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Bayesian Inference

On the Use of Bayes Factor in Frequentist Testing of a Precise Hypothesis

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This article deals with Bayes factors as useful Bayesian tools in frequentist testing of a precise hypothesis. A result and several examples are included to justify the definition of Bayes factor for point null hypotheses, without merging the initial distribution with a degenerate distribution on the null hypothesis. Of special interest is the problem of testing a proportion (joint with a natural criterion to compare different tests), the possible presence of nuisance parameters, or the influence of Bayesian sufficiency on this problem. The problem of testing a precise hypothesis under a Bayesian perspective is also considered and two alternative methods to deal with are given.

Keywords Bayes factor; Precise hypotheses.

Mathematics Subject Classification Primary 62F15, 62F03; Secondary 62B05.

1. Introduction

An interesting problem in classical statistical inference is that of testing a point (or precise) null hypothesis. Think, for instance, that it is known that \( \theta_0 \) is assumed to be the true value of the parameter \( \theta \) in a population A, and we want to “confirm” if this is also the case in the population B.

Well-known statistical methods have been designed to test precise hypotheses about most of the parameters of interest in statistics: means, proportions, Poisson parameters, variances, correlation coefficients, and so on.

From a Bayesian point of view, a major handicap arises when testing a precise hypothesis, as its final probability is zero for continuous posterior distributions. But precise hypotheses are of clear interest and Bayesians have not given up proposing other kind of solutions to this problem, usually implying a modification of the prior distribution, assigning probability \( \pi_0 > 0 \) (commonly, \( \pi_0 = 0.5 \)) to the point

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null hypothesis, and spreading the remaining $1 - \pi_0$ probability over the alternative hypothesis according to the original prior distribution (see, for instance, Berger, 1985). Choosing a specific prior distribution for a concrete testing problem could be well justified in some cases, but could be questionable in another ones from a Bayesian perspective. Two alternative Bayesian methods to deal with this subject are suggested in the next section.

Nevertheless, this article is mainly concerned with the following question: What can be done from a Bayesian point of view to obtain a frequentist answer to the problem of testing a point null hypothesis?

We consider the Bayes factor as the appropriate tool to answer this question, since it is a quantity of a clear Bayesian nature that reaches an interesting meaning from a classical perspective: in fact, it measures the evidence of the data in favor of the null hypothesis once the prior advantage of the null hypothesis has been removed, and it should be recognized as the most useful Bayesian quantity for a frequentist looking for ideas in the Bayesian framework to solve his testing problems. A recent review on the Bayes factor can be found in Kass and Raftery (1995). A classical reference on the Bayes factor is Jeffreys (1961), where it is considered as the cornerstone of Bayesian hypothesis testing.

When the prior probability of the null hypothesis is zero, its posterior probabilities are also zero (predictively a.e.), and the Bayes factor is not well defined, as it is of the form $0/0$.

For testing the point null hypothesis $\theta = \theta_0$ about the unknown parameter $\theta$ given the data $x$, rather than replacing the prior distribution $Q$ by $\pi_0 \delta_{\theta_0} + (1 - \pi_0)Q$ (where $\delta_{\theta_0}$ stands for the degenerate distribution at the point $\theta_0$) as explained above, we consider preferable from a Bayesian perspective to define its Bayes factor $\beta(x, \theta_0)$ as the limit (if it exists) when $\epsilon$ goes to 0 of the Bayes factor $\beta(x, \theta_0 + \epsilon)$ for the interval null hypothesis $[\theta_0 - \epsilon, \theta_0 + \epsilon]$. This is also accomplished in Sec. 2.

This approach is also preferable for a frequentist with a “confirmatory” problem. In the last section, some examples are included to see how this definition leads to reasonable tests; there, the Bayes factor will be considered as a test statistic, and the null hypothesis will be rejected when it is small “enough”, where “enough” depends on the distribution of the Bayes factor under the null hypothesis and the significance level desired for the test.

This provides a general method to construct new frequentist tests (or to justify, from a Bayesian setting, some well-known classical tests), and the goodness of the obtained tests could justify the proposed definition of the Bayes factor for point null hypotheses. In particular, the tests obtained by this method to contrast a point null hypothesis about a proportion or a Poisson parameter are good competitors of the equal-tailed corresponding tests.

2. Bayes Factor for Testing Point Null Hypotheses

Let us consider the simple statistical experiment of observing a sample of size $n$ from a real probability distribution $P_\theta$, $\theta$ being an unknown real parameter in a open interval $\Theta \subseteq \mathbb{R}$. In a Bayesian context, we consider the parameter space $\Theta$ endowed with a prior distribution $Q$ with continuous density function $q$. We assume the existence of a density $p_\theta$ (with respect to the Lebesgue measure) of the distribution $P_\theta$, $\theta \in \Theta$, and we denote $\ell(x, \theta) := \prod_{i=1}^n p_\theta(x_i), x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the likelihood function, that is supposed continuous in $\theta$. Hence, the posterior distribution $P_\theta^*$
exists for every \( x \) and has density \( p_\theta^*(\theta) = C(x)\xi(x, \theta)q(\theta) \), where \( C(x) \) is a suitable positive constant so that \( \int_\Theta p_\theta^*(\theta)d\theta = 1 \). The densities \( p_\theta^* \) is supposed continuous in \( \theta \). The function \( x \mapsto \int_\Theta \xi(x, \theta)q(\theta)d\theta \) is the so-called predictive density.

The Bayes factor \( \beta(x, \Theta_0) \) in favor of the null hypothesis given the data \( x \in \mathbb{R}^n \) is defined as

\[
\beta(x, \Theta_0) := \frac{P_\theta^*(\Theta_0)/P_\theta^*(\Theta_1)}{Q(\Theta_0)/Q(\Theta_1)},
\]

when the quotient is well defined. Nevertheless, for the problem of testing a null hypothesis \( \Theta_0 \subseteq \Theta \) vs. \( \Theta_1 \), the problem of testing a precise hypothesis, without modifying the prior distribution, can be considered at the beginning of the section.

It is not difficult to calculate this limit even in a more general setting than the considered at the beginning of the section.
Proposition 2.1. When \( \Theta \) is an open set of \( \mathbb{R}^k \) and the likelihood function \( \ell(x, \theta) \) and the prior density \( q(\theta) \) are continuous functions of \( \theta \), given \( \theta_0 \in \Theta \), we have that

\[
\lim_{\epsilon \to 0} \beta(x, \theta_0 \pm \epsilon) = \frac{dP^*}{dQ}(\theta_0) = C(x)\ell(x, \theta_0).
\]

The proof follows from the fact that, for a function \( f \) continuous on \( \theta_0 = (\theta_{01}, \ldots, \theta_{0k}) \) and \( \epsilon = (\epsilon_1, \ldots, \epsilon_k) \),

\[
\frac{1}{2^k} \prod_{i=1}^k \left| \int_{\Pi_i[\theta_0-\epsilon_i, \theta_0+\epsilon_i]} f(\theta) d\theta - f(\theta_0) \right| \leq \sup_{|\theta-\theta_0| \leq \epsilon_i} |f(\theta) - f(\theta_0)| \to 0.
\]

We could decide whether to accept or not the point null hypothesis according to the value of the Bayes factor: this method does not modify the prior distribution for a concrete problem, but modifies the usual Bayesian procedure of testing hypotheses. At this point, it should be noted that, for prior distributions of the form \( \pi_0 \delta_{\theta_0} + (1 - \pi_0)Q \), it usually happens that both \( \beta(x, \theta_0) \), defined as in (1), and \( \lim_{\epsilon \to 0} \beta(x, \theta_0 \pm \epsilon) \) exist and are different (see Levine and Casella, 1996, Th. 3.4), but this is due to the discontinuity introduced in the modification of the prior distribution (a good reason, perhaps, to avoid such a modification).

We include several examples to illustrate the behavior of the definition (2) of the Bayes factor for point null hypotheses.

Example 2.1. Let \( P_\theta = N(\theta, \sigma^2) \), \( \theta \in \mathbb{R} \), and \( Q = N(a, b^2) \) (\( \sigma, a, \) and \( b \) known). It is readily shown that the posterior distribution \( P^*_\theta \) given the observed sample \( x \in \mathbb{R}^k \) is normal \( N(A_x, B^2) \), where

\[
A_x = \frac{nb^2 \bar{x} + a\sigma^2}{nb^2 + \sigma^2}, \quad B^2 = \frac{\sigma^2 b^2}{nb^2 + \sigma^2}.
\]

According to the previous proposition, to test the point null hypothesis \( \theta = \theta_0 \) vs. \( \theta \neq \theta_0 \), the Bayes factor given \( x \) would be

\[
\beta(x, \theta_0) = \frac{b}{B} \frac{\phi\left(\frac{\theta_0 - A_x}{B}\right)}{\phi\left(\frac{\theta_0 - a}{b}\right)}
\]

where \( \phi \) denotes the density of the standard normal distribution. The result has a clear meaning: the more the Bayes estimate for \( \theta \), \( A_x \), differs from the point \( \theta_0 \) of the null hypothesis, the smaller is the Bayes factor in favor of the null hypothesis is.

Example 2.2. Let \( P_\theta = b_n(\theta) \) (binomial distribution), \( \theta \in ]0, 1[ \), and \( Q \) the uniform distribution on \( ]0, 1[ \). The posterior distribution \( P^*_\theta \) given \( k = 0, 1, \ldots, n \), has density

\[
dP^*_\theta(\theta) = (n + 1) \binom{n}{k} \theta^k (1 - \theta)^{n-k} d\theta.
\]
It follows that the Bayes factor for testing the point null hypotheses \( \theta = \theta_0 \) against \( \theta \neq \theta_0 \) is

\[
\beta(k, \theta_0) = (n+1) \binom{n}{k} \theta_0^k (1 - \theta_0)^{n-k}.
\]

**Example 2.3.** Let \( P_0 \) be the Poisson distribution with parameter \( \theta > 0 \), and \( Q \) the gamma distribution \( G(\theta_0, 1) \) with parameters \( \theta_0 \) and 1. The posterior distribution \( P_k^* \) given \( k = 1, 2, \ldots \), has density

\[
dP_k^*(\theta) = \frac{2^{k+\theta_0}}{\Gamma(k+\theta_0)} \theta^{k+\theta_0-1} e^{-2\theta} I_{[0,\infty]}(\theta) d\theta,
\]

and, given \( \theta_0 > 0 \), the Bayes factor for testing the point null hypotheses \( \theta = \theta_0 \) against \( \theta \neq \theta_0 \) is

\[
\beta(k, \theta_0) = \left( \frac{2}{e} \right)^{\theta_0} \frac{\Gamma(\theta_0)}{\Gamma(k+\theta_0)} (2\theta_0)^k.
\]

For a sample \((k_1, \ldots, k_n)\) of size \( n \) of a Poisson distribution of unknown parameter \( \theta > 0 \), it can be shown analogously that the Bayes factor is

\[
\beta(k_1, \ldots, k_n; \theta_0) = \left( \frac{n+1}{e^{\theta_0}} \right)^{\theta_0} \frac{\Gamma(\theta_0)}{\Gamma(k_1+\theta_0, \ldots, k_n+\theta_0)} [(n+1)\theta_0]^k,
\]

where \( k = \sum_{i=1}^n k_i \).

The idea of considering the limit of Bayes factors for small intervals \([\theta_0 - \epsilon, \theta_0 + \epsilon]\) as the Bayes factor for the point null hypothesis \( \theta = \theta_0 \) also works for non continuous posterior densities.

**Example 2.4.** Let \( P_0 = \mathcal{U}[0, \theta]^a \) and \( Q = \mathcal{U}[0, 1] \) stands for the uniform distribution on the interval \([a, b] \), i.e., we observe a sample of size \( n \) from a uniform distribution on the interval \([0, \theta]\), where the parameter \( \theta \in [0, 1] \) is supposed to be unknown. For \( x = (x_1, \ldots, x_a) \in [0, 1]^a \), we write \( x(a) = X_{(a)}(x) = \max_{1 \leq i \leq a} x_i \). The posterior distribution \( P_k^* \) given \( x \) has density

\[
dP_k^*(\theta) = C(x) \theta^{a-1} I_{[X_{(a)}, \theta]}(\theta) d\theta,
\]

where \( C(x) = (n-1)x_{(a)}^{a-1}(1-x_{(a)}^{a-1}) \). Given \( \theta_0 \in [0, 1] \) and \( 0 < \epsilon < \min(\theta_0, 1 - \theta_0) \), we write \( \Theta_\epsilon = [\theta_0 - \epsilon, \theta_0 + \epsilon] \). If \( \beta_\epsilon \) denotes the Bayes factor for testing the null hypothesis \( \Theta_\epsilon \) vs. \( \Theta_\epsilon^c := [0, 1] \setminus \Theta_\epsilon \), we have

\[
\lim_{\epsilon \to 0} \beta_\epsilon = \begin{cases} 
0 & \text{if } x(a) > \theta_0 \\
\frac{n-1}{2\theta_0(1 - \theta_0^{a-1})} & \text{if } x(a) = \theta_0 \\
\frac{(n-1)x_{(a)}^{a-1}}{\theta_0^a(1 - x_{(a)}^{a-1})} & \text{if } x(a) < \theta_0 
\end{cases}
\]

According to the previous comments, we can take this limit as the Bayes factor \( \beta(x, \theta_0) \) for testing the point null hypothesis \( \theta = \theta_0 \) against \( \theta \neq \theta_0 \). Note that the point null hypothesis \( \theta_0 \) has any chance when it is smaller that \( x(a) \), and that the
Bayes factor decreases with \( x_{i(a)} \in ]0, \theta_0[ \). Note also the discontinuity of the Bayes factor at the left of \( \theta_0 \); at this point, it is half of the limit from the left.

A similar method could work in the presence of nuisance parameters.

**Example 2.5.** Let us consider the Bayesian statistical experiment corresponding to a sample of size \( n \) from a uniform distribution on the interval \( \theta - \delta, \theta + \delta \], where the unknown parameters \( \theta \) and \( \delta \) are a priori uniformly distributed on the intervals \( ]-1, 1[ \) and \( [0, 1[ \), respectively. Take \( \theta_0 \in ]-1, 1[ \) and \( 0 < \epsilon < \min(|\theta_0 + 1|, |\theta_0 - 1|) \), and denote \( \Theta_\epsilon = [\theta_0 - \epsilon, \theta_0 + \epsilon] \). The limit when \( \epsilon \) goes to 0 of the Bayes factor for testing \( \theta \in \Theta_\epsilon \) against \( \theta \notin \Theta_\epsilon \) (in presence of the nuisance parameter \( \delta \)) will be considered as the Bayes factor \( \beta(x, \theta_0) \) for testing the point null hypothesis \( \theta = \theta_0 \). Denoting \( x_{(1)} = X_{(1)}(x_1, \ldots, x_n) = \min_{1 \leq i \leq n} x_i \), it is not difficult to see that \( \beta(x, \theta_0) = 0 \) when \( \max(x_{i(a)} - \theta_0, \theta_0 - x_{(1)}) > 1 \), and, otherwise,

\[
\beta(x, \theta_0) = \begin{cases} 
1 & \text{if } x_{(1)} + x_{(a)} > \theta_0 \\
\frac{1 - (x_{(a)} - \theta_0)^{n-1}}{2^n(n - 1)} & \text{if } \frac{x_{(1)} + x_{(a)}}{2} < \theta_0 \\
\frac{1 - (\theta_0 - x_{(1)})^{n-1}}{2^n(n - 1)} & \text{if } \frac{x_{(1)} + x_{(a)}}{2} = \theta_0 \\
\frac{1 - (\theta_0 - x_{(1)})^{n-1}}{2^{n+1}(n - 1)} & \text{if } x_{(1)} + x_{(a)} > \theta_0 \\
\frac{1 - (x_{(a)} - \theta_0)^{n-1}}{2^{n+1}(n - 1)} & \text{if } \frac{x_{(1)} + x_{(a)}}{2} < \theta_0 \\
\frac{1 - (\theta_0 - x_{(1)})^{n-1}}{2^{n+1}(n - 1)} & \text{if } \frac{x_{(1)} + x_{(a)}}{2} = \theta_0 \\
\end{cases}
\]

3. **Some Applications in Frequentist Testing Problems of Point Null Hypothesis**

Nevertheless, in this article we are mainly interested in how a Bayesian can be useful to solve the classical problem of testing precise hypotheses. The most interesting Bayesian tool in classical inference is, perhaps, the Bayes factor, because it is considered as the contribution of the data on the posterior odds of the null hypothesis, once the prior odds has been discounted (although it depends on the prior distribution, as it can be seen in the proposition above). With the idea in mind of being useful for frequentist inference from a Bayesian perspective, we can use the Bayes factor as a test statistic, and reject the null hypothesis when the Bayes factor is smaller than a certain constant determined by the desired significance level and its distribution under the null hypothesis. In these cases, acceptable prior distributions for a frequentist statistician could be objective priors, or, at least, for the case of a point null hypothesis, prior distributions whose mean is the point \( \theta_0 \) proposed by the null hypothesis. Anyway, every prior distribution leads to a frequentist test and we get a Bayesian-based general method of obtaining classical tests.

Many efforts have been made in the literature to reconcile measures of Bayesian evidence, as the Bayes factor or the posterior probability of the null hypothesis, and classical measures of evidence, as the \( P \)-value. The interested reader is referred to Berger and Sellke (1987), Casella and Berger (1990), or Gómez-Villegas et al. (2002), where a good introduction to the problem can be found. These articles usually consider families of prior distributions that are specially chosen for the null hypothesis considered. What we propose is to use the Bayes factor as a classical test statistic, and this is, in some sense, an alternative way of reconciling both points of
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view. See also Good (1992) for a brief review of several kind of Bayes/non Bayes compromise.

Example 3.1 (Testing the Mean of a Normal Population). According to the above reasoning, to test the point null hypothesis \( \theta = \theta_0 \) against \( \theta \neq \theta_0 \), a frequentist should reject when the Bayes factor is smaller than a certain constant \( C_1 \), i.e., if

\[
\frac{b \phi(\frac{\bar{x} - \theta_0}{\sigma})}{B \phi(\frac{\bar{x} - \theta_0}{\sigma})} < C_1.
\]

This is equivalent to the existence of a constant \( C_2 \) such that

\[
\frac{|\theta_0 - \bar{x}|}{B} = \frac{|nb^2(\theta_0 - \bar{x}) + a^2(\theta_0 - a)|}{\sigma b \sqrt{nb^2 + \sigma^2}} > C_2.
\]

For a frequentist, who does not admit the prior opinion, it is quite reasonable to take \( a = \theta_0 \) (he has formulated the null hypothesis that the true mean is \( \theta_0 \) and should admit that a logical value for \( a \) is \( \theta_0 \)), in which case the rejection set is of the form

\[
|\theta_0 - \bar{x}| > C_3
\]

or even

\[
\sqrt{n} \frac{\bar{x} - \theta_0}{\sigma} > C,
\]

where the constants are chosen so that the significance level will be \( \alpha \). Hence \( C = z_\alpha \), where \( \int_{-z_\alpha}^{z_\alpha} \phi(x)dx = 1 - \alpha \).

Example 3.2 (Testing a Proportion). Given \( \theta_0 \in ]0, 1[ \), to test \( \theta = \theta_0 \) vs. \( \theta \neq \theta_0 \), a frequentist could consider reasonable to use the Bayes factor of Example 3.2 as a test statistic, and reject the point null hypothesis when \( \beta(k, \theta_0) \) is small. This is equivalent to a critical region of the form

\[
T(k) := \left( \frac{n}{k} \right) \theta_0^k (1 - \theta_0)^{n-k} < C
\]

where the constant \( C \) is chosen so that the level of significance be \( \alpha \in ]0, 1[ \). If \( (k_0, k_1, \ldots, k_n) \) is a permutation of \( \{0, 1, \ldots, n\} \) such that \( T(k_0) \geq T(k_1) \geq \cdots \geq T(k_n) \), there exists \( m \in \{0, 1, \ldots, n\} \) such that

\[
\sum_{i=0}^{m-1} T(k_i) \leq 1 - \alpha < \sum_{i=0}^{m} T(k_i),
\]

and the resulting test accepts the point null hypothesis when \( k \in \{k_0, k_1, \ldots, k_{m-1}\} \), and rejects it when \( k \in \{k_{m+1}, \ldots, k_n\} \) and, also, with probability

\[
y = \frac{\sum_{i=0}^{m} T(k_i) - (1 - \alpha)}{T(k_n)}.
\]
when \( k = k_w \). This test, which will be referred as BUB-test (BUB stands for “Bayesian prior uniform binomial”), is slightly different from the exact bilateral binomial test (EBB-test), also known as equal-tail binomial test, which spreads the Type I probability error \( \alpha \) symmetrically on the two tails; namely, \( m_1, m_2 \in \{0, 1, \ldots, n\} \) can be found such that

\[
\sum_{i=0}^{m_1-1} T(i) \leq \alpha/2 < \sum_{i=0}^{m_1} T(i), \quad \sum_{i=m_2+1}^{n} T(i) \leq \alpha/2 < \sum_{i=m_2}^{n} T(i)
\]

and the EBB-test rejects the point null hypothesis when \( k \in \{0, 1, \ldots, m_1 - 1\} \cup \{m_2 + 1, \ldots, n\} \), and also, with probability

\[
\gamma_1 = \frac{\alpha/2 - \sum_{i=0}^{m_1-1} T(i)}{T(m_1)}
\]

when \( k = m_1 \), and with probability

\[
\gamma_2 = \frac{\alpha/2 - \sum_{i=m_2+1}^{n} T(i)}{T(m_2)}
\]

when \( k = m_2 \).

None of these tests has power function strictly better than the other. So, 7,350 comparisons between the two tests have been carried out, one for each possible combination of the values \( n = 1, 2, 3, \ldots, 50 \), \( p_0 = 0.01, 0.02, \ldots, 0.49 \), and \( \alpha = 0.01, 0.05, 0.1 \); for \( p_0 = 0.05 \) both tests coincide, while the behavior for greater values is the same than for \( 1 - p_0 \). To compare both tests, we have calculated the mean of their power functions, obtaining that the BUB-test is better than EBB-test in all of the 7,350 cases considered. Figure 1 below includes the power functions of both tests for \( n = 10 \), \( \alpha = 0.05 \), and \( p_0 = 0.38 \) (continuous line correspond to the BUB test); the difference of the means of the power functions in this case is 0.0165. We also include an histogram with the distribution of the differences between the means of the power functions in the 7,350 cases, and a graph where these differences are expressed as a function of \( p_0 \) for \( n = 10 \) and \( \alpha = 0.05 \).

Instead of the objective prior considered, we can use a prior distribution with mean \( \theta_0 \), as the beta distribution \( B(\theta_0, 1 - \theta_0) \); in this case, the obtained test is slightly different from the two given above: its critical region is of the form

\[
\frac{n!}{\Gamma(k + \theta_0)\Gamma(n - k - \theta_0 + 1)} \theta_0^k(1 - \theta_0)^{n-k} < C.
\]

The reader is also referred to Brown et al. (2001), where a systematic comparative study of several methods of constructing confidence intervals for binomial proportions is carried out by using the probability of coverage or the expected length as criterion.

**Example 3.3** (Testing the Parameter of a Poisson Distribution). Given \( \theta_0 \in ]0, +\infty[ \), to test \( \theta = \theta_0 \) vs. \( \theta \neq \theta_0 \) for sample size \( n = 1 \), a frequentist could consider reasonable to use the Bayes factor of the Example 3.2 as a test statistic, since the prior distribution considered to obtain it \( (G(\theta_0, 1)) \) has mean \( \theta_0 \). So the point null
hypothesis will be rejected when $\beta(k, \theta_0)$ is small. This is equivalent to a critical region of the form

$$T(k) := \frac{(2\theta_0)^{k+\theta_0-1}}{\Gamma(k+\theta_0)} e^{-2\theta_0} < C$$

where the constant $C$ is chosen so that the level of significance be $\alpha \in [0, 1]$. When $\theta_0 \in \mathbb{N}$, the critical region can be written as

$$T(k) := P_{2\theta_0}((k + \theta_0 - 1)) < C.$$
There exists $p \in \{0, 1, \ldots, m\}$ such that $\sum_{i=0}^{p-1} P_{\theta_0}(\{k_i\}) \leq 1 - \alpha < \sum_{i=0}^{p} P_{\theta_0}(\{k_i\})$. The proposed test accepts the point null hypothesis $\theta = \theta_0$ when $k \in \{k_0, k_1, \ldots, k_{p-1}\}$, and rejects it when $k \in \{k_{p+1}, \ldots, k_m\}$ or $k \not\in \{k_0, \ldots, k_m\}$, while for $k = k_p$ the point null hypothesis will be rejected with probability

$$\gamma = \frac{\sum_{i=0}^{p} P_{\theta_0}(\{k_i\}) - (1 - \alpha)}{P_{\theta_0}(\{k_p\})}$$

For arbitrary sample size $n$, the critical region could be written as

$$T(k) := e^{-(n+1)\theta_0} \left(\frac{(n+1)\theta_0}{\Gamma(k + \theta_0)}\right)^{k+\theta_0-1} \leq C,$$

or, when $\theta_0 \in \mathbb{N}$,

$$T(k) := P_{(n+1)\theta_0}(\{k + \theta_0 - 1\}) < C,$$

where $k = \sum_{i=1}^{n} k_i$. We can describe explicitly the critical region with analogous arguments to the ones used above.

**Example 3.4** (Testing the Maximum Possible Value of a Uniformly Distributed Non Negative Random Variable). Take $\theta_0 \in ]0, 1[$ and consider the problem of testing $\theta = \theta_0$ vs. $\theta \neq \theta_0$. Since we have used a objective prior, a suitable critical region for a frequentist will be of the form $\beta(x, \theta_0) < C_1$. Since

$$\frac{(n-1)x_{(n)}^{n-1}}{\theta_0^2(1-x_{(n)}^{n-1})} < C_1$$

is equivalent to $x_{(n)} < C$ for some constant $C$, the critical region of the test will be of the form $\{X_{(n)} < C\} \cup \{X_{(n)} > \theta_0\}$. The constant $C$ is to be determined in order that $P_{\theta_0}(X_{(n)} < C) = \alpha$, because $P_{\theta_0}(X_{(n)} > \theta_0) + P_{\theta_0}(X_{(n)} = \theta_0) = 0$. It is readily shown that $C = \theta_0^2t^{1/n}$.

The method could propose a solution in presence of nuisance parameters.

**Example 3.5** (A Test About a Parameter of a Uniform Distribution in Presence of a Nuisance Parameter). Analogous arguments throw a critical region for the problem considered in Example 3.2 in the previous section of the form

$$\begin{cases}
X_{(n)} - \theta_0 < C & \text{if } \frac{x_{(1)} + x_{(n)}}{2} > \theta_0 \\
\theta_0 - X_{(1)} < C & \text{if } \frac{x_{(1)} + x_{(n)}}{2} < \theta_0
\end{cases}$$

As defined in (2), the Bayes factor of a point null hypothesis only depends on data throughout a Bayesian sufficient statistic and, this way, a sensible reduction on both sample and parameter spaces could be achieved; recall that a classical sufficient statistic does not entail any reduction in the parameter space. In the next example, we consider the problem of testing a point null hypothesis about the mean of a
trivariate normal distribution with known covariance matrix, obtaining a case where the dimensions of sample and parameter spaces goes from 3 to 2 after a Bayesian sufficiency reduction, while classical sufficiency does not carry any reduction.

**Example 3.6.** We observe three independent normally distributed real random variables with variance 1 and unknown means vector \( \theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R} \); let us suppose \( \theta \) normally distributed a priori with null mean and covariance matrix

\[
V_0 = \begin{pmatrix}
  \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
  \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\
  \frac{1}{3} & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}
\]

The bi-dimensional statistic \( T(x_1, x_2, x_3) := (x_1 - x_3, x_1 + x_2 + x_3) \) is sufficient from a Bayesian point of view (see, for instance, Florens et al., 1990, p. 81, Example 2). If we write \( \mu = (\mu_1, \mu_2) = (\theta_1 - \theta_3, \theta_1 + \theta_2 + \theta_3) \), the distribution of \( T \) given \( \theta \) is \( N(\mu_1, 2) \times N(\mu_2, 3) \), and the prior distribution of the parameter \( \mu \) is a bivariate normal distribution of null mean and covariance matrix

\[
\begin{pmatrix}
  2 & 0 \\
  0 & 3
\end{pmatrix}
\]

The problem of testing \( \theta = 0 \) becomes reduced by sufficiency to the problem of testing the point null hypothesis \( \mu = 0 \) in the experiment image of \( T \). It is easy to see that the Bayes factor for the reduced problem given \( y = T(x) \) is \( \beta(y) = 2\phi\left(\frac{2}{\sqrt{6}}\right)\phi\left(\frac{2}{\sqrt{6}}\right) \), where \( \phi \) stands for the normal standard density function. So, for the original problem, the Bayes factor is \( \beta(x) = 2\phi\left(\frac{x_3 - x_2}{\sqrt{2}}\right)\phi\left(\frac{x_1 + x_2 + x_3}{\sqrt{6}}\right) \).

**References**


