$  is the only infinite dimensional separable separably injective Banach space, according to a deep result of Zippin [56]; we refer to [57] for what is perhaps the simplest proof.

The basic structural and stability properties of these classes are studied in Section 3: we will show that infinite-dimensional separably injective spaces are  $\mathcal{L}_\infty$ -spaces, contain  $c_0$  and have Pełczyński's property (V). Universally separably injective spaces, moreover, are Grothendieck spaces, contain  $\ell_\infty$  and enjoy a stronger form of Pełczyński's property (V) that we call Rosenthal's property (V). In Section 4 we provide natural examples of (universally) separably injective spaces, including the remarkable fact that ultraproducts built over countably incomplete ultrafilters are universally separably injective as long as they are  $\mathcal{L}_\infty$ -spaces, in spite of the fact that they are never injective. In Section 5 we obtain two fundamental characterizations of universally separably injective spaces: (a) a Banach space  $E$  is universally separably injective if and only if every separable subspace is contained in a copy of  $\ell_\infty$  inside  $E$ . (b) A Banach space  $E$  is universally separably injective if and only if for every separable space  $S$  one has  $\text{Ext}(\ell_\infty/S, E) = 0$ ; i.e.,  $E$  is complemented in any superspace  $Z$  such that  $Z/E = \ell_\infty/S$ . Characterization (a) allows to prove that universal separable injectivity is a 3-space property, which provides many new examples of spaces with this property. Characterization (b) leads to the result  $\text{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$ , which provides a new unexpected solution for equation  $\text{Ext}(X, X) = 0$ . Section 6 focuses on special properties of 1-separably injective spaces. This is the point in which set theory axioms enter the game. Indeed, Lindenstrauss obtained in the middle sixties what can be understood as a proof that, under the continuum hypothesis, 1-separably injective spaces are 1-universally separably injective; he left open the question in ZFC. We construct a consistent example of a Banach space of type  $C(K)$  which is 1-separably injective but not 1-universally separably injective. The final section contains several applications and open problems.

## 2. Background

Our notation is fairly standard, as in [44]. We will work with real scalars, although our results can be extended, *mutatis mutandis*, for complex scalars. Given a set  $\Gamma$  we denote by  $\ell_\infty(\Gamma)$  the space of all bounded scalar functions on  $\Gamma$ , endowed with the sup norm and  $c_0(\Gamma)$  is the closed subspace spanned by the characteristic functions of the singletons of  $\Gamma$ . A Banach space  $X$  is said to be a  $\mathcal{L}_{\infty, \lambda}$ -space if every finite dimensional subspace  $F$  of  $X$  is contained in another finite

dimensional subspace of  $X$  whose Banach–Mazur distance to the corresponding  $\ell_\infty^n$  is at most  $\lambda$ . A space is said to be a  $\mathcal{L}_\infty$ -space if it is a  $\mathcal{L}_{\infty,\lambda}$ -space for some  $\lambda \geq 1$ ; we will say that it is a  $\mathcal{L}_{\infty,\lambda+}$ -space when it is a  $\mathcal{L}_{\infty,\lambda'}$ -space for all  $\lambda' > \lambda$ . A Lindenstrauss space is one whose dual is isometric to  $L_1(\mu)$  for some measure  $\mu$ . Lindenstrauss spaces and  $\mathcal{L}_{\infty,1+}$ -spaces are identical classes.

We say that a Banach space is a  $\mathcal{C}$ -space if it is isometrically isomorphic to  $C(K)$  for some compact space  $K$ . For a neat characterization of those real Banach algebras which are isometrically isomorphic to  $C(K)$ , see [1, Theorem 4.2.5].

Throughout the paper, **ZFC** denotes the usual setting of set theory with the Axiom of Choice, while **CH** denotes the continuum hypothesis.

2.1. *The push-out and pull-back constructions*

The push-out construction appears naturally when one considers a couple of operators defined on the same space, in particular in any extension problem. Let us explain why. Given operators  $\alpha : Y \rightarrow A$  and  $\beta : Y \rightarrow B$ , the associated push-out diagram is

$$\begin{array}{ccc}
 Y & \xrightarrow{\alpha} & A \\
 \beta \downarrow & & \downarrow \beta' \\
 B & \xrightarrow{\alpha'} & \text{PO}
 \end{array} \tag{1}$$

Here, the push-out space  $\text{PO} = \text{PO}(\alpha, \beta)$  is quotient of the direct sum  $A \oplus_1 B$  (with the sum norm) by the closure of the subspace  $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$ . The map  $\alpha'$  is given by the inclusion of  $B$  into  $A \oplus_1 B$  followed by the natural quotient map  $A \oplus_1 B \rightarrow (A \oplus_1 B)/\overline{\Delta}$ , so that  $\alpha'(b) = (0, b) + \overline{\Delta}$  and, analogously,  $\beta'(a) = (a, 0) + \overline{\Delta}$ . The diagram (1) is commutative:  $\beta'\alpha = \alpha'\beta$ . Moreover, it is ‘minimal’ in the sense of having the following universal property: if  $\beta'' : A \rightarrow C$  and  $\alpha'' : B \rightarrow C$  are operators such that  $\beta''\alpha = \alpha''\beta$ , then there is a unique operator  $\gamma : \text{PO} \rightarrow C$  such that  $\alpha'' = \gamma\alpha'$  and  $\beta'' = \gamma\beta'$ . Clearly,  $\gamma((a, b) + \overline{\Delta}) = \beta''(a) + \alpha''(b)$  and one has  $\|\gamma\| \leq \max\{\|\alpha''\|, \|\beta''\|\}$ . The basic properties of push-out diagrams are gathered below; see [7, Lemma 2.1].

**Lemma 2.1.**

- (a)  $\max\{\|\alpha'\|, \|\beta'\|\} \leq 1$ .
- (b) If  $\alpha$  is an isomorphic embedding, then  $\Delta$  is closed.
- (c) If  $\alpha$  is an isometric embedding and  $\|\beta\| \leq 1$  then  $\alpha'$  is an isometric embedding.
- (d) If  $\alpha$  is an isomorphic embedding then  $\alpha'$  is an isomorphic embedding.
- (e) If  $\|\beta\| \leq 1$  and  $\alpha$  is an isomorphism then  $\alpha'$  is an isomorphism and  $\|(\alpha')^{-1}\| \leq \max\{1, \|\alpha\|\}$ .

The pull-back construction is the dual of that of push-out in the sense of categories, that is, “reversing arrows”. Indeed, let  $\alpha : A \rightarrow Z$  and  $\beta : B \rightarrow Z$  be operators acting between Banach spaces. The associated pull-back diagram is

$$\begin{array}{ccc}
 \text{PB} & \xrightarrow{\beta'} & A \\
 \alpha' \downarrow & & \downarrow \alpha \\
 B & \xrightarrow{\beta} & Z
 \end{array} \tag{2}$$

Here, the pull-back space is  $PB = PB(\alpha, \beta) = \{(a, b) \in A \oplus_{\infty} B : \alpha(a) = \beta(b)\}$ . The arrows after primes are the restriction of the projections onto the corresponding factor. Needless to say (2) is minimally commutative in the sense that if the operators  $''\beta : C \rightarrow A$  and  $''\alpha : C \rightarrow B$  satisfy  $\alpha \circ ''\beta = \beta \circ ''\alpha$ , then there is a unique operator  $\gamma : C \rightarrow PB$  such that  $''\alpha = '\alpha\gamma$  and  $''\beta = '\beta\gamma$ . Clearly,  $\gamma(c) = (''\beta(c), ''\alpha(c))$  and  $\|\gamma\| \leq \max\{\|''\alpha\|, \|''\beta\|\}$ . Quite clearly  $'\alpha$  is onto if  $\alpha$  is.

2.2. Exact sequences

A short exact sequence of Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \longrightarrow 0 \tag{3}$$

where  $Y, X$  and  $Z$  are Banach spaces and the arrows are operators in such a way that the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem  $\iota$  embeds  $Y$  as a closed subspace of  $X$  and  $Z$  is isomorphic to the quotient  $X/\iota(Y)$ . We say that  $0 \rightarrow Y \rightarrow X_1 \rightarrow Z \rightarrow 0$  is equivalent to (3) if there exists an operator  $t : X \rightarrow X_1$  making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \parallel & & \downarrow t & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0. \end{array} \tag{4}$$

This is a true equivalence relation since such a  $t$  has to be an isomorphism. The sequence (3) is said to be trivial if it is equivalent to the direct sum sequence  $0 \rightarrow Y \rightarrow Y \oplus Z \rightarrow Z \rightarrow 0$ . This happens if and only if it splits, that is, there is an operator  $p : X \rightarrow Y$  such that  $p\iota = \mathbf{1}_Y$  ( $\iota(Y)$  is complemented in  $X$ ); equivalently, there is an operator  $s : Z \rightarrow X$  such that  $\pi s = \mathbf{1}_Z$ . For every pair of Banach spaces  $Z$  and  $Y$ , we denote by  $\text{Ext}(Z, Y)$  the space of all exact sequences  $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$  modulo equivalence. We write  $\text{Ext}(Z, Y) = 0$  to indicate that every sequence of the form (3) is trivial. The reason for this notation is that  $\text{Ext}(Z, Y)$  has a natural linear structure [18,24] for which the (class of the) trivial extension is the zero element.

Suppose we are given an exact sequence (3) and an operator  $t : Y \rightarrow B$ . Consider the push-out of the couple  $(\iota, t)$  and draw the corresponding arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t' & & & & \\ & & B & \xrightarrow{t'} & \text{PO} & & & & \end{array}$$

By Lemma 2.1(a),  $t'$  is an isomorphic embedding. Now, the operator  $\pi : X \rightarrow Z$  and the zero operator  $0 : B \rightarrow Z$  satisfy the identity  $\pi\iota = 0t = 0$ , and thus the universal property of the push-out gives a unique operator  $\varpi : \text{PO} \rightarrow Z$  – defined as  $\varpi((x, b) + \Delta) = \pi(x)$  – making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t' & & \parallel & & \\ 0 & \longrightarrow & B & \xrightarrow{t'} & \text{PO} & \xrightarrow{\varpi} & Z & \longrightarrow & 0 \end{array} \tag{5}$$

Elementary considerations show that the lower sequence in the preceding diagram is exact: check commutativity, and discard everything but the definition of PO. That sequence will we referred to as the push-out sequence. Actually, the universal property of the push-out makes this diagram unique, in the sense that for any other commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
 & & \downarrow \iota & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0
 \end{array}$$

the lower row turns out to be equivalent to the push-out sequence in (5). For this reason we usually refer to a diagram like that as a push-out diagram. The universal property of the push-out diagram immediately yields the following.

**Lemma 2.2.** *With the above notations, the push-out sequence splits if and only if  $t$  extends to  $X$ , that is, there is an operator  $T : X \rightarrow B$  such that  $T\iota = t$ .*

Proceeding dually one obtains the pull-back sequence. Consider again (3) and an operator  $u : A \rightarrow Z$ . Let us form the pull-back diagram of the couple  $(\pi, u)$  thus:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
 & & & & \uparrow \iota' & & \uparrow u & & \\
 & & & & \text{PB} & \xrightarrow{\iota'\pi} & A & & 
 \end{array}$$

Recalling that  $\iota'\pi$  is onto and taking  $J(y) = (0, \iota(y))$ , it is easily seen that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
 & & \parallel & & \uparrow \iota' & & \uparrow u & & \\
 0 & \longrightarrow & Y & \xrightarrow{J} & \text{PB} & \xrightarrow{\iota'\pi} & A & \longrightarrow & 0.
 \end{array} \tag{6}$$

The lower sequence is exact, and we shall referred to it as the pull-back sequence. The splitting criterion is now as follows.

**Lemma 2.3.** *With the above notations, the pull-back sequence splits if and only if  $u$  lifts to  $X$ , that is, there is an operator  $U : A \rightarrow X$  such that  $\pi U = u$ .*

Given an exact sequence  $0 \rightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \rightarrow 0$  and another Banach space  $B$ , taking operators with values in  $B$  one gets the exact sequence

$$0 \rightarrow \mathcal{L}(Z, B) \xrightarrow{\pi^*} \mathcal{L}(X, B) \xrightarrow{\iota^*} \mathcal{L}(Y, B)$$

that can be continued to form a “long exact sequence”

$$\begin{aligned}
 0 \rightarrow \mathcal{L}(Z, B) &\xrightarrow{\pi^*} \mathcal{L}(X, B) \xrightarrow{\iota^*} \mathcal{L}(Y, B) \xrightarrow{\beta} \text{Ext}(Z, B) \rightarrow \text{Ext}(X, B) \\
 &\rightarrow \text{Ext}(Y, B).
 \end{aligned}$$

Here we just indicate the action of  $\beta$ : it takes  $t \in \mathcal{L}(Y, B)$  into (the class in  $\text{Ext}(Z, B)$  of) the lower extension of the push-out diagram (5). A detailed description of homology sequences can be seen in [18].

### 3. Basic properties of (universally) separably injective spaces

**Definition 3.1.** A Banach space  $E$  is separably injective if for every separable Banach space  $X$  and each subspace  $Y \subset X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : X \rightarrow E$ . If some extension  $T$  exists with  $\|T\| \leq \lambda\|t\|$  we say that  $E$  is  $\lambda$ -separably injective.

It is easy to check that every separably injective space  $E$  is  $\lambda$ -separably injective for some  $\lambda$  since every sequence of norm-one operators  $t_n : Y_n \rightarrow E$  induces a norm-one operator  $t : \ell_1(Y_n) \rightarrow E$ . Separable injective spaces can be characterized as follows.

**Proposition 3.2.** For a Banach space  $E$  the following properties are equivalent.

- (a)  $E$  is separably injective.
- (b) Every operator from a subspace of  $\ell_1$  into  $E$  extends to  $\ell_1$ .
- (c) For every Banach space  $X$  and each subspace  $Y$  such that  $X/Y$  is separable, every operator  $t : Y \rightarrow E$  extends to  $X$ .
- (d) If  $X$  is a Banach space containing  $E$  and  $X/E$  is separable, then  $E$  is complemented in  $X$ .
- (e) For every separable space  $S$  one has  $\text{Ext}(S, E) = 0$ .

Moreover,

- (1) The space  $E$  is  $\lambda$ -complemented in every  $Z$  such that  $Z/E$  is separable if and only if every operator  $t : Y \rightarrow E$  admits an extension  $T : X \rightarrow E$  with  $\|T\| \leq \lambda\|t\|$ , whenever  $X/Y$  is separable.
- (2) If  $E$  is  $\lambda$ -separably injective, then for every operator  $t : Y \rightarrow E$  there exists an extension  $T : X \rightarrow E$  of  $T$  with  $\|T\| \leq 3\lambda\|t\|$ , whenever  $X/Y$  is separable.

**Proof.** It is clear that (c)  $\Rightarrow$  (a)  $\Rightarrow$  (b) and (c)  $\Rightarrow$  (d)  $\Leftrightarrow$  (e). Moreover, (1) shows that (d)  $\Rightarrow$  (c) and (2) shows that (a)  $\Rightarrow$  (c). The remaining implication (b)  $\Rightarrow$  (a) follows from the proof of (2) below.

For the sufficiency statement in (1) simply consider  $t$  as the identity on  $E$ . For the necessity statement, given an operator  $t : Y \rightarrow E$  from the associated push-out diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{t} & X & \xrightarrow{\pi} & X/Y & \longrightarrow & 0 \\
 & & \downarrow t & & \downarrow t' & & \parallel & & \\
 0 & \longrightarrow & E & \xrightarrow{t'} & \text{PO} & \longrightarrow & \text{PO}/E & \longrightarrow & 0.
 \end{array}$$

Since  $\text{PO}/E = X/Y$  is separable, there is a projection  $p : \text{PO} \rightarrow E$  with norm at most  $\lambda$ , and thus, recalling that  $\|t'\| \leq 1$ , the composition  $pt' : X \rightarrow E$  yields an extension of  $t$  with norm at most  $\lambda$ .

The proof for (2) is a little more tricky. Let  $q$  be a surjective map from  $\ell_1 \rightarrow X/Y$ . The lifting property of  $\ell_1$  provides an operator  $Q : \ell_1 \rightarrow X$ . Consider thus the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker q & \xrightarrow{J} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\
 & & \phi \downarrow & & Q \downarrow & & \parallel & & \\
 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0
 \end{array}$$

Let us construct the true push-out of the couple  $(\phi, j)$  and the corresponding complete diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker q & \xrightarrow{J} & \ell_1 & \xrightarrow{q} & X/Y \longrightarrow 0 \\
 & & \phi \downarrow & & \downarrow \phi' & & \parallel \\
 0 & \longrightarrow & Y & \xrightarrow{J'} & \text{PO} & \longrightarrow & X/Y \longrightarrow 0.
 \end{array}$$

We can consider without loss of generality that  $\|\phi\| = 1$ . Let  $S : \ell_1 \rightarrow E$  be an extension of  $t\phi$  with  $\|S\| \leq \lambda\|t\phi\| \leq \lambda\|t\|$ . By the universal property of the push-out, there exists an operator  $L : \text{PO} \rightarrow E$  such that  $L\phi' = S$  and  $\|L\| \leq \max\{\|t\|, \|S\|\} \leq \lambda\|t\|$ . Again by the universal property of the push-out, there is a diagram of equivalent exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{J'} & \text{PO} & \longrightarrow & X/Y \longrightarrow 0 \\
 & & \parallel & & \gamma \downarrow & & \parallel \\
 0 & \longrightarrow & Y & \xrightarrow{t} & X & \xrightarrow{p} & X/Y \longrightarrow 0,
 \end{array}$$

where the isomorphism  $\gamma$  is defined as  $\gamma((y, u) + \Delta) = J(y) + Q(u)$  is such that  $\|\gamma\| \leq \max\{\|J\|, \|Q\|\} \leq 1$ . The desired extension of  $t$  to  $X$  is  $T = L\gamma^{-1}$ , where  $\gamma^{-1}$  comes defined by

$$\gamma^{-1}(x) = (x - s(px), s(px)) + \Delta,$$

where  $s : X/Y \rightarrow \ell_1$  is a homogeneous bounded selection for  $q$  with  $\|s\| \leq 1$ . One clearly has  $\|\gamma^{-1}\| \leq 3$ , and therefore  $\|T\| \leq 3\lambda$ .  $\square$

We are especially interested in the following subclass of separably injective spaces.

**Definition 3.3.** A Banach space  $E$  is said to be universally separably injective if for every Banach space  $X$  and each separable subspace  $Y \subset X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : X \rightarrow E$ . If some extension  $T$  exists with  $\|T\| \leq \lambda\|t\|$  we say that  $E$  is universally  $\lambda$ -separably injective.

It is easy to check that a Banach space  $E$  is universally separably injective if and only if every  $E$ -valued operator with separable range extends to any superspace. It is also easy to show that every universally separably injective space is  $\lambda$ -universally separably injective for some  $\lambda$ .

Recall that a Banach space  $X$  has Pełczyński’s property  $(V)$  if every operator defined on  $X$  is either weakly compact or it is an isomorphism on a subspace isomorphic to  $c_0$ . It is well-known that Lindenstrauss spaces (i.e.,  $\mathcal{L}_{\infty,1+}$ -spaces) have this property [39]. Not all  $\mathcal{L}_{\infty}$ -spaces have Pełczyński’s property  $(V)$ : for example, the  $\mathcal{L}_{\infty}$ -spaces without copies of  $c_0$  constructed by Bourgain and Delbaen [14]; or those that can be obtained from Bourgain–Pisier [15]; or the space  $\Omega$  constructed in [21] as a twisted sum  $0 \rightarrow C[0, 1] \rightarrow \Omega \rightarrow c_0 \rightarrow 0$  with strictly singular quotient map. Other examples appear in the more recent papers [6,32].

**Definition 3.4.** We will say that  $X$  has Rosenthal’s property  $(V)$  if every operator defined on  $X$  is either weakly compact or it is an isomorphism on a subspace isomorphic to  $\ell_{\infty}$ .

Recall that a Banach space  $X$  is said to be a Grothendieck space if every operator from  $X$  to a separable Banach space (or to  $c_0$ ) is weakly compact. Clearly, a Banach space with Pełczyński’s property  $(V)$  is a Grothendieck space if and only if it has no complemented subspace isomorphic to  $c_0$ . It is well-known that  $\ell_{\infty}$  is a Grothendieck space. One moreover has the following.

**Proposition 3.5.**

- (a) A separably injective space is of type  $\mathcal{L}_\infty$ , has Pełczyński's property (V) and, when it is infinite dimensional, contains copies of  $c_0$ .
- (b) A universally separably injective space is a Grothendieck space of type  $\mathcal{L}_\infty$ , has Rosenthal's property (V) and, when it is infinite dimensional, contains  $\ell_\infty$ .

**Proof.** (a) Let  $E$  be a  $\lambda$ -separably injective space. We want to see that if  $Y$  is a subspace of any Banach space  $X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : X \rightarrow E^{**}$  with  $\|T\| \leq \lambda\|t\|$ . This implies that  $E^{**}$  is  $\lambda$ -injective, by an old result of Lindenstrauss [42, Theorem 2.1]. Since  $E^{**}$  is infinite-dimensional, it is then a  $\mathcal{L}_{\infty, 9\lambda^+}$  space and so is  $E$ . Let  $t : Y \rightarrow E$  be an operator. Given a finite-dimensional subspace  $F$  of  $X$ , let  $T_F : F \rightarrow E$  be any operator extending the restriction of  $t$  to  $Y \cap F$ . Let  $\mathcal{F}$  be the set of finite-dimensional subspaces of  $X$ , ordered by inclusion, let  $\mathcal{U}$  be any ultrafilter refining the Fréchet filter on  $\mathcal{F}$ , that is, containing every set of the form  $\{G \in \mathcal{F} : F \subset G\}$  for fixed  $F \in \mathcal{F}$ . Then, define  $T : X \rightarrow E^{**}$  taking

$$T(x) = \text{weak}^*\text{-}\lim_{\mathcal{U}(F)} T_F(1_{F(x)}x).$$

It is easily seen that  $T$  is a linear extension of  $t$ , with  $\|T\| \leq \lambda\|t\|$ .

To show that  $E$  contains  $c_0$  and has property (V), let  $T : E \rightarrow X$  be a non-weakly compact operator ( $E$  being an infinite dimensional  $\mathcal{L}_\infty$  space that cannot be reflexive). Choose a bounded sequence  $(x_n)$  in  $E$  such that  $(Tx_n)$  has no weakly convergent subsequences and let  $Y$  be the subspace spanned by  $(x_n)$  in  $E$ . As  $Y$  is separable we can regard it as a subspace of  $C[0, 1]$ . Let  $J : C[0, 1] \rightarrow E$  be any operator extending the inclusion of  $Y$  into  $E$ . Since  $TJ : C[0, 1] \rightarrow E$  is not weakly compact,  $TJ$  is an isomorphism on some subspace isomorphic to  $c_0$ ; and the same occurs to  $T$ .

(b) If, in addition to that,  $E$  is universally separably injective we may take  $T : E \rightarrow Z$  and  $Y \subset E$  as before but this time we consider  $Y$  as a subspace of  $\ell_\infty$ . If  $J : \ell_\infty \rightarrow E$  is any extension of the inclusion of  $Y$  into  $E$ , then  $TJ : \ell_\infty \rightarrow Z$  is not weakly compact. Hence it is an isomorphism on some subspace isomorphic to  $\ell_\infty$  and so is  $T$ .  $\square$

Several modifications on the proof of Ostrovskii [48] yield the following.

**Proposition 3.6.** A  $\lambda$ -separably injective space with  $\lambda < 2$  is either finite-dimensional or has density character at least  $\mathfrak{c}$ .

**Proof.** Let  $X$  be an infinite dimensional  $\lambda$ -separably injective space for  $\lambda < 2$ . We have shown that  $X$  contains  $c_0$ , and thus by a result of James [36] it contains, for each  $\varepsilon > 0$ , an  $(1 + \varepsilon)$ -isomorphic copy of  $c_0$ . With a standard renorming [50, Proposition 1] we may assume  $X$  contains  $c_0$  isometrically and it is  $\lambda'$ -separably injective, still with  $\lambda' < 2$ . So, let  $u : c_0 \rightarrow X$  be an isometric embedding and let  $u_n = u(e_n)$ , where  $(e_n)$  is the unit basis of  $c_0$ . Consider the push-out space

$$\begin{array}{ccc} c_0 & \xrightarrow{u} & X \\ \iota \downarrow & & \downarrow \\ \ell_\infty & \longrightarrow & \text{PO} \end{array}$$

in which  $\iota$  is the natural inclusion map. In the rest of the proof we identify the elements of  $c_0$ ,  $\ell_\infty$  and  $X$  with their images in PO. For each element  $f \in \ell_\infty$  with all coordinates  $\pm 1$ , let



$P_f : X + [f] \rightarrow X$  be a projection with norm at most  $\lambda'$ . If  $\text{dens } X < \mathfrak{c}$ , then, for each  $\varepsilon > 0$ , there exist  $f \neq g$  such that  $\|P_f(f) - P_g(g)\| < \varepsilon$ . Pick  $n$  so that  $f(n) = 1$  and  $g(n) = -1$ . One has  $\|u_n - \frac{1}{2}f\|_{\text{PO}} = \|u_n + \frac{1}{2}g\|_{\text{PO}} = \frac{1}{2}$ . This yields

$$\begin{aligned} 2 = 2\|u_n\| &< \|u_n + P_f(f/2) - P_g(g/2) + u_n\| + \varepsilon \\ &\leq \|P_f(u_n - (f/2))\| + \|P_g(u_n + (g/2))\| + \varepsilon \\ &\leq \|P_f\|\|u_n - f/2\| + \|P_g\|\|u_n + g/2\| + \varepsilon \\ &\leq \lambda' + \varepsilon < 2, \end{aligned}$$

a contradiction.  $\square$

Recall that a class of Banach spaces is said to have the 3-space property if whenever  $X/Y$  and  $Y$  belong to the class, then so  $X$  does. The monograph [24] contains a rather complete study of 3-space properties.

- Proposition 3.7.** (1) *The class of separably injective spaces has the 3-space property.*  
 (2) *The quotient of two separably injective spaces is separably injective.*  
 (3) *The class of universally separably injective spaces has the 3-space property.*  
 (4) *The quotient of a universally separably injective space by a separably injective space is universally separably injective.*

**Proof.** The simplest proof for the 3-space property (1) follows from characterization (2) in Proposition 3.2: let us consider an exact sequence  $0 \rightarrow F \rightarrow E \xrightarrow{\pi} G \rightarrow 0$  in which both  $F$  and  $G$  are separably injective. Let  $\phi : K \rightarrow E$  be an operator from a subspace  $\iota : K \rightarrow \ell_1$  of  $\ell_1$ ; then  $\pi\phi$  can be extended to an operator  $\Phi : \ell_1 \rightarrow G$ , which can in turn be lifted to an operator  $\Psi : \ell_1 \rightarrow E$ . The difference  $\phi - \Psi\iota$  takes values in  $F$  and can thus be extended to an operator  $e : \ell_1 \rightarrow F$ . The desired operator is  $\Psi + e$ . (A proof that properties having the form  $\text{Ext}(X, S) = 0$  are always 3-space properties of  $X$  can be found in [18].)

To prove (2) and (4) let us consider an exact sequence  $0 \rightarrow F \rightarrow E \xrightarrow{\pi} G \rightarrow 0$  in which  $F$  is separably injective and  $E$  is (universally) separably injective. Let  $\phi : Y \rightarrow G$  be an operator from a separable space  $Y$  which is a subspace of a separable (arbitrary) space  $X$ . Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \xrightarrow{q} & G & \longrightarrow & 0 \\ & & & & \parallel & & \uparrow \phi & & \uparrow \phi \\ 0 & \longrightarrow & F & \longrightarrow & \text{PB} & \xrightarrow{Q} & Y & \longrightarrow & 0 \end{array}$$

Since  $F$  is separably injective, the lower exact sequence splits, so  $Q$  has a selection operator  $s : Y \rightarrow \text{PB}$ . By the injectivity assumption about  $E$ , there exists an operator  $T : X \rightarrow E$  agreeing with  $Qs$  on  $Y$ . Then  $qT : X \rightarrow G$  is the desired extension of  $\phi$ .

The proof for (3) has to wait until Theorem 5.2 when a suitable characterization of universally separably injective spaces will be presented.  $\square$

Several variations of these results can be seen in [26]. It is obvious that if  $(E_i)_{i \in I}$  is a family of  $\lambda$ -separably injective Banach spaces, then  $\ell_\infty(I, E_i)$  is  $\lambda$ -separably injective. The non-obvious fact that also  $c_0(I, E_i)$  is separably injective can be considered as a vector valued version of Sobczyk’s theorem. Proofs for this result have been obtained by Johnson–Oikhberg [38], Rosenthal [52], Cabello Sánchez [16] and Castillo–Moreno [25], each with its own estimate for the constant. These are  $2\lambda^2$  (implicitly),  $\lambda(1 + \lambda)^+$ ,  $(3\lambda^2)^+$  and  $6(1 + \lambda)$ , respectively.

**Remark 3.8.** Let  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  be a short exact sequence of Banach spaces. We know from Proposition 3.7 that  $E$  is separably injective if the other two relevant spaces are; and the same happens with  $G$ . What about  $F$ ? Bourgain [12] constructed an uncomplemented copy of  $\ell_1$  in  $\ell_1$ , which yields an exact sequence  $0 \rightarrow \ell_1 \rightarrow \ell_1 \rightarrow B \rightarrow 0$  that does not split. By Lindenstrauss’ lifting  $B$  is not a  $\mathcal{L}_1$  space. Its dual sequence  $0 \rightarrow B^* \rightarrow \ell_\infty \rightarrow \ell_\infty \rightarrow 0$  shows that the kernel of a quotient mapping between two injective spaces may fail to be even a  $\mathcal{L}_\infty$ -space.

#### 4. Examples

All injective spaces are universally separably injective. Sobczyk theorem states that  $c_0$  is 2-separably injective in its natural supremum norm and so is  $c_0(\Gamma)$  for any index set  $\Gamma$  (see [22] for a rather detailed survey). They are not universally separably injective since they do not contain  $\ell_\infty$ .

##### 4.1. Twisted sums

The 3-space property yields that twisted sums of separably injective are also separably injective. In particular:

- Twisted sums of  $c_0$  and  $c_0(\Gamma)$ . This includes the Johnson–Lindenstrauss spaces  $JL_\infty(\mathcal{M})$  [37] obtained by taking the closure of the linear span in  $\ell_\infty$  of  $c_0$  and the characteristic functions of the sets of an uncountable almost disjoint family  $\mathcal{M}$  of subsets of  $\mathbb{N}$ . Marciszewski and Pol answer in [47] a question of Koszmider [40, Question 5] showing that there exist  $2^c$  almost disjoint families  $\mathcal{M}$  generating mutually non-isomorphic Johnson–Lindenstrauss spaces, all of them  $\mathcal{C}$ -spaces.
- Twisted sums of two nonseparable  $c_0(\Gamma)$  spaces. This includes variations of the previous construction using the Sierpinski–Tarski [41] generalization of the construction of almost-disjoint families; the Ciesielski–Pol space (see [29]); the WCG nontrivial twisted sums of  $c_0(\Gamma)$  obtained independently by Argyros, Castillo, Granero, Jimenez and Moreno [5] and by Marciszewski [46].
- Twisted sums of  $c_0$  and  $\ell_\infty$ , as those constructed in [20].
- Benyamini constructed in [11] an  $\mathcal{M}$ -space which is not complemented in any  $\mathcal{C}$ -space. This space turns out to be a twisted sum of  $c_0$  and  $c_0(\ell_\infty/c_0)$  (see [24, p. 104] or [26, Prop.3.3]) and so it is separably injective by Proposition 3.7(a).

It is not hard to prove that none of the preceding examples can be universally separably injective.

##### 4.2. The space $\ell_\infty^c(\Gamma)$

A typical 1-universally separably injective space is the space  $\ell_\infty^c(\Gamma)$  of countably supported bounded functions  $f : \Gamma \rightarrow \mathbb{R}$ , where  $\Gamma$  is an uncountable set. One has the following.

**Proposition 4.1.** *The space  $\ell_\infty^c(\Gamma)$  is: (a) universally 1-separably injective; (b) not isometric to a  $\mathcal{C}$ -space; (c) isomorphic to a  $\mathcal{C}$ -space; (d) not injective.*

**Proof.** (a) Every separable subspace of  $\ell_\infty^c(\Gamma)$  is contained in an isometric copy of  $\ell_\infty$ . (b) The unit ball of every  $C(K)$  has extreme points and quite clearly, the ball of  $\ell_\infty^c(\Gamma)$  has no extreme

point. (c) Consider the unitization of  $\ell_\infty^c(\Gamma)$  inside  $\ell_\infty(\Gamma)$ , that is,

$$\ell_\infty^c(\Gamma)_+ = \{f \in \ell_\infty(\Gamma) : f = \lambda 1_\Gamma + g : \lambda \in \mathbb{R}, g \in \ell_\infty^c(\Gamma)\}.$$

It is clear that  $\ell_\infty^c(\Gamma)$  is 2-isomorphic to  $\ell_\infty^c(\Gamma)_+$ . As every unital subalgebra of  $\ell_\infty(\Gamma)$  the latter is isometrically isomorphic to the algebra of all continuous real-valued functions on certain compact space  $K$ . (d)  $\ell_\infty^c(\Gamma)$  contains  $c_0(\Gamma)$ . But Rosenthal proved in [51] that an injective space containing  $c_0(I)$  must also contain  $\ell_\infty(I)$ . Thus, if  $\ell_\infty^c(\Gamma)$  were injective, it should contain a subspace isomorphic to  $\ell_\infty(\Gamma)$ . This is obviously impossible if the cardinality of  $\Gamma$  satisfies  $|\Gamma|^{\aleph_0} < 2^{|\Gamma|}$  by comparison of their density characters. But indeed the result holds even if  $|\Gamma|$  does not satisfy that inequality. To see this, let us show that, given any uncountable  $\Gamma$ , there is no injective operator  $T : \ell_\infty(\aleph_1) \rightarrow \ell_\infty^c(\Gamma)$ . With a slight abuse of notation,  $\ell_\infty(\aleph_1)^*$  is the space of all bounded finitely additive measures on the family  $\wp(\aleph_1)$  of all subsets of  $\aleph_1$ . Consider the measures  $\nu_\gamma = T^*(\delta_\gamma)$ ,  $\gamma \in \Gamma$ , for an operator  $T$  as above. They have the property that for every  $A \subset \aleph_1$ ,  $\{\gamma \in \Gamma : \nu_\gamma(A) \neq 0\}$  is countable. Let  $f : \aleph_1 \rightarrow 2^\mathbb{N}$  be a one-to-one map. The set  $\Delta$  of those  $\gamma \in \Gamma$  for which there is a clopen  $U \subset 2^\mathbb{N}$  with  $\nu_\gamma(f^{-1}(U)) \neq 0$  is countable, by the above property and the fact that there are only countably many clopens. For all  $\gamma \notin \Delta$  we have that  $\nu_\gamma\{\alpha\} = 0$  for all  $\alpha \in \aleph_1$ . On the other hand, since  $\Delta$  is countable,  $\nu_\gamma\{\alpha\} = 0$  for all  $\gamma \in \Delta$  and all but countably many  $\alpha$ 's in  $\aleph_1$ . In particular, there exist  $\alpha \neq \beta$ ,  $\alpha, \beta \in \aleph_1$  such that  $\nu_\gamma\{\alpha\} = \nu_\gamma\{\beta\} = 0$  for all  $\gamma \in \Gamma$ . This means that  $\delta_\gamma(T1_\alpha) = \delta_\gamma(T1_\beta)$  for all  $\gamma$ , that implies that  $T1_\alpha = T1_\beta$  which contradicts that  $T$  is one-to-one.  $\square$

The space  $\ell_\infty^c(\Gamma)$  shows that, contrarily to what happens in the injective case, universally 1-separably injective spaces need not to be isometric to any  $\mathcal{C}$ -space. Actually there exist universally 1-separably injective spaces which are not even complemented in any  $\mathcal{C}$ -space; see [7] for an account.

### 4.3. The space $\ell_\infty/c_0$

Since  $\ell_\infty$  is injective and  $c_0$  is separably injective, it follows from Proposition 3.7 that  $\ell_\infty/c_0$  is universally separably injective, although the constant thus obtained is not optimal. It will follow from Proposition 4.6(a) that  $\ell_\infty/c_0$  is 1-universally separably injective. Moreover, it can also be proved that every separable subspace of  $\ell_\infty/c_0$  is contained in a subalgebra of  $\ell_\infty/c_0$  isometrically isomorphic to  $\ell_\infty$ .

It is well-known that  $\ell_\infty/c_0$  is not injective. The simplest proof appears in Rosenthal [51]: the space  $\ell_\infty/c_0$  contains  $c_0(\mathfrak{c})$  while it cannot contain  $\ell_\infty(\mathfrak{c})$ . This proof is quite rough in a sense: it says that  $\ell_\infty/c_0$  is uncomplemented in its bidual, a huge superspace. Denoting  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ , Amir had shown in [2] that  $C(\mathbb{N}^*)$  is not complemented in  $\ell_\infty(2^\mathfrak{c})$ , which provides another proof that  $\ell_\infty/c_0$  is not injective. It can be even shown [27] that  $C(\mathbb{N}^*)$  contains an uncomplemented copy  $Y$  of itself. Proposition 5.6 yields that the corresponding quotient  $C(\mathbb{N}^*)/Y$  cannot be isomorphic to a quotient of  $\ell_\infty$  by a separable subspace.

### 4.4. Other $C(K)$ -spaces

Recall that a compact Hausdorff space  $K$  is said to be an  $F$ -space if disjoint open  $F_\sigma$  sets (equivalently, cozeros) have disjoint closures. Equivalently, if any continuous function  $f : K \rightarrow \mathbb{R}$  can be decomposed as  $f = u|f|$  for some continuous function  $u : K \rightarrow \mathbb{R}$ . It is an immediate consequence of Tietze's extension theorem that a closed subset of an  $F$ -space is an  $F$ -space. In particular,  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$  is an  $F$ -space. One has the following.

**Proposition 4.2.** *A compact space  $K$  is an  $F$ -space if and only if the Banach space  $C(K)$  is 1-separably injective.*

The sufficiency is a particular case of the implication (ii)  $\Rightarrow$  (iv) in [43]; while the necessity follows from (iv)  $\Rightarrow$  (i) in the same paper. See also [9] for several generalized forms of this result. Simple examples show that  $K$  may fail to be an  $F$ -space if the space  $C(K)$  is merely isomorphic even to a 1-injective space. This can be deduced from [54, Theorem 2.6], or else from [53]: identify two points  $u, v \in \mathbb{N}^*$  that we may consider as two free ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$  on  $\mathbb{N}$  and let us call  $\beta(u, v)$  the corresponding quotient space of  $\beta\mathbb{N}$ . The space  $C(\beta(u, v)) = \{f \in C(\beta\mathbb{N}) : f(u) = f(v)\}$  is a closed hyperplane of  $C(\beta\mathbb{N})$  and thus it is 2-isomorphic to  $\ell_\infty$ . However,  $\beta(u, v)$  is not an  $F$ -space: pick  $U \in \mathcal{U} \setminus \mathcal{V}$ , so that  $V = \mathbb{N} \setminus U$  belongs to  $\mathcal{V}$ . Set the function  $f : \mathbb{N} \rightarrow \mathbb{R}$  given by

$$f(n) = \frac{1_U(n) - 1_V(n)}{n}$$

and extend it to a continuous function on  $\beta\mathbb{N}$  denoted again  $f$ . As  $f(u) = f(v) = 0$  we have  $f \in C(\beta(u, v))$ . However there is no factorization  $f = g|f|$  with  $g \in C(\beta(u, v))$  since in this case we would have  $g(u) = 1$  and  $g(v) = -1$ .

Given a compact space  $K$ , we write  $K'$  for its derived set, that is, the set of its accumulation points. This process can be iterated to define  $K^{(n+1)}$  as  $(K^{(n)})'$ . We say that  $K$  has height  $n$  if  $K^{(n)} = \emptyset$ . We say that  $K$  has finite height if it has height  $n$  for some  $n \in \mathbb{N}$ .

**Proposition 4.3.** *If  $K$  is a compact space of height  $n$ , then  $C(K)$  is  $(2n - 1)$ -separably injective. Consequently, if  $K$  is a compact space of finite height then  $C(K)$  is separably injective although it is not universally separably injective unless  $K$  is finite.*

**Proof.** Let  $Y \subset X$  with  $X$  separable and let  $t : Y \rightarrow C(K)$  be a norm one operator. The range of  $t$  is separable and every separable subspace of a  $C(K)$  is contained in an isometric copy of  $C(L)$ , where  $L$  is the quotient of  $K$  after identifying  $k$  and  $k'$  when  $y(k) = y(k')$  for all  $y \in Y$ . This  $L$  is metrizable because  $Y$  is separable. Moreover, if  $K$  has height  $n$ , then  $L$  has height at most  $n$  and so it is homeomorphic to  $[0, \omega^r \cdot k]$  with  $r < n, k < \omega$ . Since  $C[0, \omega^r \cdot k]$  is  $(2r + 1)$ -separably injective [10], our operator can be extended to an operator  $T : X \rightarrow C(K)$  with norm

$$\|T\| \leq (2r + 1)\|t\| \leq (2n - 1)\|t\|,$$

concluding the proof.  $\square$

Using arguments from Amir [2], Baker [10] showed that  $(2n - 1)$  is the best constant for separable injectivity when  $K$  is a metrizable compact of finite height  $n$ . There are some difficulties in generalizing those arguments for nonmetrizable compact spaces, so we do not know if it could exist a nonmetrizable compact space  $K$  of height  $n$  such that  $C(K)$  is  $\lambda$ -separably injective for some  $\lambda < 2n - 1$ .

**Proposition 4.4.** *The space of all bounded Borel (respectively, Lebesgue) measurable functions on the line is 1-separably injective in the sup norm.*

**Proof.** The given spaces are (unital) closed subalgebras of  $\ell_\infty(\mathbb{R})$ . Thus they can be represented as  $C(K)$  spaces. On the other hand, each measurable function can be decomposed as  $f = u|f|$ , with  $u$  (and  $|f|$ , of course) measurable. This clearly implies that the corresponding compacta are  $F$ -spaces.  $\square$

Argyros proved in [4] that none of the spaces in the above example is injective. This is very simple in the Borel case: the characteristic functions of the singletons generate a copy of  $c_0(\mathbb{R})$  in the space of bounded Borel functions. The density character of the latter space is the continuum, as there are  $\mathfrak{c}$  Borel subsets. Therefore it cannot contain a copy of  $\ell_\infty(\mathbb{R})$ , whose density character is  $2^{\mathfrak{c}}$ .

#### 4.5. $M$ -ideals

A closed subspace  $J \subset X$  is called an  $M$ -ideal [33, Definition 1.1] if its annihilator  $J^\perp = \{x^* \in X^* : \langle x^*, x \rangle = 0 \text{ for all } x \in J\}$  is an  $L$ -summand in  $X^*$ . This just means that there is a linear projection  $P$  on  $X^*$  whose range is  $J^\perp$  and such that  $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$  for all  $x^* \in X^*$ . The easier examples of  $M$ -ideals are just ideals in  $C(K)$ -spaces. In particular, if  $H$  is a closed subset of the compact space  $K$  and  $L = K \setminus H$  then  $C_0(L)$  is an  $M$ -ideal in  $C(K)$ , which is straightforward from the Riesz representation of  $C(K)^*$ . A remarkable generalization of the Borsuk–Dugundji theorem for  $M$ -ideals was provided by Ando [3] and, independently, Choi and Effros [28]. In order to state it let us recall that a Banach space  $Z$  has the  $\lambda$ -approximation property ( $\lambda$ -AP) if, for every  $\varepsilon > 0$  and every compact subset  $K$  of  $Z$ , there exists a finite rank operator  $T$  on  $Z$ , with  $\|T\| \leq \lambda$ , such that  $\|Tz - z\| < \varepsilon$ , for every  $z \in K$ . We say that  $Z$  has the bounded approximation property (BAP) if it has the  $\lambda$ -AP, for some  $\lambda$ . We refer the reader to [23] for background and basic information about approximation properties.

**Theorem 4.5** ([33], Theorem 2.1). *Let  $J$  be an  $M$ -ideal in the Banach space  $E$  and  $\pi : E \rightarrow E/J$  the natural quotient map. Let  $Y$  be a separable Banach space and  $t : Y \rightarrow E/J$  be an operator. Assume further that one of the following conditions is satisfied.*

- (1)  $Y$  has the  $\lambda$ -AP.
- (2)  $J$  is a Lindenstrauss space.

*Then  $t$  can be lifted to  $E$ , that is, there is an operator  $T : Y \rightarrow E$  such that  $\pi T = t$ . Moreover one can get  $\|T\| \leq \lambda\|t\|$  under the assumption (1) and  $\|T\| = \|t\|$  under (2).*

One has the following.

**Proposition 4.6.** *Let  $J$  be an  $M$ -ideal in a Banach space  $E$ .*

- (a) *If  $E$  is  $\lambda$ -(universally) separably injective, then  $E/J$  is  $\lambda^2$ -(universally) separably injective.*
- (b) *If  $E$  is  $\lambda$ -separably injective, then  $J$  is  $2\lambda^2$ -separably injective.*

*When  $J$  is a Lindenstrauss space (which is always the case if  $E$  is), then the exponent 2 can be eliminated. In particular, if  $H$  is a closed subset of the compact space  $K$  and  $L = K \setminus H$  one has the following.*

- (c) *If  $C(K)$  is  $\lambda$ -(universally) separably injective, then so is  $C(H)$ .*
- (d) *If  $C(K)$  is  $\lambda$ -separably injective, then  $C_0(L)$  is  $2\lambda$ -separably injective.*

**Proof.** (a) By (the proof of) Proposition 3.5,  $E^{**}$  is  $\lambda$ -injective and so it has the  $\lambda$ -AP. Since  $E^{**} = J^{**} \oplus_\infty (E/J)^{**}$  we see that also  $J^{**}$  and  $(E/J)^{**}$  have the  $\lambda$ -AP. Hence both  $J$  and  $(E/J)$  have the  $\lambda$ -AP. Let  $Y$  be a separable subspace of  $X$  and  $t : Y \rightarrow E/J$  an operator. Let  $S$  be a separable subspace of  $E/J$  containing the image of  $t$ . By [23, Theorem 9.7] we may assume  $S$  has the  $\lambda$ -AP. Let  $s : S \rightarrow E$  be the lifting provided by Theorem 4.5, so that  $\|s\| \leq \lambda$ . Now, if  $T : X \rightarrow E$  is an extension of  $st$ , then  $\pi T : X \rightarrow E/J$  is an extension of  $t$ , and this can be achieved with  $\|\pi T\| = \|T\| \leq \lambda^2\|t\|$ .

(d) – and (b) –. Let us remark that if  $S$  is a subspace of  $C(K)$  containing  $C_0(L)$  and  $S/C_0(L)$  is separable, then there is a projection  $p : S \rightarrow C_0(L)$  of norm at most 2. Indeed,  $S/C_0(L)$  is a separable subspace of  $C(K_1)$  and there is a lifting  $s : S/C_0(L) \rightarrow C(K)$ , with  $\|s\| = 1$ , and  $p = id_S - sr$  is the required projection. Now, let  $t : Y \rightarrow C_0(L)$  be an operator, where  $Y$  is a subspace of a separable Banach space  $X$ . Considering  $t$  as taking values in  $C(K)$ , there is an extension  $T : X \rightarrow C(K)$  with  $\|T\| \leq \lambda\|t\|$ . Let  $S$  denote the least closed subspace of  $C(K)$  containing the range of  $T$  and  $C_0(L)$  and  $p : S \rightarrow C_0(L)$  a projection with  $\|p\| \leq 2$ . The composition  $pT : X \rightarrow C_0(L)$  is an extension of  $t$  and clearly,  $\|pT\| \leq 2\lambda\|t\|$ .  $\square$

#### 4.6. Ultraproducts of type $\mathcal{L}_\infty$

Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer the reader to Heinrich’s survey paper [34] or Sims’ notes [55]. Let  $I$  be a set,  $\mathcal{U}$  be an ultrafilter on  $I$ , and  $(X_i)_{i \in I}$  a family of Banach spaces. Then  $\ell_\infty(I, X_i)$ , endowed with the supremum norm, is a Banach space, and  $c_0^{\mathcal{U}}(X_i) = \{(x_i) \in \ell_\infty(I, X_i) : \lim_{\mathcal{U}(i)} \|x_i\| = 0\}$  is a closed subspace of  $\ell_\infty(I, X_i)$ . The ultraproduct of the family  $(X_i)_{i \in I}$  following  $\mathcal{U}$  is defined as the quotient

$$[X_i]_{\mathcal{U}} = \ell_\infty(X_i) / c_0^{\mathcal{U}}(X_i).$$

We denote by  $[(x_i)]$  the element of  $[X_i]_{\mathcal{U}}$  which has the family  $(x_i)$  as a representative. It is not difficult to show that  $\|[(x_i)]\| = \lim_{\mathcal{U}(i)} \|x_i\|$ . In the case  $X_i = X$  for all  $i$ , we denote the ultraproduct by  $X_{\mathcal{U}}$ , and call it the ultrapower of  $X$  following  $\mathcal{U}$ . If  $T_i : X_i \rightarrow Y_i$  is a uniformly bounded family of operators, the ultraproduct operator  $[T_i]_{\mathcal{U}} : [X_i]_{\mathcal{U}} \rightarrow [Y_i]_{\mathcal{U}}$  is given by  $[T_i]_{\mathcal{U}}[(x_i)] = [T_i(x_i)]$ . Quite clearly,  $\|[T_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}(i)} \|T_i\|$ .

**Definition 4.7.** An ultrafilter  $\mathcal{U}$  on a set  $I$  is countably incomplete if there is a sequence  $(I_n)$  of subsets of  $I$  such that  $I_n \in \mathcal{U}$  for all  $n$ , and  $\bigcap_{n=1}^\infty I_n = \emptyset$ .

It is obvious that any countably incomplete ultrafilter is non-principal and also that every non-principal (or free) ultrafilter on  $\mathbb{N}$  is countably incomplete. Assuming all free ultrafilters countably incomplete is consistent with ZFC, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible. From now on we will assume that all ultrafilters we use are countably incomplete.

The class of  $\mathcal{L}_{\infty, \lambda+}$  spaces is stable under ultraproducts [13, Proposition 1.22]. In the opposite direction, a Banach space is a  $\mathcal{L}_{\infty, \lambda+}$  space if and only if some (or every) ultrapower is. In particular, a Banach space is a Lindenstrauss space if and only if so are its ultrapowers. It is possible however to produce Lindenstrauss spaces as ultraproducts of families containing no  $\mathcal{L}_\infty$ -space. Indeed, if  $p(i) \rightarrow \infty$  along  $\mathcal{U}$ , then the ultraproduct  $[L_{p(i)}]_{\mathcal{U}}$  is a Lindenstrauss space (and, in fact, an abstract  $\mathcal{M}$ -space; see [17, Lemma 3.2]). The following result about the structure of separable subspaces of ultraproducts of type  $\mathcal{L}_\infty$  will be fundamental for us.

**Lemma 4.8.** Suppose  $[X_i]_{\mathcal{U}}$  is a  $\mathcal{L}_{\infty, \lambda+}$ -space. Then each separable subspace of  $[X_i]_{\mathcal{U}}$  is contained in a subspace of the form  $[F_i]_{\mathcal{U}}$ , where  $F_i \subset X_i$  is finite dimensional and  $\lim_{\mathcal{U}(i)} d(F_i, \ell_\infty^{k(i)}) \leq \lambda$ , with  $k(i) = \dim F_i$ .

**Proof.** Let us assume  $S$  is an infinite-dimensional separable subspace of  $[X_i]_{\mathcal{U}}$ . Let  $(s^n)$  be a linearly independent sequence spanning a dense subspace in  $S$  and, for each  $n$ , let  $(s_i^n)$  be a fixed representative of  $s^n$  in  $\ell_\infty(X_i)$ . Let  $S^n = \text{span}\{s^1, \dots, s^n\}$ . Since  $[X_i]_{\mathcal{U}}$  is a  $\mathcal{L}_{\infty, \lambda+}$ -space there

is, for each  $n$ , a finite dimensional  $F^n \subset [X_i]_{\mathcal{U}}$  containing  $S^n$  with  $d(F^n, \ell_\infty^{\dim F^n}) \leq \lambda + 1/n$ . For fixed  $n$ , let  $(f^m)$  be a basis for  $F^n$  containing  $s^1, \dots, s^n$ . Choose representatives  $(f_i^m)$  such that  $f_i^m = s_i^\ell$  if  $f^m = s^\ell$ . Moreover, let  $F_i^n$  be the subspace of  $X_i$  spanned by  $f_i^m$  for  $1 \leq m \leq \dim F^n$ .

Let  $(I_n)$  be a decreasing sequence of subsets  $I_n \in \mathcal{U}$  such that  $\bigcap_{n=1}^\infty I_n = \emptyset$ . For each integer  $n$  put

$$J'_n = \{i \in I : d(F_i^n, \ell_\infty^{\dim F^n}) \leq \lambda + 2/n\} \cap I_n$$

and  $J_m = \bigcap_{n \leq m} J'_n$ . All these sets are in  $\mathcal{U}$  and  $\bigcap_m J_m = \emptyset$ . We define a function  $k : I \rightarrow \mathbb{N}$  as

$$k(i) = \sup\{n : i \in J_n\}.$$

For each  $i \in I$ , take  $F_i = F_i^{k(i)}$ . This is a finite-dimensional subspace of  $X_i$  whose Banach–Mazur distance to the corresponding  $\ell_\infty^{k(i)}$  is at most  $\lambda + 2/k(i)$ . It is clear that  $[F_i]_{\mathcal{U}}$  contains  $S$  and also that  $k(i) \rightarrow \infty$  along  $\mathcal{U}$ , which completes the proof.  $\square$

**Lemma 4.9.** *For every function  $k : I \rightarrow \mathbb{N}$ , the space  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$  is universally 1-separably injective.*

**Proof.** Let  $\Gamma$  be the disjoint union of the sets  $\{1, 2, \dots, k(i)\}$  viewed as a discrete set. Now observe that  $c_0^{\mathcal{U}}(\ell_\infty^{k(i)})$  is an ideal in  $\ell_\infty(\ell_\infty^{k(i)}) = \ell_\infty(\Gamma) = C(\beta\Gamma)$  and apply Proposition 4.6(a).  $\square$

**Theorem 4.10.** *Let  $(X_i)_{i \in I}$  be a family of Banach spaces such that  $[X_i]_{\mathcal{U}}$  is a  $\mathcal{L}_{\infty, \lambda+}$ -space. Then  $[X_i]_{\mathcal{U}}$  is universally  $\lambda$ -separably injective.*

**Proof.** It is clear that a Banach space is universally  $\lambda$ -separably injective if and only if every separable subspace is contained in some larger universally  $\lambda$ -separably injective subspace. Combine now the last two lemmata.  $\square$

**Corollary 4.11.** *Let  $(X_i)_{i \in I}$  be a family of Banach spaces. If  $[X_i]_{\mathcal{U}}$  is a Lindenstrauss space, then it is universally 1-separably injective.*

### 5. Two characterizations of universally separably injective spaces

In Proposition 3.5 (b) it was proved that universally separably injective spaces contain  $\ell_\infty$ . Much more is indeed true.

**Definition 5.1.** We say that a Banach space  $X$  is  $\ell_\infty$ -upper-saturated if every separable subspace of  $X$  is contained in some (isomorphic) copy of  $\ell_\infty$  inside  $X$ .

It is easy to see that if  $X$  is  $\ell_\infty$ -upper-saturated, then there is  $\lambda \geq 1$  so that every separable subspace of  $X$  is contained in a  $\lambda$ -isomorphic copy of  $\ell_\infty$  inside  $X$ : if  $X_n$  is a separable subspace of  $X$  that can only be embedded into  $n$ -isomorphic copies of  $\ell_\infty$  then  $\bigcup X_n$  spans a separable subspace of  $X$  that cannot be embedded into any copy of  $\ell_\infty$ .

**Theorem 5.2.** *An infinite-dimensional Banach space is universally separably injective if and only if it is  $\ell_\infty$ -upper-saturated.*

**Proof.** The sufficiency is a consequence of the injectivity of  $\ell_\infty$ . In order to show the necessity, let  $Y$  be a separable subspace of a universally separably injective space  $X$ . We consider a subspace  $Y_0$  of  $\ell_\infty$  isomorphic to  $Y$  and an isomorphism  $t : Y_0 \rightarrow Y$ . We can find projections  $p$  on  $X$  and  $q$  on  $\ell_\infty$  such that  $Y \subset \ker p$ ,  $Y_0 \subset \ker q$ , and both  $p$  and  $q$  have range isomorphic to  $\ell_\infty$ . Indeed, let  $\pi : X \rightarrow X/Y$  be the quotient map. Since  $X$  contains  $\ell_\infty$  and  $Y$  is separable,  $\pi$  is not weakly compact so, by Proposition 3.5(b), there exists a subspace  $M$  of  $X$  isomorphic to  $\ell_\infty$  where  $\pi$  is an isomorphism. Now  $X/Y = \pi(M) \oplus N$ , with  $N$  a closed subspace. Hence  $X = M \oplus \pi^{-1}(N)$ , and it is enough to take  $p$  as the projection with range  $M$  and kernel  $\pi^{-1}(N)$ . Since  $\ker p$  and  $\ker q$  are universally separably injective spaces, we can take operators  $u : X \rightarrow \ker q$  and  $v : \ell_\infty \rightarrow \ker p$  such that  $v = t$  on  $Y_0$  and  $u = t^{-1}$  on  $Y$ .

Let  $w : \ell_\infty \rightarrow \text{ran } p$  be an operator satisfying  $\|w(x)\| \geq \|x\|$  for all  $x \in \ell_\infty$ . We will show that the operator

$$T = v + w(\mathbf{1}_{\ell_\infty} - uv) : \ell_\infty \longrightarrow X$$

is an isomorphism (into). This suffices to end the proof since  $\text{ran } T$  is isomorphic to  $\ell_\infty$  and both  $T$  and  $v$  agree with  $t$  on  $Y_0$ , so  $Y \subset \text{ran } T \subset X$ . Since  $\text{ran } v \subset \ker p$  and  $\text{ran } w \subset \text{ran } p$ , there exists  $C > 0$  such that for all  $x \in \ell_\infty$  one has

$$\|Tx\| \geq C \max\{\|v(x)\|, \|w(\mathbf{1}_{\ell_\infty} - uv)x\|\}.$$

Now, if  $\|vx\| < (2\|u\|)^{-1}\|x\|$ , then  $\|uvx\| < \frac{1}{2}\|x\|$ ; hence

$$\|w(\mathbf{1}_{\ell_\infty} - uv)x\| \geq \|(\mathbf{1}_{\ell_\infty} - uv)x\| > \frac{1}{2}\|x\|.$$

Thus  $\|Tx\| \geq C(2\|u\|)^{-1}\|x\|$  for every  $x \in X$ .  $\square$

We can now complete the proof of Proposition 3.7(3) and show that the class of universally separably injective spaces has the 3-space property.

**Proposition 5.3.** *The class of universally separably injective spaces has the 3-space property.*

**Proof.** By Theorem 5.2 one has to show that being  $\ell_\infty$ -upper-saturated is a 3-space property. Let  $0 \rightarrow Y \rightarrow X \xrightarrow{q} Z \rightarrow 0$  be an exact sequence in which both  $Y, Z$  are  $\ell_\infty$ -upper-saturated, and let  $S$  be a separable subspace of  $X$ . It is not hard to find separable subspaces  $S_0, S_1$  of  $X$  such that  $S \subset S_1$  and  $S_1/S_0 = [q(S)]$ . Let  $Y_\infty$  be a copy of  $\ell_\infty$  inside  $Y$  containing  $S_0$ . By the injectivity of  $\ell_\infty$ ,  $S$  is contained in the subspace  $Y_\infty \oplus [q(S)]$  of  $X$ . And since there exists a copy  $Z_\infty$  of  $\ell_\infty$  containing  $[q(S)]$ ,  $S$  is therefore contained in the subspace  $Y_\infty \oplus Z_\infty$  of  $X$ , which is isomorphic to  $\ell_\infty$ .  $\square$

A homological characterization of universally separably injective spaces is also possible. We need first to show the following.

**Proposition 5.4.** *If  $U$  is a universally separably injective space then  $\text{Ext}(\ell_\infty, U) = 0$ .*

**Proof.** Partington’s distortion theorem [49, Theorem 3] asserts that a Banach space containing  $\ell_\infty$  contains an almost isometric copy of  $\ell_\infty$  (see also Dowling [31]). This last copy will therefore be, say, 2-complemented. Let  $\Gamma$  denote the set of all the 2-isomorphic copies of  $\ell_\infty$  inside  $\ell_\infty$ . For each  $E \in \Gamma$  let  $\iota_E : E \rightarrow \ell_\infty$  be the canonical embedding,  $p_E$  a projection onto  $E$  of norm



at most 2 and  $u_E : E \rightarrow \ell_\infty$  an isomorphism, with  $\|u_E\| \|u_E^{-1}\| \leq 2$ . Assume that a nontrivial exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow \ell_\infty \longrightarrow 0$$

exists. We consider, for each  $E \in \Gamma$ , a copy of the preceding sequence, and form the product of all these copies  $0 \longrightarrow \ell_\infty(\Gamma, U) \longrightarrow \ell_\infty(\Gamma, X) \longrightarrow \ell_\infty(\Gamma, \ell_\infty) \longrightarrow 0$ . Let us consider the embedding  $J : \ell_\infty \rightarrow \ell_\infty(\Gamma, \ell_\infty)$  defined as  $J(x)(E) = u_E p_E(x)$  and then form the pull-back sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \ell_\infty(\Gamma, X) & \longrightarrow & \ell_\infty(\Gamma, \ell_\infty) & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow J & & \\ 0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \text{PB} & \xrightarrow{q} & \ell_\infty & \longrightarrow & 0. \end{array}$$

Let us show that  $q$  cannot be an isomorphism on a copy of  $\ell_\infty$ . Otherwise, it would be an isomorphism on some  $E \in \Gamma$  and thus the new pull-back sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \text{PB} & \xrightarrow{q} & \ell_\infty & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow u_E & & \\ 0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \text{PB}_E & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

would split. And therefore the same would be true making push-out with the canonical projection  $\pi_E : \ell_\infty(\Gamma, U) \rightarrow U$  onto the  $E$ -th copy of  $U$ :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \text{PB}_E & \longrightarrow & E & \longrightarrow & 0 \\ & & \pi_E \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & U & \longrightarrow & \text{PO}_E & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

But it is not hard to see that new pull-back with  $u_E^{-1}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & \text{PO}_E & \longrightarrow & E & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow u_E^{-1} & & \\ 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \ell_\infty & \longrightarrow & 0 \end{array}$$

produces exactly the starting sequence which, by assumption, was nontrivial.

However, the space PB should be universally separably injective by Proposition 3.7(3); hence it must have Rosenthal’s property (V), by Proposition 3.5(b), and therefore  $q$  should be an isomorphism on some copy of  $\ell_\infty$ . This contradiction shows that the starting nontrivial sequence cannot exist.  $\square$

We are thus ready to establish our second characterization of universal separable injectivity. Please note that every separable Banach space embeds into  $\ell_\infty$  in exactly one way up to automorphisms of  $\ell_\infty$ , by the Lindenstrauss–Rosenthal theorem [44, Theorem 2.f.12].

**Theorem 5.5.** *A Banach space  $U$  is universally separably injective if and only if for every separable Banach space  $S$  one has  $\text{Ext}(\ell_\infty/S, U) = 0$ .*

**Proof.** Let  $S$  be a separable subspace of  $\ell_\infty$  and let  $U$  be universally separably injective. The homology sequence obtained by applying  $\mathfrak{L}(-, U)$  to the sequence  $0 \rightarrow S \rightarrow \ell_\infty \rightarrow \ell_\infty/S \rightarrow 0$  yields an exact sequence

$$\dots \rightarrow \mathfrak{L}(\ell_\infty, U) \rightarrow \mathfrak{L}(S, U) \rightarrow \text{Ext}(\ell_\infty/S, U) \rightarrow \text{Ext}(\ell_\infty, U).$$

Since  $\text{Ext}(\ell_\infty, U) = 0$ , one obtains that every exact sequence  $0 \rightarrow U \rightarrow X \rightarrow \ell_\infty/S \rightarrow 0$  fits in a push-out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \ell_\infty/S \longrightarrow 0. \end{array}$$

Since  $U$  is universally separably injective, the lower sequence splits.

The converse is clear: every operator  $t : S \rightarrow U$  from a separable Banach space into a space  $U$  produces a push-out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S \longrightarrow 0 \\ & & t \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & U & \longrightarrow & \text{PO} & \longrightarrow & \ell_\infty/S \longrightarrow 0. \end{array}$$

The lower sequence splits by the assumption  $\text{Ext}(\ell_\infty/S, U) = 0$  and so  $t$  extends to  $\ell_\infty$ , according to the splitting criterion for push-out sequences.  $\square$

This leads to the unexpected proposition given below.

**Proposition 5.6.** *Every short exact sequence  $0 \rightarrow \ell_\infty/c_0 \rightarrow X \rightarrow \ell_\infty/c_0 \rightarrow 0$  splits, that is,  $\text{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$ .*

Essentially, only three solutions of the equation  $\text{Ext}(X, X) = 0$  were previously known, namely: the injective spaces (by the very definition),  $c_0$  (by the Sobczyk theorem), and the spaces  $L_1(\mu)$  (by Lindenstrauss’ lifting).

It is not however true that  $\text{Ext}(U, U) = 0$  for all universally separably injective spaces  $U$ : if  $V$  is a universally separably injective non-injective space (such as  $\ell_\infty^c(\Gamma)$  or the ultraproducts appearing in Theorem 4.10 or Corollary 4.11) then every exact sequence  $0 \rightarrow V \rightarrow \ell_\infty(\Gamma) \rightarrow \ell_\infty(\Gamma)/V \rightarrow 0$  is not trivial. By Proposition 3.7  $W = \ell_\infty(\Gamma)/V$  is universally separably injective and, obviously,  $\text{Ext}(W, V) \neq 0$ . The product space  $U = V \times W$  is universally separably injective and  $\text{Ext}(U, U) \neq 0$ .

In rough contrast with Proposition 5.3, one has the following.

**Corollary 5.7.** *Rosenthal’s property (V) is not a 3-space property.*

**Proof.** With the same construction as above, start with a nontrivial exact sequence  $0 \rightarrow \ell_2 \rightarrow E \rightarrow \ell_\infty \rightarrow 0$  (see [20, Section 4.2]) and construct an exact sequence

$$0 \longrightarrow \ell_\infty(\Gamma, \ell_2) \longrightarrow X \xrightarrow{q} \ell_\infty \longrightarrow 0,$$

where  $q$  cannot be an isomorphism on a copy of  $\ell_\infty$ . So  $X$  lacks Rosenthal’s property (V). The space  $\ell_\infty(\Gamma, \ell_2)$  has Rosenthal’s property (V) as a quotient of  $\ell_\infty(\Gamma, \ell_\infty) = \ell_\infty(\Gamma \times \mathbb{N})$ , since the property obviously passes to quotients.  $\square$

### 6. 1-separably injective spaces

Our first result here establishes a major difference between 1-separably injective and general separably injective spaces: 1-separably injective spaces must be Grothendieck (hence they cannot be separable or WCG) while a 2-separably injective space, such as  $c_0$ , can be even separable. The following lemma due to Lindenstrauss [43, p. 221, proof of (i)  $\Rightarrow$  (v)] provides a quite useful technique.

**Lemma 6.1.** *Let  $E$  be a 1-separably injective space and  $Y$  a separable subspace of  $X$ , with  $\text{dens } X = \aleph_1$ . Then every operator  $t : Y \rightarrow E$  can be extended to an operator  $T : X \rightarrow E$  with the same norm.*

This yields the following.

**Proposition 6.2.** *Under CH every 1-separably injective Banach space is universally 1-separably injective and therefore a Grothendieck space.*

**Proof.** Let  $E$  be 1-separably injective,  $X$  an arbitrary Banach space and  $t : Y \rightarrow E$  an operator, where  $Y$  is a separable subspace of  $X$ . Let  $[t(Y)]$  be the closure of the image of  $t$ . This is a separable subspace of  $E$  and so there is an isometric embedding  $u : [t(Y)] \rightarrow \ell_\infty$ . As  $\ell_\infty$  is 1-injective there is an operator  $T : X \rightarrow \ell_\infty$  whose restriction to  $Y$  agrees with  $ut$ . Thus it suffices to extend the inclusion of  $[t(Y)]$  into  $E$  to  $\ell_\infty$ . But, under CH, the density character of  $\ell_\infty$  is  $\aleph_1$  and the preceding Lemma applies. The ‘therefore’ part is now a consequence of Proposition 3.5(b).  $\square$

The “therefore” part survives in ZFC.

**Proposition 6.3.** *Every 1-separably injective space is a Grothendieck and a Lindenstrauss space.*

**Proof.** The proof of Proposition 3.5 yields that 1-separably injective spaces are of type  $\mathcal{L}_{\infty,1+}$ , that is, Lindenstrauss spaces; indeed, during that proof it was shown that the bidual of a 1-separably injective space  $E$  is 1-injective, hence a Lindenstrauss space as well as  $E$  itself. It remains to prove that a 1-separably injective space  $E$  must be Grothendieck. It suffices to show that  $c_0$  is not complemented in  $E$ , so let  $J : c_0 \rightarrow E$  be an embedding. Consider an almost-disjoint family  $\mathcal{M}$  of size  $\aleph_1$  formed by infinite subsets of  $\mathbb{N}$  and construct the associated Johnson–Lindenstrauss twisted sum space

$$0 \longrightarrow c_0 \longrightarrow JL_\infty(\mathcal{M}) \longrightarrow c_0(\aleph_1) \longrightarrow 0.$$

The space  $JL_\infty(\mathcal{M})$  has density character  $\aleph_1$  and, by Lemma 6.1, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & JL_\infty(\mathcal{M}) & \longrightarrow & c_0(\aleph_1) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & c_0 & \xrightarrow{J} & E & \longrightarrow & E/J(c_0) & \longrightarrow & 0. \end{array}$$

If  $c_0$  was complemented in  $E$  then it would be complemented in  $JL_\infty(\mathcal{M})$  as well, which is not.  $\square$

Proposition 6.2 leads to the question about the necessity of the continuum hypothesis. We will prove now that it cannot be dropped.

6.1. A 1-separably injective but not 1-universally separably injective  $C(K)$

**Lemma 6.4.** *Let  $K, L, M$  be compact spaces and let  $f : K \rightarrow M$  be a continuous map, with  $J = f^\circ : C(M) \rightarrow C(K)$  its induced operator, and let  $\iota : C(M) \rightarrow C(L)$  be a positive norm one operator. Suppose that  $S : C(L) \rightarrow C(K)$  is an operator with  $\|S\| = 1$  and  $S\iota = J$ . Then  $S$  is a positive operator.*

**Proof.** Obviously  $S \geq 0$  if and only if  $S^*\delta_x \geq 0$  for all  $x \in K$ , where  $\delta_x$  is the unit mass at  $x$  and  $S^* : C(K)^* \rightarrow C(L)^*$  is the adjoint operator. Fix  $x \in K$ . By the Riesz theorem we have that  $S^*\delta_x = \mu$  is a measure of total variation  $\|\mu\| \leq 1$ . Let  $\mu = \mu^+ - \mu^-$  be the Hahn–Jordan decomposition of  $\mu$ , so that  $\|\mu\| = \|\mu^+\| + \|\mu^-\|$ , with  $\mu^+, \mu^- \geq 0$ . We have that  $\delta_{f(x)} = J^*\delta_x = \iota^* S^*\delta_x = \iota^* \mu$ ; thus

$$\delta_{f(x)} = \iota^* \mu^+ - \iota^* \mu^- \quad \text{and} \quad \|\delta_{f(x)}\| = \|\iota^* \mu^+\| + \|\iota^* \mu^-\|.$$

Since  $\iota$  is a positive operator these imply that the above is the Hahn–Jordan decomposition of  $\delta_{f(x)}$  and so  $\iota^* \mu^- = 0$ , hence  $\mu^- = 0$ .  $\square$

**Definition 6.5.** Let  $L$  be a zero-dimensional compact space. An  $\aleph_2$ -Lusin family on  $L$  is a family  $\mathcal{F}$  of pairwise disjoint nonempty clopen subsets of  $L$  with  $|\mathcal{F}| = \aleph_2$ , such that whenever  $\mathcal{G}$  and  $\mathcal{H}$  are subfamilies of  $\mathcal{F}$  with  $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$ , then

$$\overline{\bigcup\{G \in \mathcal{G}\}} \cap \overline{\bigcup\{G \in \mathcal{H}\}} \neq \emptyset.$$

The following lemma shows the consistency of the existence of an  $\aleph_2$ -Lusin family on  $\mathbb{N}^*$ . This is rather folklore of set-theory, but we did not find a reference so we state it and give a hint of the proof.

**Lemma 6.6.** *Under MA and the assumption  $\mathfrak{c} = \aleph_2$  there exists an  $\aleph_2$ -Lusin family on  $\mathbb{N}^*$ .*

**Proof.** By Stone duality, since the Boolean algebra associated to  $\mathbb{N}^*$  is  $\wp(\mathbb{N})/\text{fin}$ , an  $\aleph_2$ -Lusin family on  $\mathbb{N}^*$  is all the same as an almost disjoint family  $\{A_\alpha\}_{\alpha < \omega_2}$  of infinite subsets of  $\mathbb{N}$  such that for every  $B \subset \mathbb{N}$  either  $\{\alpha : |A_\alpha \setminus B| \text{ is finite}\}$  or  $\{\alpha : |A_\alpha \cap B| \text{ is finite}\}$  has cardinality  $< \aleph_2$ . Let  $\{B_\alpha : \alpha < \omega_2\}$  be an enumeration of all infinite subsets of  $\mathbb{N}$ . We construct the sets  $A_\alpha$  inductively on  $\alpha$ . Suppose  $A_\gamma$  has been constructed for  $\gamma < \alpha$ . We define a forcing notion  $\mathbb{P}$  whose conditions are pairs  $p = (f_p, F_p)$  where  $f_p$  is a  $\{0, 1\}$ -valued function on a finite subset  $\text{dom}(f_p)$  of  $\mathbb{N}$  and  $F_p$  is a finite subset of  $\alpha$ . The order relation is that  $p < q$  if  $f_p$  extends  $f_q$ ,  $F_p \supset F_q$  and  $f_p$  vanishes in  $A_\gamma \setminus \text{dom}(f_q)$  for  $\gamma \in F_q$ . One checks that this forcing is ccc. Hence, by MA, using a big enough generic filter the forcing provides an infinite set  $A_\alpha \subset \mathbb{N}$  such that, for all  $\gamma < \alpha$ ,

- (1)  $A_\alpha \cap A_\gamma$  is finite, and
- (2) if  $B_\gamma$  is not contained in any finite union of  $A_\delta$ 's, then  $A_\alpha \cap B_\gamma$  is infinite.  $\square$

**Theorem 6.7.** *It is consistent that there exists a compact space  $K$  for which the Banach space  $C(K)$  is 1-separably injective but not universally 1-separably injective.*

**Proof.** We will suppose that  $\mathfrak{c} = \aleph_2$  and that there exists an  $\aleph_2$ -Lusin family in  $\mathbb{N}^*$ . Under these hypotheses, let  $K$  be the Stone dual compact space of the Cohen–Parovičenko Boolean algebra of [30, Theorem 5.6]. The definition of that Boolean algebra implies that  $K$  is an  $F$ -space and thus

$C(K)$  is 1-separably injective by Proposition 4.2. We show that it is not universally 1-separably injective. The argument follows the scheme of [30, Theorem 5.10], where they prove that  $K$  does not map onto  $\beta\mathbb{N}$ , but we use  $\aleph_2$ -Lusin families instead of  $\omega_2$ -chains because they fit better in the functional analytic context.

Let  $\{U_n : n \in \mathbb{N}\}$  be a sequence of pairwise disjoint clopen subsets of  $K$ , and let  $U = \bigcup_n U_n$ . Let  $c \subset \ell_\infty$  be the Banach space of convergent sequences, and  $t : c \rightarrow C(K)$  be the operator given by  $t(z)(x) = z_n$  if  $x \in U_n$  and  $t(z)(x) = \lim z_n$  if  $x \notin U$ .

If  $C(K)$  were universally 1-separably injective, we should have an extension  $T : \ell_\infty \rightarrow C(K)$  of  $t$  with  $\|T\| = 1$ . We shall derive a contradiction from the existence of such operator.

Notice that the conditions of Lemma 6.4 are applied, so  $T$  is positive (observe that  $c = C(\mathbb{N} \cup \{\infty\})$  and  $T = f \circ$  where  $f : K \rightarrow \mathbb{N} \cup \{\infty\}$  is given by  $f(x) = n$  if  $x \in U_n$  and  $f(x) = \infty$  if  $x \notin U$ ).

For every  $A \subset \mathbb{N}$  we will denote  $[A] = \overline{A}^{\beta\mathbb{N}} \setminus \mathbb{N}$ . The clopen subsets of  $\mathbb{N}^*$  are exactly the sets of the form  $[A]$ , and we have that  $[A] = [B]$  if and only if  $(A \setminus B) \cup (B \setminus A)$  is finite.

Let  $\mathcal{F}$  be an  $\aleph_2$ -Lusin family in  $\mathbb{N}^*$ . For  $F = [A] \in \mathcal{F}$  and  $0 < \varepsilon < \frac{1}{2}$ , let

$$F_\varepsilon = \{x \in K \setminus U : T(1_A)(x) > 1 - \varepsilon\}.$$

Let us remark that  $F_\varepsilon$  depends only on  $F$  and not on the choice of  $A$ . This is because if  $[A] = [B]$ , then  $1_A - 1_B \in c_0$ ; hence  $T(1_A - 1_B) = t(1_A - 1_B)$  which vanishes out of  $U$ , so  $T(1_A)|_{K \setminus U} = T(1_B)|_{K \setminus U}$ .

**Claim 1.** *If  $\delta < \varepsilon$  and  $F \in \mathcal{F}$ , then  $\overline{F_\delta} \subset F_\varepsilon$ .*

**Claim 2.**  *$F_\varepsilon \cap G_\varepsilon = \emptyset$  for every  $F \neq G$ .*

**Proof of Claim 2.** Since  $F \cap G = \emptyset$  we can choose  $A, B \subset \mathbb{N}$  such that  $F = [A], G = [B]$  and  $A \cap B = \emptyset$ . If  $x \in F_\varepsilon \cap G_\varepsilon, \tilde{T}(1_A + 1_B)(x) > 2 - 2\varepsilon > 1$  which is a contradiction because  $1_A + 1_B = 1_{A \cup B}$  and  $\|\tilde{T}(1_{A \cup B})\| \leq \|\tilde{T}\| \|1_{A \cup B}\| = 1$ .  $\square$

For every  $F \in \mathcal{F}$ , let  $\tilde{F}$  be a clopen subset of  $K \setminus U$  such that  $\overline{F_{0.2}} \subset \tilde{F} \subset F_{0.3}$ . By the preceding claims, this is a disjoint family of clopen sets. It follows from Proposition 2.6 and Corollary 5.12 in [30] that  $K \setminus U$  does not contain any  $\aleph_2$ -Lusin family. Therefore we can find  $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$  with  $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$  such that

$$\overline{\bigcup\{\tilde{G} : G \in \mathcal{G}\}} \cap \overline{\bigcup\{\tilde{H} : H \in \mathcal{H}\}} = \emptyset.$$

Now, for every  $n \in \mathbb{N}$  choose a point  $p_n \in U_n$ . Let  $g : \beta\mathbb{N} \rightarrow K$  be a continuous function such that  $g(n) = p_n$ .

**Claim 3.** *For  $u \in \beta\mathbb{N}, A \subset \mathbb{N}, T(1_A)(g(u)) = \begin{cases} 1, & \text{if } u \in [A]; \\ 0, & \text{if } u \notin [A]. \end{cases}$*

**Proof of Claim 3.** It is enough to check it for  $u = n \in \mathbb{N}$ . This is a consequence of the fact that  $T$  is positive, because if  $m \in A, n \notin A$ , then  $0 \leq t(1_m) \leq T(1_A) \leq t(1_{\mathbb{N} \setminus \{n\}}) \leq 1$ .  $\square$

The function  $g$  is one-to-one because

$$\overline{\{p_n : n \in A\}} \cap \overline{\{p_n : n \notin A\}} = \emptyset$$

for every  $A \subset \mathbb{N}$ , as the function  $T(1_A)$  separates these sets. On the other hand, as a consequence of Claim 3 above, for every  $F \in \mathcal{F}$  and every  $\varepsilon, g^{-1}(F_\varepsilon) = F$ , and also  $g^{-1}(\tilde{F}) = F$ . These facts make the families  $\mathcal{H}$  and  $\mathcal{G}$  above to contradict that  $\mathcal{F}$  is an  $\aleph_2$ -Lusin family in  $\mathbb{N}^*$ .  $\square$

### 7. Concluding remarks

#### 7.1. An application to duality

Now, we give an application which uses both ultraproducts and  $M$ -ideals. As a preparation, let us check the following.

**Lemma 7.1.** *Let  $(E_i)_{i \in I}$  be a family of Banach spaces and  $\mathcal{U}$  an ultrafilter on  $I$ . Then  $c_0^{\mathcal{U}}(E_i)$  is an  $M$ -ideal in  $\ell_\infty(E_i)$ .*

**Proof.** It is very difficult to manage the dual of  $\ell_\infty(E_i)$  and so we need a different approach avoiding duality. It is proved in [33, Theorem 2.2] that  $J$  is an  $M$ -ideal in  $X$  if and only if it satisfies the following condition: given a finite family of closed balls  $B(x^k, r_k)$  in  $X$  such that  $B(x^k, r_k) \cap J \neq \emptyset$  for all  $k$  and

$$\bigcap_k B(x^k, r_k) \neq \emptyset,$$

one has

$$\bigcap_k B(x^k, r_k + \varepsilon) \cap J \neq \emptyset$$

for each  $\varepsilon > 0$ .

Let us check this condition for  $c_0^{\mathcal{U}}(E_i)$ . Let  $B(x^k, r_k)$  be the corresponding balls and take  $x = (x_i)$  in their intersection. Also, for each  $k$ , pick  $y^k \in B(x^k, r_k) \cap c_0^{\mathcal{U}}(E_i)$ .

Now, given  $\varepsilon > 0$ , as  $\|y_i^k\| \rightarrow 0$  along  $\mathcal{U}$  we may find  $I_\varepsilon$  in  $\mathcal{U}$  such that  $\|y_i^k\| \leq \varepsilon$  for all  $k$  and all  $i \in I_\varepsilon$ . We define  $y = (y_i)$  taking

$$y_i = \begin{cases} 0 & \text{for } i \in I_\varepsilon \\ x_i & \text{otherwise.} \end{cases}$$

It is clear that  $y \in \bigcap_k B(x^k, r_k + \varepsilon) \cap c_0^{\mathcal{U}}(E_i)$ . □

A Banach space  $E$  has the Uniform Approximation Property (UAP) if and only if all its ultrapowers have the BAP. This is not the genuine definition, but an equivalent condition (see [23, Section 7]). One has the following.

**Proposition 7.2.** *Let  $X$  be a separable Banach space and let  $E$  be a Banach space. Suppose either  $X$  has the BAP or  $E$  has the UAP. If  $\text{Ext}(X, E) = 0$  then  $\text{Ext}(X, E_{\mathcal{U}}) = 0$  for all ultrafilters  $\mathcal{U}$ .*

**Proof.** We write the proof in the case where  $X$  has the BAP and leave the case where  $E$  has the UAP to the reader. Let  $\pi : \ell_1 \rightarrow X$  be any quotient map and set  $K = \ker \pi$ . It is well-known that, given a Banach space  $Y$ , the condition  $\text{Ext}(X, Y) = 0$  is equivalent to: every operator  $v : K \rightarrow Y$  has an extension  $\tilde{v} : \ell_1 \rightarrow Y$ . If so, this can be done with  $\|\tilde{v}\| \leq C\|v\|$ , by the open mapping theorem.

So, let  $u : K \rightarrow E_{\mathcal{U}}$  be an operator. We know from Lusky [45] that  $K$  has the BAP when  $X$  has the BAP. In view of Lemma 7.1, Theorem 4.5 (1) applies and so  $u$  lifts to an operator  $\tilde{u} : K \rightarrow \ell_\infty(I, E)$  we may write  $\tilde{u} = (u_i)$ , with  $u_i \in L(K, E)$ . As  $\text{Ext}(X, E)$  vanishes we can extend each  $u_i$  to an operator  $v_i : \ell_1 \rightarrow E$ , with  $\|v_i\| \leq C\|u_i\| \leq C\|\tilde{u}\|$ . This gives an operator  $\ell_1 \rightarrow \ell_\infty(I, E)$  and composing with the quotient map defining  $E_{\mathcal{U}}$  we get the required extension  $v : \ell_1 \rightarrow E_{\mathcal{U}}$ . □

It would be interesting to know if the BAP (or the UAP) is truly necessary in Proposition 7.2. The separability of  $X$  is necessary since ultraproducts via countably incomplete ultrafilters are never injective (see [35, Theorem 2.6] or [55, Section 8]; but also [8]). Thus, there must be some Banach space  $X$  for which  $\text{Ext}(X, (\ell_\infty)_{\mathcal{U}}) \neq 0$ , while one obviously has  $\text{Ext}(X, \ell_\infty) = 0$ . An unexpected consequence of Proposition 7.2 is the following.

**Corollary 7.3.** *Let  $X$  be a separable Banach space and  $Y$  a Banach space. Suppose either  $X$  has the BAP or  $Y$  has the UAP. Then  $\text{Ext}(X, Y) = 0$  implies  $\text{Ext}(Y^*, X^*) = 0$ .*

**Proof.** Since  $\text{Ext}(X, Y^*) = \text{Ext}(Y, X^*)$  then  $\text{Ext}(Y^*, X^*) = \text{Ext}(X, Y^{**})$  for all Banach spaces  $X$  and  $Y$ . Now, if  $\text{Ext}(X, Y) = 0$  we know from Proposition 7.2 that  $\text{Ext}(X, Y_{\mathcal{U}}) = 0$  for all ultrafilters  $\mathcal{U}$ . Since  $Y^{**}$  is a complemented subspace of a suitable ultrapower of  $Y$ , the result follows.  $\square$

## 7.2. Open problems

There are many questions that arise naturally from the content of this paper and we have been unable to resolve.

It would be interesting to characterize (universally) separably injective  $C(K)$  in terms of topological properties of  $K$ . As for concrete examples we do not know if separable injectivity passes to injective tensor products and, in fact, we do not know whether  $\ell_\infty \check{\otimes}_\varepsilon \ell_\infty = C(\beta\mathbb{N} \times \beta\mathbb{N})$  is separably injective or not.

We do not know if 1-separably injective spaces  $X$  must be universally separably injective in ZFC even if  $X = C(K)$ , with  $K$  an  $F$ -space. See Proposition 6.2 and Theorem 6.7.

It is an open question to decide if  $\text{Ext}(X, Y) = 0$  implies  $\text{Ext}(Y^*, X^*) = 0$  for all Banach spaces  $X$  and  $Y$ . The main difficulty for a direct proof is that, as it was shown in [19, Theorem 5], there are elements in  $\text{Ext}(Y^*, X^*)$  which are not duals of elements of  $\text{Ext}(X, Y)$ .

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## References

- [1] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, in: Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
- [2] D. Amir, Projections onto continuous function spaces, Proc. Amer. Math. Soc. 15 (1964) 396–402.
- [3] T. Ando, Closed range theorems for convex sets and linear liftings, Pacific J. Math. 44 (1973) 393–410.
- [4] S.A. Argyros, On the space of bounded measurable functions, Quart. J. Math. Oxford Ser. (2) 34 (1983) 129–132.
- [5] S.A. Argyros, J.M.F. Castillo, A.S. Granero, M. Jiménez, J.P. Moreno, Complementation and embeddings of  $c_0(I)$  in Banach spaces, Proc. Lond. Math. Soc. 85 (2002) 742–772.
- [6] S.A. Argyros, R.G. Haydon, A hereditarily indecomposable  $\mathcal{L}_\infty$ -space that solves the scalar-plus-compact problem, Acta Math. 206 (1) (2011) 1–54.
- [7] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno, Banach spaces of universal disposition, J. Funct. Anal. 261 (2011) 2347–2361.

- [8] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno, On ultraproducts of  $\mathcal{L}_\infty$ -spaces, 2012, Preprint.
- [9] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno,  $\aleph$ -injective Banach spaces and  $\aleph$ -projective compacta, 2012, Preprint.
- [10] J.W. Baker, Projection constants for  $C(S)$  spaces with the separable projection property., *Proc. Amer. Math. Soc.* 41 (1973) 201–204.
- [11] Y. Benyamini, An  $M$ -space which is not isomorphic to a  $C(K)$ -space, *Israel J. Math.* 28 (1977) 98–104.
- [12] J. Bourgain, A counterexample to a complementation problem, *Compo. Math.* 43 (1981) 133–144.
- [13] J. Bourgain, New classes of  $\mathcal{L}^p$ -spaces, in: *Lecture Notes in Math.*, vol. 889, Springer-Verlag, 1981.
- [14] J. Bourgain, F. Delbaen, A class of special  $\mathcal{L}_\infty$  spaces, *Acta Math.* 145 (1980) 155–176.
- [15] J. Bourgain, G. Pisier, A construction of  $\mathcal{L}_\infty$ -spaces and related Banach spaces, *Bol. Soc. Bras. Mat.* 14 (1983) 109–123.
- [16] F. Cabello Sánchez, Yet another proof of Sobczyk’s theorem, in: *Methods in Banach Space Theory*, in: *London Math. Soc. Lecture Notes*, vol. 337, Cambridge Univ. Press, 2006, pp. 133–138.
- [17] F. Cabello Sánchez, Transitivity of  $M$ -spaces and Wood’s conjecture, *Math. Proc. Cambridge Philos. Soc.* 124 (1998) 513–520.
- [18] F. Cabello Sánchez, J.M.F. Castillo, The long homology sequence for quasi-Banach spaces, with applications, *Positivity* 8 (2004) 379–394.
- [19] F. Cabello Sánchez, J.M.F. Castillo, Duality and twisted sums of Banach spaces, *J. Funct. Anal.* 175 (2000) 1–16.
- [20] F. Cabello Sánchez, J.M.F. Castillo, Uniform boundedness and twisted sums of Banach spaces, *Houston J. Math.* 30 (2004) 523–536.
- [21] F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton, D.T. Yost, Twisted sums with  $C(K)$ -spaces, *Trans. Amer. Math. Soc.* 355 (2003) 4523–4541.
- [22] F. Cabello Sánchez, J.M.F. Castillo, D.T. Yost, Sobczyk’s theorems from A to B, *Extracta Math.* 15 (2000) 391–420.
- [23] P.G. Casazza, Approximation properties, in: *Handbook of the Geometry of Banach Spaces*, vol. I, North-Holland, Amsterdam, 2001, pp. 271–316.
- [24] J.M.F. Castillo, M. González, Three-space problems in Banach space theory, in: *Lecture Notes in Math.*, vol. 1667, Springer-Verlag, 1997.
- [25] J.M.F. Castillo, Y. Moreno, Sobczyk’s theorem and the bounded approximation property, *Studia Math.* 201 (2010) 1–19.
- [26] J.M.F. Castillo, Y. Moreno, J. Suárez, On Lindenstrauss–Pelczyński spaces, *Studia Math.* 174 (2006) 213–231.
- [27] J.M.F. Castillo, A. Plichko, Banach spaces in various positions, *J. Funct. Anal.* 259 (2010) 2098–2138.
- [28] M.-D. Choi, E.G. Effros, Lifting problems and the cohomology of  $C^*$ -algebras, *Canadian J. Math.* 29 (1977) 1092–1111.
- [29] R. Deville, G. Godefroy, V. Zizler, Smoothness and renormings in Banach spaces, in: *Pitman Monographs and Surveys in Pure and Applied Mathematics*, vol. 64, 1993.
- [30] A. Dow, K.P. Hart, Applications of another characterization of  $\beta\mathbb{N} \setminus \mathbb{N}$ , *Topology Appl.* 122 (2002) 105–133.
- [31] P.N. Dowling, On  $\ell_\infty$ -subspaces of Banach spaces, *Collect. Math.* 51 (3) (2000) 255–260.
- [32] I. Gasparis, A new isomorphic  $\ell_1$  predual not isomorphic to a complemented subspace of a  $C(K)$  space, [arXiv:1205.4317](https://arxiv.org/abs/1205.4317).
- [33] P. Harmand, D. Werner, W. Werner,  $M$ -ideals in Banach spaces and Banach algebras, in: *Lecture Notes in Math.*, vol. 1547, Springer-Verlag, 1993.
- [34] S. Heinrich, Ultraproducts in Banach space theory, *J. Reine Angew. Math.* 313 (1980) 72–104.
- [35] C.W. Henson, L.C. Moore, Nonstandard hulls of the classical Banach spaces, *Duke Math. J.* 41 (1974) 277–284.
- [36] R.C. James, Uniformly nonsquare Banach spaces, *Ann. of Math.* 80 (1964) 542–550.
- [37] W.B. Johnson, J. Lindenstrauss, Some remarks on weakly compactly generated Banach spaces, *Israel J. Math.* 17 (1974) 219–230.
- [38] W.B. Johnson, T. Oikhberg, Separable lifting property and extensions of local reflexivity, *Illinois J. Math.* 45 (2001) 123–137.
- [39] W.B. Johnson, M. Zippin, Separable  $L_1$  preduals are quotients of  $C(\Delta)$ , *Israel J. Math.* 16 (1973) 198–202.
- [40] P. Koszmider, On decomposition of Banach spaces of continuous functions on Mrówka’s spaces, *Proc. Amer. Math. Soc.* 133 (2005) 2137–2146.
- [41] K. Kuratowski, A. Mostowski, *Set Theory*, P. W. N., Warszawa, 1968.
- [42] J. Lindenstrauss, On the extension of compact operators, *Mem. Amer. Math. Soc.* 48 (1964).
- [43] J. Lindenstrauss, On the extension of operators with range in a  $C(K)$  space, *Proc. Amer. Math. Soc.* 15 (1964) 218–225.



- [44] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces Vol. I*, Springer-Verlag, 1977.
- [45] W. Lusky, Three-space problems and bounded approximation property, *Studia Math.* 159 (2003) 417–434.
- [46] W. Marciszewski, On Banach spaces  $C(K)$  isomorphic to  $c_0(I)$ , *Studia Math.* 156 (2003) 295–302.
- [47] W. Marciszewski, R. Pol, On Banach spaces whose norm-open sets are  $F_\sigma$  sets in the weak topology, *J. Math. Anal. Appl.* 350 (2009) 708–722.
- [48] M.I. Ostrovskii, Separably injective Banach spaces, *Functional Anal. i. Prilozhen* 20 (1986) 80–81. English transl.: *Functional Anal. Appl.* 20 (1986) 154–155.
- [49] J.R. Partington, Subspaces of certain Banach sequence spaces, *Bull. London Math. Soc.* 13 (1981) 162–166.
- [50] A. Pełczyński, Projections in certain Banach spaces, *Studia Math.* 19 (1960) 209–228.
- [51] H.P. Rosenthal, On relatively disjoint families of measures, with some applications to Banach space theory, *Studia Math.* 37 (1970) 13–36.
- [52] H.P. Rosenthal, The complete separable extension property, *J. Operator Theory* 43 (2000) 329–374.
- [53] W. Schachermayer, On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras, *Dissertationes Math.* 214 (1982).
- [54] G.L. Seever, Measures on F-spaces, *Trans. Amer. Math. Sc.* 133 (1968) 267–280.
- [55] B. Sims, Ultra-techniques in Banach space theory, in: *Queen's Papers in Pure and Applied Mathematics*, vol. 60, Kingston, ON, 1982.
- [56] M. Zippin, The separable extension problem, *Israel J. Math.* 26 (1977) 372–387.
- [57] M. Zippin, Extension of bounded linear operators, in: *Handbook of the Geometry of Banach Spaces*, Vol. II, North-Holland, Amsterdam, 2003, pp. 1703–1741.