Criteria for transience of branching Markov chains

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Abstract

A branching Markov chain (BMC) is a system of particles in discrete time. The BMC starts with one particle in an arbitrary starting position \(x\). At each time particles split up in offspring particles independently according to some probability distributions that may depend on the locations of the particles. The new particles then move independently according to a Markov chain.

An irreducible Markov chain is either recurrent or transient: either all or none states are visited infinitely often. It turns out that this dichotomy breaks down for BMC and that one can classify BMCs in three different types. Let \(\alpha(x)\) be the probability that, starting the BMC in \(x\), the state \(x\) is hit infinitely often by some particles. There are three possible regimes: transient \((\alpha(x) = 0 \ \forall x)\), weakly recurrent \((0 < \alpha(x) < 1 \ \forall x)\) and strongly recurrent \((\alpha(x) = 1 \ \forall x)\). We give equivalent criteria for transience of BMC and discuss some interesting consequences.

1. Branching Markov chains

The study of BMCs goes back at least to Crump and Mode [4] and Joffe and Moncayo [6]. Many basic results are obtained by Hammersley, Biggins, and Kingman in the 70's. Despite the fact that the definitions of BMCs given in these works differ, e.g. discrete or continuous state space, only positive increments in the case of age-dependent processes, the main ideas and questions concerning the theory of branching Markov chains are developed in these articles. After this body of work the number of articles on BMCs exploded and the theory of BMCs became an active field of research with various applications. Here we point out the paper of Menshikov and Volkov [7] that gives first qualitative characteristics for BMCs. In [7] criteria for recurrence and transience are treated and first equivalent conditions in terms of functional equations are given. An important and fruitful variation of BMCs are tree-indexed Markov chains introduced by Benjamini and Peres [1], [2].

1.1. Definition

We introduce the model of branching Markov chains (BMCs). Let \((X, P)\) be an irreducible and infinite Markov chain in discrete time. For all
$x \in X$ we define an **offspring distribution**: let

$$\mu(x) = (\mu_k(x))_{k \geq 1}$$

be a sequence of nonnegative numbers satisfying

$$\sum_{k=1}^{\infty} \mu_k(x) = 1 \text{ and } m(x) := \sum_{k=1}^{\infty} k \mu_k(x) < \infty.$$  

We define the BMC $(X, P, \mu)$ with underlying Markov chain $(X, P)$ and branching distribution $\mu = (\mu(x))_{x \in X}$ following [7]. At time 0 we start with one particle in an arbitrary starting position $x \in X$. At time 1 this particle splits up in $k$ offspring particles with probability $\mu_k(x)$. Still at time $n = 1$, these $k$ offspring particles then move independently according to the Markov chain $(X, P)$. The process is defined inductively: At each time each particle in position $x$ splits up according to $\mu(x)$ and the offspring particles move according to $(X, P)$. At any time, all particles move and branch independently of the other particles and the previous history of the process. Let $\eta(n)$ be the total number of particles at time $n$ and let $x_i(n)$ denote the position of the $i$th particle at time $n$. Denote $P_x(\cdot) = P(\cdot | x_{0}(1) = x)$ the probability measure for a BMC started with one particle in $x$. A priori it is not clear in which sense transience and recurrence of Markov chains can be generalized to BMC. One possibility is to say a BMC is recurrent if with probability 1 at least one particle returns to the starting position. This approach was followed in [7]. We choose a different one, e.g. compare [1] and [5].

**Definition 1.1** Let

$$\alpha(x) := P_x \left( \sum_{n=1}^{\infty} \sum_{i=1}^{\eta(n)} 1\{x_i(n) = x\} = \infty \right).$$

A BMC is recurrent, if $\alpha(x) > 0$ for some (⇔ all) $x \in X$, strongly recurrent, if $\alpha(x) = 1$ for some (⇔ all) $x \in X$ and transient otherwise. We write $\alpha > 0$ if $\alpha(x) > 0$ for all $x \in X$ and $\alpha \equiv 1$ and $\alpha \equiv 0$, respectively.

### 2. Criteria for transience and recurrence

The Green function $G(x, y|m) = \sum_{n=1}^{\infty} p^{(n)}(x, y)m^n$ gives the expected number of particles that visits $y$ when we start the BMC with constant mean offspring $m$ in $x$. Due to this interpretation, $G(x, y|m) < \infty$ implies the transience of a BMC with constant mean offspring $m$. Observe, that $G(x, y|m) < \infty$ either for all or none $x$. In general, the converse is not true, compare with Corollaries 3.3 and 3.5. Hence, the Green function is not a good characteristic for
transience and recurrence. In order to obtain a good characteristic we introduce a modified version of the BMC. We fix some position $o \in X$, which we denote the origin of $X$. The new process is like the original BMC at time $n = 1$, but is different for $n > 1$. After the first time step we conceive the origin as freezing: if a particle reaches the origin, it stays there forever and stops splitting up. We denote this new process with $\text{BMC}^*$. Let $\eta(n, o)$ be the number of particles at position $o$ at time $n$. We define the random variable $\nu(o)$ as

$$
\nu(o) := \lim_{n \to \infty} \eta(n, o) \in \{0, 1, \ldots\} \cup \{\infty\}.
$$

We write $E_x \nu(o)$ for the expectation of $\nu(o)$ given that $x_0(1) = x$.

The next Theorem, due to [9], gives several sufficient and necessary conditions for transience that can be seen as a generalization of conditions for transience of Markov chains.

**Theorem 2.1** A BMC $(X, P, \mu)$ with $m(y) > 1$ for some $y$ is transient if and only if the three equivalent conditions hold:

(i) $E_o \nu(o) \leq 1$ for some (⇒ all) $o \in X$.

(ii) $E_x \nu(o) < \infty$ for all $x, o \in X$.

(iii) There exists a strictly positive function $f(\cdot)$ such that

$$
P f(x) := \sum_y p(x, y) f(y) \leq \frac{f(x)}{m(x)} \quad \forall x \in X.
$$

**Remark 2.2** The function $f(x) := E_x \nu(o)$ satisfies $P f(x) = \frac{f(x)}{m(x)}$ for all $x \in X$ if and only if $E_x \nu(o) = 1$ for all $x \sim o$, i.e., $p(x, o) > 0$.

In particular, if the mean offspring is constant, i.e., $m(x) = m \forall x \in X$, we have the following criterion of [5] in terms of the spectral radius $\rho(P) := \lim \sup_n (p^{(n)}(x, x))^{1/n}$ of the underlying Markov chain.

**Theorem 2.3** The BMC $(X, P, \mu)$ with constant mean offspring $m > 1$ is transient if and only if $m \leq 1/\rho(P)$.

3. Consequences

In this section we give some consequences of Theorems 2.1 and 2.3. In Corollary 3.1 we see that amenable groups can be characterized with the help of transience and recurrence of BMC. Denote $X_n$ the position of the Markov chain $(X, P)$ at time $n$ and $T_y = \inf_{n \geq 1} \{X_n = y\}$. We give a probabilistic interpretation of the generating function $U(x, y|z) := \sum_{n=1}^{\infty} P_x(T_y = n) z^n$ in
terms of BMC and deduce a criterion for \(\rho\)-recurrence of Markov chains in terms of BMC, see Proposition 3.2. The remaining part is devoted to the question in which situations the finiteness of the Green function \(G(x,x|m)\) is equivalent to the transience of the BMC with constant mean offspring \(m\), see Corollaries 3.3, 3.4 and 3.5. The proofs can be found in [9].

### 3.1. Amenability and BMC

We obtain a variation of Proposition 1.5. in [2], which characterizes amenable groups via BMC. Let \(\Gamma\) be the Cayley graph of a finitely generated group with generating set \(S\) and denote \(Q\) the transition probabilities of the simple random walk on \(\Gamma\), i.e., \(q(x,y) = 1/|S|\) if \(yx^{-1} \in S\) and 0 otherwise.

**Corollary 3.1** A finitely generated group \(\Gamma\) is amenable if and only if every BMC \((\Gamma,Q,\mu)\) with constant mean offspring \(m > 1\) is recurrent.

To see this, we merely need to combine Theorem 2.3 with the well-known result of Kesten stating that every irreducible and symmetric random walk on a finitely generated group has spectral radius 1 if and only if it is amenable.

### 3.2. Green function

The generating function \(G(x,x|m)\) is the expected number of particles returning to the starting position \(x\) of a BMC with constant mean offspring \(m\). The generating function \(U(x,y|z) = \sum_{n=1}^{\infty} P_x(T_y = n)z^n\) can be interpreted in terms of BMC, too. Observe that \(P_x(T_y = n)m^n\) is the expected number of particles visiting \(y\) at time \(n\) for the first time in their ancestry line. Hence we can express \(U\) in terms of the process BMC*:

\[
U(x,y|m) = E_x\nu(y),
\]

where \(E_x\) corresponds to a BMC* with constant mean offspring \(m\).

**Proposition 3.2** Let \(o \in X\). The Markov Chain \((X,P)\) is \(\rho\)-recurrent if and only if \(f(x) := E_x\nu(o)\) is \(\rho\)-harmonic.

We have already mentioned that the finiteness of the Green function is in general only a sufficient condition for transience of BMCs. We will discuss the natural question when \(G(x,y|m) = \infty\) does imply the recurrence of the BMC, see also [1] and [2] for results for tree-indexed Markov chains. We give an answer in terms of \(\rho\)-transience and \(\rho\)-recurrence that follows immediately from Theorem 2.1 and the definition of \(\rho\)-transience.

**Corollary 3.3** Let \((X,P)\) be \(\rho\)-transient. Then the BMC \((X,P,\mu)\) is transient if and only if \(G(x,y|m) < \infty\).

In the case of simple random walks on nonamenable groups the Green function gives a sufficient and necessary condition for transience, since any
irreducible simple random walk on a nonamenable discrete group is $\rho$-transient, i.e.,
\[ G(x, y|1/\rho) < \infty. \]

**Corollary 3.4** Let $(X, P)$ be a simple random walk on a nonamenable graph. Then the BMC $(X, P, \mu)$ with constant mean offspring $m$ is transient if and only if
\[ G(e, e|m) < \infty. \]

**Corollary 3.5** For every $\rho$-recurrent Markov chain $(X, P)$ we have $G(x, y|1/\rho) = \infty$ and that the corresponding BMC $(X, P, \mu)$ with mean $m = 1/\rho$ is transient.

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4. Bibliography


