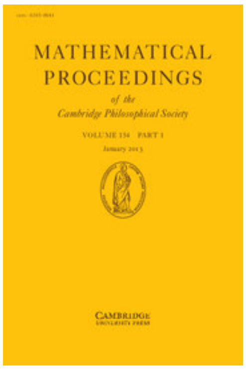

On amenability of the Banach algebras l$_{\text{[infty infinity]}}$(S, [fraktur U])

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Mathematical Proceedings of the Cambridge Philosophical Society / Volume 132 / Issue 02 / March 2002, pp 319 - 322
DOI: 10.1017/S0305004102005443, Published online: 15 March 2002

Math. Proc. Camb. Phil. Soc. (2002), **132**, 319 © Cambridge Philosophical Society
DOI: 10.1017/S0305004102005443 *Printed in the United Kingdom*

On amenability of the Banach algebras $\ell_\infty(S, \mathfrak{A})$

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(*Received* 24 *March* 2000; *revised* 8 *September* 2000)

Let $\mathfrak{A}$ be an associative Banach algebra. Given a set S, we write $\ell_\infty(S, \mathfrak{A})$ for the Banach algebra of all bounded functions $f: S \to \mathfrak{A}$ with the usual norm $\|f\|_\infty = \sup_{s\in S} \|f(s)\|_{\mathfrak{A}}$ and pointwise multiplication. When S is countable, we simply write $\ell_\infty(\mathfrak{A})$.

In this short note, we exhibit examples of amenable (resp. weakly amenable) Banach algebras $\mathfrak{A}$ for which $\ell_\infty(S, \mathfrak{A})$ fails to be amenable (resp. weakly amenable), thus solving a problem raised by Gourdeau in [**7**] and [**8**]. We refer the reader to [**4**, **9**, **10**] for background on amenability and weak amenability. For basic information about the Arens product in the second dual of a Banach algebra the reader can consult [**5**, **6**]. Here we only recall that, given a bilinear operator $B: \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ acting between Banach spaces, there is a bilinear extension (the first Arens extension of B) $B'': \mathfrak{X}'' \times \mathfrak{Y}'' \to \mathfrak{Z}''$ given by

$$B''(x'', y'') = w^* - \lim_x \Big(w^* - \lim_y B(x, y) \Big) \quad (x'' \in \mathfrak{X}'', y'' \in \mathfrak{Y}''),$$

where the iterated limits are taken first for $x \in \mathfrak{X}$ converging to x'' in the weak* topology of $\mathfrak{X}''$ and then for $y \in \mathfrak{Y}$ converging to y'' in the weak* topology of $\mathfrak{Y}''$. In particular, if $\mathfrak{A}$ is a Banach algebra, then the bidual space $\mathfrak{A}''$ is always a Banach algebra under the (first) Arens product

$$a'' \cdot b'' = w^* - \lim_a \Big(w^* - \lim_b a \cdot b \Big) \quad (a'', b'' \in \mathfrak{A}''),$$

where the iterated limits are taken for a and b in $\mathfrak{A}$ converging respectively to a'' and b'' in the weak* topology of $\mathfrak{A}''$.

Our main result reads as follows.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra satisfying one of the following conditions*:

(a) *the product $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is jointly weakly continuous on bounded sets*;

(b) *$\mathfrak{A}$ is a unital C^*-algebra.*

Then there exists a surjective homomorphism from $\ell_\infty(S, \mathfrak{A})$ onto $\mathfrak{A}''$ for some proper choice of the set S. If $\mathfrak{A}'$ is a separable Banach space, then a countable S suffices.

* Supported in part by DGICYT project PB97-0377.
† Supported in part by Junta de Extremadura IPOOA075.

Thus, if $\mathfrak{A}$ is an algebra satisfying either (a) or (b) and such that the bidual algebra $\mathfrak{A}''$ is commutative and fails to be (weakly) amenable, then $\ell_\infty(S, \mathfrak{A})$ cannot be (weakly) amenable, even if $\mathfrak{A}$ is.

The decisive step in the proof of Theorem 1 is the equivalence (a) $\Leftrightarrow$ (c) in Lemma 1, first proved by Aron, Hervés and Valdivia in [**2**]. For the sake of completeness, we include a direct, simpler proof.

LEMMA 1 ([**2**]). *For a bilinear operator* $B\colon \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ *the following conditions are equivalent*:

(a) *B is jointly weakly continuous on bounded sets*;

(b) *given weakly Cauchy bounded nets* $(x_\alpha)_\alpha$ *in* $\mathfrak{X}$ *and* $(y_\alpha)_\alpha$ *in* $\mathfrak{Y}$, *one has* $w-\lim_\alpha B(x_\alpha, y_\alpha) = 0$ *provided either* $(x_\alpha)_\alpha$ *or* $(y_\alpha)_\alpha$ *is weakly null*;

(c) *B is jointly weakly uniformly continuous on bounded sets*;

(d) *B'' is jointly weakly* (uniformly) continuous on bounded sets.*

Proof. To prove that (a) implies (b), it obviously suffices to see that 0 is a weak cluster point of $B(x_\alpha, y_\alpha)$. We may assume and do that $(x_\alpha)_\alpha$ is weakly null. Fix $z' \in \mathfrak{Z}'$ and $\varepsilon > 0$. Then for every α there is $\beta(\alpha) \geqslant \alpha$ so that $|\langle z', B(x_{\beta(\alpha)}, y_\alpha)\rangle| < \varepsilon$. One then has

$$\begin{aligned}|\langle z', B(x_{\beta(\alpha)}, y_{\beta(\alpha)}\rangle| &\leqslant \varepsilon + |\langle z', B(x_{\beta(\alpha)}, y_{\beta(\alpha)}) - B(x_{\beta(\alpha)}, y_\alpha)\rangle| \\ &= \varepsilon + |\langle z', B(x_{\beta(\alpha)}, y_{\beta(\alpha)} - y_\alpha)\rangle|.\end{aligned}$$

Now, since both $(x_{\beta(\alpha)})_\alpha$ and $(y_{\beta(\alpha)} - y_\alpha)_\alpha$ are weakly null bounded nets, we have $w-\lim_\alpha B(x_{\beta(\alpha)}, y_{\beta(\alpha)} - y_\alpha) = 0$, so that $|\langle z', B(x_{\beta(\alpha)}, y_{\beta(\alpha)} - y_\alpha)\rangle| < \varepsilon$ for α large enough. Since $(\beta(\alpha))_\alpha$ is cofinal, the implication is proved.

We now prove that (b) implies (c). We must show that $(B(x_\alpha, y_\alpha))_\alpha$ is weakly Cauchy whenever (x_α) and (y_α) are weakly Cauchy bounded nets. In other words, one has to prove that $w-\lim_{\alpha,\beta}(B(x_\alpha, y_\alpha) - B(x_\beta, y_\beta)) = 0$. (Here, the set of pairs (α, β) is directed taking $(\alpha, \beta) \leqslant (\alpha', \beta')$ if and only if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$.) Since both $(x_\alpha - x_\beta)_{\alpha,\beta}$ and $(y_\alpha - y_\beta)_{\alpha,\beta}$ are weakly null and $(y_\alpha)_{\alpha,\beta}$ and $(x_\beta)_{\alpha,\beta}$ are weakly Cauchy nets, one has

$$w-\lim_{\alpha,\beta}(B(x_\alpha, y_\alpha) - B(x_\beta, y_\beta)) = w-\lim_{\alpha,\beta} B(x_\alpha - x_\beta, y_\alpha) + w-\lim_{\alpha,\beta} B(x_\beta, y_\alpha - y_\beta) = 0,$$

as desired.

Finally, we show that (c) implies (d). Define a bilinear operator $\tilde{B}\colon \mathfrak{X}'' \times \mathfrak{Y}'' \to \mathfrak{Z}''$ taking

$$\tilde{B}(x'', y'') = w^* - \lim_\alpha B(x_\alpha, y_\alpha),$$

where $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ are bounded nets in $\mathfrak{X}$ and $\mathfrak{Y}$ with $x'' = w^* - \lim_\alpha x_\alpha$ and $y'' = w^* - \lim_\alpha y_\alpha$. The definition of $\tilde{B}$ makes sense because the restriction to $\mathfrak{X}$ (resp. to $\mathfrak{Y}$) of the weak* topology of $\mathfrak{X}''$ (resp. of $\mathfrak{Y}''$) is the weak topology of $\mathfrak{X}$ (resp. of $\mathfrak{Y}$). That $\tilde{B}$ is jointly weakly* uniformly continuous on bounded sets of $\mathfrak{X}'' \times \mathfrak{Y}''$ is obvious. We prove that $\tilde{B}$ is the Arens extension of B. That $\tilde{B}$ and B'' coincide on $\mathfrak{X} \times \mathfrak{Y}''$ is clear. That they are the same now follows from the fact that both $\tilde{B}$ and B'' are separately weakly* continuous in the first argument. Thus, as (d) obviously implies (a), the whole proof is complete.

COROLLARY 1. *The product of a Banach algebra is jointly weakly continuous on* *bounded sets if and only if the Arens product in the bidual algebra is jointly weakly* continuous on bounded sets.*

This provides the 'intrinsic condition' missing in [**5**, theorem 2] and solves the second problem of Duncan and Hosseinium's list [**5**, section 6]. We pass to the proof of Theorem 1.

Proof of Theorem 1. We give the proof in case (a). Let D be a dense subset of the unit ball of $\mathfrak{A}'$ and let S be the net of all finite subsets of D ordered by inclusion. Let U be an ultrafilter on S refining the Fréchet filter. Define $\Psi\colon \ell_\infty(S,\mathfrak{A}) \to \mathfrak{A}''$ by $\Psi(f) = w^* - \lim_{U(s)} f(s)$. This definition makes sense because of the weak* compactness of balls in $\mathfrak{A}''$. Clearly, Ψ is linear and bounded, with $\|\Psi\| = 1$.

We show that Ψ is surjective. Take $a'' \in \mathfrak{A}''$. Then, for every $s \in S$ there is $f(s) \in \mathfrak{A}$ such that $\langle a', f(s)\rangle = \langle a'', a'\rangle$ for each $a' \in s$, with $\|f(s)\|_{\mathfrak{A}} \leqslant \|a''\|_{\mathfrak{A}''} + \varepsilon$. It follows that $(f(s))_s$ converges weakly* to a'' with respect to the Fréchet filter on S and, therefore, $\Psi(f) = a''$.

It remains to prove that Ψ is a homomorphism or, in other words, that

$$w^* - \lim_{U(s)} f(s)\cdot g(s) = \Big(w^* - \lim_{U(s)} f(s)\Big)\cdot\Big(w^* - \lim_{U(s)} g(s)\Big)$$

holds true for all f and g in $\ell_\infty(S,\mathfrak{A})$. But this clearly follows from the (joint) weak* continuity of the Arens product on bounded sets and our choice of U. This completes the proof in case (a).

Part (b) immediately follows from the results in [**6**, section II].

We now present the examples. Let us recall the trivial fact that if $\mathfrak{A}_1 \to \mathfrak{A}_2$ is a bounded homomorphism with dense range and $\mathfrak{A}_1$ is (commutative and weakly) amenable, then so is $\mathfrak{A}_2$.

Example 1. An amenable C^*-algebra $\mathfrak{A}$ such that $\ell_\infty(\mathfrak{A})$ fails to be amenable.

Proof. Take $\mathfrak{A} = \mathscr{K}(\mathscr{H})$, the space of all compact operators on a separable Hilbert space. It is well known that the dual space $\mathscr{K}(\mathscr{H})'$ is isometrically isomorphic to $\mathscr{H}\hat{\otimes}\mathscr{H}$ (which is separable) via the pairing

$$\langle T, u\rangle = \sum_{n=1}^{\infty}\langle x_n|Ty_n\rangle,$$

where $T \in \mathscr{K}(\mathscr{H})$ and $u \in \mathscr{H}\hat{\otimes}\mathscr{H}$, with $u = \sum_{n=1}^{\infty} x_n \otimes y_n$. It is easily seen that the product of $\mathscr{K}(\mathscr{H})$ is jointly weakly continuous. Finally observe that $\mathscr{K}(\mathscr{H})$ is amenable (see [**4**]) but $\mathscr{K}(\mathscr{H})''$ (with the Arens product) is not, since it is isometrically isomorphic to the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$. Hence $\ell_\infty(\mathscr{K}(\mathscr{H}))$ cannot be amenable. If you prefer a unital counterexample, simply take $\mathfrak{A}$ as the algebra generated in $\mathscr{L}(\mathscr{H})$ by $\mathscr{K}(\mathscr{H})$ and the identity on $\mathscr{H}$.

As a curious by-product of Example 1, we obtain that the algebras $\ell_\infty\check{\otimes}\mathscr{K}(\mathscr{H})$ and $\ell_\infty(\mathscr{K}(\mathscr{H}))$ are not isomorphic, in striking contrast with the scalar case. (Notice that $\ell_\infty\check{\otimes}\mathscr{K}(\mathscr{H})$ is amenable since $\ell_\infty\hat{\otimes}\mathscr{K}(\mathscr{H})$ is and the natural map $\ell_\infty\hat{\otimes}\mathscr{K}(\mathscr{H}) \to \ell_\infty\check{\otimes}\mathscr{K}(\mathscr{H})$ has dense range.)

Example 2. A weakly amenable Lipschitz algebra $\mathfrak{A}$ such that $\ell_\infty(\mathfrak{A})$ fails to be weakly amenable.

Proof. Let K be a compact metric space with metric $d(\cdot,\cdot)$ and let $0 < \alpha < 1$. Then $\mathrm{lip}_\alpha(K)$ is the algebra of all complex-valued functions on K for which

$$\varrho_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

is finite and $\mathrm{lip}_\alpha(K)$ is the subalgebra of those f such that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as} \quad d(x,y) \to 0.$$

Both algebras are equipped with the norm $\|f\|_\alpha = \|f\|_\infty + \varrho_\alpha(f)$. Bade, Curtis and Dales proved in [**3**] that the algebra $\mathrm{lip}_\alpha(K)''$ is isometrically isomorphic to $\mathrm{lip}_\alpha(K)$. On the other hand, $\mathrm{lip}_\alpha(K)$ is weakly amenable for all $0 < \alpha < \frac{1}{2}$, while $\mathrm{lip}_\alpha(K)$ has point derivations for every infinite K. Thus, the proof will be complete if we show that K can be chosen in such a way that $\mathrm{lip}_\alpha(K)$ satisfies the hypothesis (a) of Theorem 1. Take $K = \mathbb{T}$, the unit circle. Then the Banach space $\mathrm{lip}_\alpha(K)$ turns out to be isomorphic (in the pure linear sense) to c_0, the space of all null sequences. This implies that every bilinear operator from $\mathrm{lip}_\alpha(K) \times \mathrm{lip}_\alpha(K)$ into any Banach space is jointly weakly continuous on bounded sets and completes the proof.

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