Determining vaccination policies through age-dependent branching models

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Abstract The aim of this paper is to define the optimal proportion of susceptible individuals into a given population, which has to be vaccinated in order to eliminate an infectious disease. To this end, the spread of infection is modelled through an age-dependent branching process and the distribution of the time to extinction of infection, depending on the proportion of the immune individuals into the population, is investigated. From this study, we suggest two vaccination policies based on the quantiles and on the mean of the infection extinction time. Finally, we provide a simulation-based method to determine the optimal vaccination policies.

Keywords Age-dependent branching process · extinction time · vaccination policies · Monte-Carlo method.

Mathematics Subject Classification (2000) 60J80 · 92D30

1 Introduction

One simplified way to describe the spread of infectious disease among the individuals of a given population is as follows. Let us assume that there exist three types of individuals in the population: infected, healthy but susceptible to catch the infection (susceptible individuals), and healthy and immune
to this disease. The disease is spreading when an infected individual is in contact with susceptible individuals. We denote by $p_k$ the probability that one infected individual contacts $k$ healthy individuals, $k \geq 0$, and by $\alpha$ (0 $\leq$ $\alpha$ $\leq$ 1) the proportion of immune individuals in the population. We assume that the population size is fixed and large enough so that $\alpha$ and the contact distribution law, $\{p_k\}_{k \geq 0}$, can be considered stable along time. Then, the probability that an infected individual transmits the disease to $k$ susceptible individuals is given by

$$p_{\alpha,k} = \sum_{j=k}^{\infty} \binom{j}{k} \alpha^j (1 - \alpha)^k p_j,$$

i.e., the infected individual has been in contact with $j$ healthy individuals and among them there have been $k$ susceptible individuals. We call $\{p_{\alpha,k}\}_{k \geq 0}$ the infection distribution law when the proportion of immune individuals in the population is $\alpha$. Notice that if every individual is non-immune, $\alpha = 0$, then every individual will be infected whenever contacts an infected one, that is, $p_{0,k} = p_k$, for all $k \geq 0$. On the other hand, if all individuals are immune, $\alpha = 1$, then the infection does not spread, that is, $p_{1,k} = 0$, for all $k \geq 0$. Following this spreading scheme along time, infected individuals pass on the disease to other susceptible individuals and so on.

Branching processes have been used to model situations similar to that described above (see [7], [8] and [11]), mainly because of their two important features. Firstly, they are well-fitted as models of population’s evolution, which develop by multiplication of individuals and secondly in any of the three possible ways (according to the classical critical classification of the branching processes) of the evolution of these processes there is a positive probability of the population to become extinct. In most of the cases we are aware of, the branching models used to describe the evolution of infectious diseases are in discrete-time (e.g. [3], [4], [6]). We suggest here a more accurate approach to this problem using continuous-time processes. We propose to model the number of individuals having the disease into the population by means of age-dependent branching processes (see Chapter 4 in [2] for details on such processes). More precisely, we consider an age-dependent branching process, such that its offspring law is determined by the infection distribution law $\{p_{\alpha,k}\}_{k \geq 0}$ and the distribution function (d.f.) of the life-length of an infected individual is given by an arbitrary d.f. $G(.)$ of a non-negative random variable (r.v.). Notice that the life-length of an infected individual does not depend on the proportion of immune individuals into the population.

When the infection is strongly detrimental for the population, such that it becomes an epidemic, then a vaccination policy should be applied to prevent the susceptible individuals and terminate the epidemic. Without lost of generality, let us assume that all healthy individuals before vaccination are susceptible. Our target is to determine what proportion of these individuals might be vaccinated/immunized to guarantee the extinction of the disease, possibly in a given period of time. We call this proportion vaccination level. Let us assume that every susceptible individual becomes immune instantaneously when he/she is vaccinated and the vaccination takes place at an
arbitrary time (the same for all of them). This paper deals with the problem to determine the optimal vaccination level depending not only on the speed of the transmission of the disease, expressed in terms of infection distribution law \( \{p_{\alpha,k}\}_{k \geq 0} \), but also on the time till the epidemic becomes extinct. To this end, in Section 2, we study the extinction time distribution of the epidemic depending on the proportion of immune individuals into the population. In Section 3, we propose two different ways of defining the optimal vaccination level, one based on the quantiles of the extinction time distribution of the disease and the other one on its mean. A simulation-based method to calculate the proportion of susceptible individuals to be vaccinated is also provided at the end of this section jointly with an illustrative example. The proofs are consigned to the end of the paper in Section 4.

2 The extinction time of the epidemic

In this section we will investigate the behaviour of the time elapsed by an infectious disease becomes extinct depending on the proportion of the immune individuals into the population when the contact distribution law is \( \{p_k\}_{k \geq 0} \). As we have modeled the spreading of the disease by means of an age-dependent branching process, then the problem reduces to study the distribution of the extinction time of such kind of processes. In general, in the theory of branching processes this question hasn’t received considerable attention (see [1], [6] and [10]). Consequently, in what follows, our goal is to investigate the distribution of the extinction time of an age-dependent branching process depending on the vaccination level \( \alpha \). To this end, for each \( \alpha \) such that \( 0 \leq \alpha \leq 1 \), we denote by \( T_\alpha \) the extinction time of an age-dependent branching process initiated at time 0 with a single infected individual, with offspring law \( \{p_{\alpha,k}\}_{k \geq 0} \) and with d.f. of the life-length \( G(\cdot) \). Intuitively, \( T_\alpha \) is the maximal time that the infection survives into the population when the proportion of immune individuals is \( \alpha \). From now on, we denote by \( v_\alpha(\cdot) \) the d.f. of the extinction time \( T_\alpha \), that is \( v_\alpha(t) = P(T_\alpha \leq t) \) for all \( t \in \mathbb{R} \) and by \( f_\alpha(\cdot) \) the probability generating function (p.g.f.) of \( \{p_{\alpha,k}\}_{k \geq 0} \). Moreover, we suppose that \( G(0^+) = 0 \), that is, there is null probability of instantaneous death and consequently \( v_\alpha(0) = 0 \). Then, from [2] (see p. 139), we deduce that \( v_\alpha(\cdot) \) is the unique bounded function such that

\[ v_\alpha(t) = \begin{cases} 0, & t < 0 \\ \int_0^t f_\alpha(v_\alpha(t - s))dG(s), & t \geq 0. \end{cases} \]  

(2)

This expression plays an important role in our study, together with the following relation between \( \alpha \) and the contact distribution law. Let \( m \) be the mean of contacts of an infected individual and let \( m_\alpha \) be the mean of susceptible individuals, which are infected by a contagious individual, given that the proportion of immune individuals in the population is \( \alpha \). Then, from (1) it is easy to calculate that

\[ m_\alpha = (1 - \alpha)m. \]  

(3)
Also, if we denote by \( f(\cdot) \) the p.g.f. of \( \{p_k\}_{k \geq 0} \), then it is easy to obtain that
\[
f_\alpha(s) = f(\alpha + (1 - \alpha)s), \quad 0 \leq s \leq 1.
\] (4)

Moreover, let \( q_\alpha \) be the extinction probability of an age-dependent branching process with reproduction law \( \{p_{\alpha,k}\}_{k \geq 0} \). It is clear that \( q_\alpha = P(T_\alpha < \infty) \) and is also well known that \( q_\alpha = 1 \) if \( m_\alpha \leq 1 \). So that, for such an \( \alpha \) for which \( m_\alpha > 1 \), \( v_\alpha(\cdot) \) is the d.f. of a non-proper r.v. because \( P(T_\alpha < \infty) < 1 \). In any case, it follows that
\[
\tilde{v}_\alpha(t) = P(T_\alpha \leq t|T_\alpha < \infty) = \frac{v_\alpha(t)}{q_\alpha}, \quad t \geq 0,
\]
and from (2), we obtain that \( \tilde{v}_\alpha(\cdot) \) also satisfies the similar type of equation, i.e.
\[
\tilde{v}_\alpha(t) = \int_0^t g_\alpha(\tilde{v}_\alpha(t-s))dG(s), \quad t \geq 0,
\]
where \( g_\alpha(s) = q_\alpha^{-1}f_\alpha(q_\alpha s) \) is a p.g.f. of a reproduction law with mean less than one.

Therefore, from now on, we consider \( \alpha \) such that the extinction time \( T_\alpha \) is a finite r.v., i.e. \( m_\alpha \leq 1 \), which implies that the infectious disease becomes extinct almost surely (a.s.). Taking into account (3), \( m_\alpha \leq 1 \) is equivalent to \( \max\{0,1-m^{-1}\} \leq \alpha \leq 1 \), which depends on the mean of contacts of an infected individual. In order to simplify the notations, from now on we denote by \( \alpha_{m} = \max\{0,1-m^{-1}\} \) the smallest proportion of immune individuals, so that the infectious disease becomes extinct a.s. Notice that the corresponding mean \( m_{\alpha_{m}} \) is the greatest mean number of susceptible individuals catching the disease by an infected individual, so that it is guaranteed that the disease becomes extinct a.s. Moreover, \( m_1 = 0 \), that is, the infectious disease does not spread to any susceptible individual.

The d.f. \( v_\alpha(\cdot) \) inherits some properties of the d.f. \( G(\cdot) \). Both of them have support on the non-negative real numbers. Moreover, when \( G(\cdot) \) is a d.f. of a discrete life-length, then obviously the extinction time d.f. is too. And, \( v_\alpha(\cdot) \) will be also an absolutely continuous d.f. if \( G(\cdot) \) is. Indeed, for all \( t \geq 0 \),
\[
v_\alpha(t) = \int_0^t f_\alpha(v_\alpha(t-s))dG(s) \\
= f_\alpha(0)G(t) + (1 - f_\alpha(0)) \int_0^t F_\alpha(t-s)dG(s),
\] (5)

with \( F_\alpha(y) = (1 - f_\alpha(0))^{-1}(f_\alpha(v_\alpha(y)) - f_\alpha(0)) \) for \( y \geq 0 \). Taking into account that \( f_\alpha(\cdot) \) is a p.g.f. and \( v_\alpha(\cdot) \) is a d.f., then \( F_\alpha(\cdot) \) is also a d.f. on non-negative real numbers, and therefore
\[
\int_0^t F_\alpha(t-s)dG(s) = \int_0^\infty F_\alpha(t-s)dG(s) = (F_\alpha * G)(t),
\]
is the convolution of \( F_\alpha(\cdot) \) and \( G(\cdot) \). Since \( G(\cdot) \) is an absolutely continuous d.f., it is well-known that \( F_\alpha * G(\cdot) \) is also an absolutely continuous d.f.
Vaccination policies

Fig. 1 Behaviour of \( v_\alpha(\cdot) \) depending on \( \alpha \).

(see [5], p. 272). Therefore, since \( v_\alpha(\cdot) \) is a convex linear combination of two absolutely continuous d.f., then it is also an absolutely continuous d.f.

Furthermore, from (2) we obtain that when every individual is immune, i.e. \( \alpha = 1 \), the extinction time is given by the life-length of the initial infected individual, that is, \( v_1(t) = G(t) \) for all \( t \geq 0 \). It stands to reason that if there are not immune individuals into the population, then it is probable that the infectious disease takes more time to become extinct. In the following result, we show this fact investigating the behaviour of \( v_\alpha(\cdot) \) depending on the parameter \( \alpha \) and when the contact distribution law is fixed.

**Theorem 1** If \( \alpha_{inf} \leq \alpha_1 < \alpha_2 \leq 1 \), then \( v_{\alpha_1}(t) \leq v_{\alpha_2}(t) \), for all \( t \geq 0 \).

Intuitively, it is clear that the greater is the proportion of the immune individuals, the more probable is that the infectious disease disappears faster. Consequently, for any \( \alpha \) with \( \alpha_{inf} \leq \alpha \leq 1 \), the d.f. \( v_\alpha(\cdot) \) is upper bounded by \( v_1(\cdot) = G(\cdot) \) and lower bounded by \( v_{\alpha_{inf}}(\cdot) \). Furthermore, all of them are lower bounded by \( v_0(\cdot) \), which is not necessary to be a proper d.f. In Fig. 1, we show this behaviour depending on \( \alpha \), for an arbitrary d.f. \( v_\alpha(\cdot) \). Moreover, we obtain that minor change in the proportion of the immune individuals generates minor change in the extinction time.

**Theorem 2** Let \( \alpha \) be such that \( m_\alpha < m_{\alpha_{inf}} \). For each \( \varepsilon > 0 \) there exist \( \eta = \eta(\varepsilon, \alpha) > 0 \) and \( \alpha^* \), with \( m_{\alpha^*} \leq 1 \) and \( |\alpha - \alpha^*| \leq \eta \), such that

\[
\sup_{0 \leq t < \infty} |v_\alpha(t) - v_{\alpha^*}(t)| \leq \varepsilon.
\]

More specifically, we have proved the continuity of the d.f. \( v_\alpha(\cdot) \) depending on \( \alpha \), for \( \alpha_{inf} < \alpha \leq 1 \). Notice that \( \alpha_{inf} \) has been excluded, which matches with \( m_\alpha = \min\{1, m\} \). Moreover, the continuity is uniform along the time. Then, from the previous theorems, for a fixed \( t \geq 0 \), we have that the function \( v_\alpha(t) \) is non-decreasing and continuous on \( \alpha \), for \( \alpha_{inf} < \alpha \leq 1 \). In Fig. 2 we show this behaviour for an arbitrary d.f. \( v_\alpha(\cdot) \).
Furthermore, some parameters of $T_\alpha$ inherit these properties of $v_\alpha(\cdot)$. In what follows we investigate the monotony and the continuity properties, first of the quantiles and then of the mean, both of the distribution of the infection extinction time, depending on the proportion of the immune individuals into the population.

For fixed $\alpha$ and $p$, with $\alpha_{\text{inf}} \leq \alpha < 1$ and $0 < p < 1$, we denote by $t_\alpha^p = \inf\{t : v_\alpha(t) \geq p\}$, that is, the quantile of order $p$ of the variable $T_\alpha$. Since $v_\alpha(\cdot) = G(\cdot)$, then $t_\alpha^p$ is the quantile of order $p$ of the life-length defined by the d.f. $G(\cdot)$. Moreover, for every $p$, we have the following result.

**Theorem 3** Let $p$ be such that $0 < p < 1$.

1. If $\alpha_{\text{inf}} \leq \alpha_1 < \alpha_2 < 1$ then $t_{\alpha_1}^p \leq t_{\alpha_2}^p$.

2. If $\alpha$ is such that $0 < m_\alpha < m_{\alpha_{\text{inf}}}$, then $\lim_{\alpha \to \alpha^{-}} t_\alpha^p = t_1^p$.

Moreover,  
(a) If $v_\alpha(t_\alpha^p) = p$, then $t_\alpha^p \leq \lim_{\alpha \to \alpha^{-}} t_\alpha^p \leq t^*$, with $t^* = \sup\{t : v_\alpha(t) = p\}$.  
(b) If $v_\alpha(t_\alpha^p) > p$, then $\lim_{\alpha \to \alpha^{-}} t_\alpha^p = t_\alpha^p$.  
(c) If $v_\alpha(\cdot)$ is an increasing and absolutely continuous function, then $\lim_{\alpha \to \alpha^{-}} t_\alpha^p = t_\alpha^p$.

Therefore, for any fixed probability $p$, the increasing of the vaccination level will cause the decreasing of the time to extinction of infectious disease. Consequently, for any $\alpha$ with $\alpha_{\text{inf}} \leq \alpha < 1$, $t_\alpha^p$ is upper bounded by $t_{\alpha_{\text{inf}}}^p$ and lower bounded by $t_1^p$. This behaviour is shown in the left graphic of Fig. 3. Moreover, for $\alpha_{\text{inf}} < \alpha < 1$, we have obtained that the function $t_\alpha^p$, depending on $\alpha$ is right-continuous for fixed $p$. On the other hand, its left-hand limit depends on the behaviour of $v_\alpha(\cdot)$ close to $t_\alpha^p$. Notice that if $G(\cdot)$ is an increasing and absolutely continuous function defined on the non-negative real numbers, we deduce from (5) that $v_\alpha(\cdot)$ is too of the same
Vaccination policies

Fig. 3 Behaviour of $t^\alpha_p$ depending on $\alpha$ for a fixed $p$.

type and therefore, for $\alpha_{inf} < \alpha \leq 1$, $t^\alpha_p$ is a continuous function depending on $\alpha$. In the right graphic of Fig. 3 the behaviour of $t^\alpha_p$ depending on $\alpha$ for a fixed $p$ and, for an arbitrary increasing and absolutely continuous function $G(\cdot)$ is presented.

Finally, we denote by $\mu_\alpha$ the mean of time to extinction of infectious disease when the proportion of immune individuals is $\alpha$. Since $T_\alpha$ is a non-negative r.v., then

$$\mu_\alpha = E[T_\alpha] = \int_0^\infty (1 - v_\alpha(t))dt, \quad (6)$$

The function $\mu_\alpha$ has the following monotony and continuity properties, depending on $\alpha$.

**Theorem 4**

1. If $\alpha_{inf} < \alpha_1 < \alpha_2 \leq 1$, then $\mu_{\alpha_2} \leq \mu_{\alpha_1}$.
2. If $\bar{\alpha}$ is such that $0 < m_{\bar{\alpha}} < m_{\alpha_{inf}}$ and $\sup\{\mu_\alpha : \bar{\alpha} < \alpha \leq 1\} < \infty$, then $\mu_{\bar{\alpha}}$ is finite and $\mu_{\bar{\alpha}} = \lim_{\bar{\alpha} \to \alpha} \mu_\alpha$. Moreover, for all $\alpha$ with $\bar{\alpha} < \alpha \leq 1$, it follows that $\lim_{\bar{\alpha} \to \alpha} \mu_{\bar{\alpha}} = \mu_\alpha$.

Since it is more probable that the infectious disease becomes extinct faster when the proportion of the immune individuals is greater, then the mean of time to extinction is lesser. Therefore, the mean of time to extinction for any proportion of immune individuals is lower bounded by the mean of life-length defined by $G(\cdot)$. When the expectation $\mu_\alpha$ is finite we have proved right-hand continuity on $\alpha$. Moreover, if $\mu_{\alpha_{inf}}$ is finite, then we have continuity for $\alpha_{inf} < \alpha \leq 1$.

3 Determining vaccination policies

When the infectious disease is strongly detrimental for the population, then it is of interest to eliminate the disease almost surely as far as possible. To this
end, we control the spread of the disease by immunizing some proportion of susceptible individuals, which in other words means we are applying a vaccination policy. However, this proportion of susceptible individuals to be vaccinated depends on the time that we allow the infectious disease to survive after vaccination.

In this section we propose two methods of obtaining optimal proportion of susceptible individuals to be immunized. Let us remind that we model the spread of disease by an age-dependent branching process as follows. Without lost generality, we suppose that before vaccination, every healthy individual which is in contact with an infected individual is not immune, i.e. the contact always produces the infection. Then, before the vaccination with probability \( p_k \) an infected individual passes the disease on \( k \) susceptible individuals. At an arbitrary time \( t_0 \) after the infection occurred into the population, we vaccinate a proportion \( \alpha \) of susceptible individuals. We suppose that the vaccination process is instantaneous and that every vaccinated individual is immune to the infectious disease from this instant of time. Then after the vaccination with probability \( p_{\alpha,k} \) (see (1)) an infected individual transmits the disease on \( k \) susceptible individuals. In the left graphic of Fig. 4, we show a simulated evolution of the number of infected individuals before and after vaccination moment.

In order to guarantee the extinction of the disease a.s., \( \alpha \) should be at least equal to \( \alpha_{inf} \). Taking into account the results of the previous section, we concluded that the increasing of the vaccination level leads to the decreasing of the extinction time of the infection. Obviously, the best is to vaccinate all the population, but it is not reasonable from practical standpoint in most of cases. That is why, we are trying to propose two possible ways of defining optimal proportion of vaccinated individuals, to guarantee that the infection terminates by given instant of time. One is based on the quantiles of the extinction time \( T_\alpha \) and the other one is based on the mean of \( T_\alpha \). Furthermore, we propose simulation-based method to apply these two vaccination policies.
3.1 Based on the quantiles of the extinction time distribution

In this paragraph, for fixed $p$ and $t$, with $0 < p < 1$ and $t > 0$, vaccination policies which guarantee that infectious disease becomes extinct, with probability greater than or equal to $p$, no later than time $t$ after vaccination, will be investigated. We determine some of these vaccination policies applying the results of the previous section as follows. Let us suppose that at the vaccination moment $t_0$, there is a single infected individual into the population and we vaccinate a proportion $\alpha$ of susceptible individuals. Since the infected individual might have already lived some time before vaccination time (see Fig. 4, the right graphic), then the probability that the disease becomes extinct no later than time $t_0 + t$ is greater than or equal to $v_\alpha(t)$. In Appendix A a mathematical justification of this fact is provided.

If there are $z$ infected individuals into the population at vaccination moment and since each individual reproduces independently of all others, then the d.f. of the extinction time of the disease is bounded by $v_\alpha(t)^z$. Therefore, any vaccination level $\alpha$ such that $v_\alpha(t) \geq p(z)$ or equivalently $t_\alpha^{p(z)} \leq t$, with $p(z) = p^{1/z}$ could be used. Taking this fact into account, we propose as optimal vaccination policy that one, which corresponds to the smallest $\alpha$ of all of them, that is,

$$\alpha_q = \alpha_q(p, t, z) = \inf\{\alpha : \alpha_{inf} \leq \alpha \leq 1, v_\alpha(t) \geq p(z)\}$$

$$= \inf\{\alpha : \alpha_{inf} \leq \alpha \leq 1, t_\alpha^{p(z)} \leq t\}.$$

Applying the results of the previous section we have that $v_{\alpha_q}(t) \geq p(z)$ and $t_\alpha^{p(z)} \leq t$ if $\alpha_q > \alpha_{inf}$. So, vaccinating a proportion $\alpha_q$ of susceptible individuals, with probability $p$ we guarantee that the infectious disease becomes extinct no latter than time $t$ after vaccination. Notice that since $(v_\alpha(t))^z$ is a lower bound of the probability of interest, then some $\alpha$ less than $\alpha_q$ could also be followed to this aim. In the left graphic of Fig. 5 we determine $\alpha_q$ from the behaviour of $v_\alpha(t)$ for fixed $t$ and in the right graphic of Fig. 5 we determine $\alpha_q$ from the behaviour of $t_\alpha^{p(z)}$ for fixed $p$. Notice that, although $t$ and $p$ have been fixed arbitrarily, in order to find a solution of the problem, it is necessary that $t \geq t_1^{p(z)}$ or equivalently $p(z) \leq v_1(t)$. Moreover, since $0 < p < 1$, then $p \leq p^{(z)}$ and $p^{(z)}$ is closed to 1 if $z$ is large enough. In Fig. 6 we show the behaviour of $\alpha_q$ depending on $t$ and $p$. Taking into account the monotony properties of the functions $v_\alpha(t)$ and $t_\alpha^{p(z)}$ (depending on $\alpha$), we obtain when $p$ increases or $t$ decreases then the optimal proportion to be vaccinated increases.

3.2 Based on the mean value of the extinction time distribution

Now, for fixed $t > 0$, we are interested in vaccination policies, which guarantee that the average time to extinction of infection after vaccination is less or equal to $t$. Using a similar argument as above, we obtain that the mean of extinction time is bounded by $z\mu_0$ when the vaccinated proportion is $\alpha$ and there are $z$ infected individuals at the moment of vaccination. Therefore, any
vaccination level $\alpha$ such that $\mu_\alpha \le t^{(z)}$, with $t^{(z)} = \frac{z^{-1}}{t}$ could be followed. The optimal vaccination policy is that one which corresponds to the smallest $\alpha$ of all of them, that is,

$$\alpha_m(\alpha_m(t, z) = \inf \{ \alpha : \alpha_{inf} \le \alpha \le 1, \mu_\alpha \le t^{(z)} \}. $$

Taking into account the results of the previous section we have that $\mu_{\alpha_m} \le t^{(z)}$ if $\alpha_m > \alpha_{inf}$. Therefore, vaccinating a proportion $\alpha_m$ of susceptible individuals, the infectious disease becomes extinct in average, no latter than time $t$ after vaccination. Notice that since $z\mu_\alpha$ is an upper bound of the mean of interest, then some $\alpha$ less than $\alpha_m$ could also be followed. Moreover,
although $t$ has been chosen arbitrarily, in order to find a solution of the problem, it is necessary that $t^{(z)} \geq \mu_1$. If $z$ is large enough compared to $t$, then $t^{(z)}$ is closed to 0 and $\alpha_m$ is closed to 1.

3.3 A simulation-based method for determining vaccination policies

In the previous paragraphs we have proposed vaccination policies defined by $\alpha_q$ and $\alpha_m$, respectively. These vaccination policies depend on the d.f. of time to extinction. Therefore, to calculate $\alpha_q$ and $\alpha_m$, it is necessary to know $v_q(\cdot)$ for $\alpha$ such that $\alpha_{inf} \leq \alpha \leq 1$. Although $v_q(\cdot)$ satisfies (2), in general it is not possible to obtain this function in a closed-form. Nowadays, more and more numeric and simulation methods have been provided in order to approximate the function satisfying (2) (see [9]). In this section we determine $\alpha_q$ and $\alpha_m$ approximating $v_q(\cdot)$ by means of a simulation-based method when $\{p_k\}_{k \geq 0}$ and $G(\cdot)$ are known. When $\alpha$ is fixed, such that $\alpha_{inf} \leq \alpha \leq 1$, we apply the Monte-Carlo method to estimate the empirical d.f. of extinction time when the proportion of immune individuals is $\alpha$. Taking different $\alpha$’s sufficiently closed, then we approach $\alpha_q$ and $\alpha_m$ from their definitions.

This method is illustrated by the following example. Let us suppose that an infectious disease is strongly damaging for the population, so that the vaccination is necessary even if there are few infected individuals into the population. We suppose that at the vaccination time there are 3 infected individuals, that is, $z = 3$. Taking $p = 0.9$ and $t = 7$, we have that $p^{(z)} = 0.965$ and $t^{(z)} = 2.333$. In order to simulate the spread of the disease when the proportion of immune individuals is $\alpha$, it is enough to know $G(\cdot)$ and $\{p_k\}_{k \geq 0}$. Let the life-length of an infected individual follows a gamma distribution with parameters equal to one and the contact distribution law follows Poisson distribution with parameter $m$. This type of distribution has been related to such kind of problems (see [6]). From (4) we have that

$$f_\alpha(s) = f(\alpha + (1 - \alpha)s) = e^{-m(1 - \alpha)(1 - \alpha)s} = e^{-m \alpha (1 - s)}, \quad 0 \leq s \leq 1,$$

which means that infection distribution law also follows Poisson distribution with parameter $m_\alpha = (1 - \alpha)m$, which is the expectation of susceptible individuals, catching the disease from infected individuals. Notice that, for fixed $m$, $\alpha$ is determined one-to-one by $m_\alpha$. Therefore, instead of calculating $\alpha_q$ and $\alpha_m$, we determine $m_q = (1 - \alpha_q)m$ and $m_m = (1 - \alpha_m)m$. Since $(1 - \alpha)m$ is a decreasing function on $\alpha$, from the definition of $\alpha_q$ and $\alpha_m$, we obtain that

$$m_q = m_q(t, p, z) = \sup\{m_p : 0 \leq m_p \leq 1, u_{m_p}(t) \geq p^{(z)}\},$$

$$m_m = m_m(t, z) = \sup\{m_p : 0 \leq m_p \leq 1, \nu_{m_p} \leq t^{(z)}\},$$

where $u_{m_p}(\cdot)$ and $\nu_{m_p}$ are the d.f. and the mean of the extinction time, respectively, when infection distribution law follows Poisson distribution with parameter $m_p$. Notice that $v_q(\cdot) = u_{m_q}(\cdot)$, and that $m_q$ and $m_m$ are independent of the magnitude of $m$. 

Vaccination policies
Therefore, to approximate $m_q$ and $m_m$, we only need to obtain the empirical distribution $u_{m_p}()$, for $0 \leq m_p \leq 1$, using the Monte-Carlo method. To this end, for each fixed $m_p$, 10,000 processes have been simulated and their extinction time have been calculated. In Fig. 7 the behaviour of empirical d.f. $u_{m_p}()$ for several $m_p$’s is showed. Notice that the behaviour is similar to that presented in Fig. 2 for $v_{\alpha}()$. Now, increasing $m_p$ the extinction time also increases (stochastically).

The behaviour of estimated value of $u_{m_p}(7)$, jointly with an upper confidence bound at level 95%, depending on $m_p$ is given in the left graphic of Fig. 8. It is illustrated that given $p^{(z)} = 0.965$, an approximation of $m_q(7,0.9,3)$ is 0.56. In the right graphic of Fig. 8 we present the surface of $m_q(t,p,3)$ depending on $t$ and $p$. This behaviour is similar to that shown in Fig. 6 for $\alpha_q$.

The estimation of $m_m$, jointly with an upper confidence bound at level 95%, depending on $m_p$ is given in the left graphic of Fig. 9. It is illustrated that an approximation of $m_m(7,3)$ is 0.66, given that $t^{(z)} = 2.333$. Notice that $u_{0.66}(7) = 0.834 < p^{(z)}$. Therefore, the policy based on the quantiles suggests us to vaccinate greater proportion of the population than that based on the mean, in this example. Finally, in the right graphic of Fig. 9, we illustrate the proportion of individuals to be vaccinated depending on $m$ and taking into account $m_m(t,3)$. Notice that, if the mean $m$ of contacts per individual is closed to 1.32, then we need to vaccinate the half of the population in order to guarantee that the infectious disease becomes extinct, in mean, no latter than time $t = 7$ after vaccination.

**Remark 1** For the computer simulation, we used the language and environment for statistical computing and graphics R (“GNU S”) (see [12]).
Vaccination policies

Fig. 8 Left graphic: Behaviour of estimated value of $u_{mp}(7)$, jointly with an upper confidence bound at level 95%, depending on $m_p$ (dotted line). Right graphic: Behaviour of $m_q(t, p, 3)$ depending on $t$ and $p$.

Fig. 9 Left graphic: Estimation of $m_m$, jointly with an upper confidence bound at level 95%, depending on $m_p$ (dotted line). Right graphic: Proportion of the population to be vaccinated depending on $m$ and taking into account $m_m(7, 3) = 0.66$.

4 Proofs

In this section we provide the proofs of the results of the paper. For each $\alpha$ such that $\alpha_{inf} \leq \alpha \leq 1$, we introduce the functional operator $H_\alpha(\cdot)$, defined on a distribution function $u(\cdot)$ of a non-negative r.v., as follows

$$H_\alpha(u)(t) = \int_0^t f_\alpha(u(t-s))dG(s), \ t \geq 0.$$ 

Also, for all $n \geq 1$, we denote by $H_\alpha^n(\cdot)$ the $n$-th composition of the operator $H_\alpha(\cdot)$. With this notation, (2) can be rewritten as $v_\alpha(t) = H_\alpha(v_\alpha)(t), \ t \geq 0$. 
Proof of Theorem 1

Let $\alpha_1$, $\alpha_2$ be such that $\alpha_{\text{inf}} \leq \alpha_1 < \alpha_2 \leq 1$. In order to obtain the result it is enough to prove the following five statements:

S1. For all $t \geq 0$, $H_{\alpha_1}(v_{\alpha_1})(t) \leq H_{\alpha_2}(v_{\alpha_1})(t)$.

S2. $H_{\alpha_2}(v_{\alpha_1})(\cdot)$ is a distribution function.

S3. For all $n \geq 1$ and $t \geq 0$, $H_{\alpha_2}^n(v_{\alpha_1})(t) \leq H_{\alpha_2}^{n+1}(v_{\alpha_1})(t)$.

S4. For all $t \geq 0$, there exists $u(t) = \lim_{n \to \infty} H_{\alpha_2}^n(v_{\alpha_1})(t)$ and $u(\cdot)$ is a distribution function.

S5. $u(\cdot) = H_{\alpha_2}(u)(\cdot)$ and $u(\cdot) = v_{\alpha_2}(\cdot)$.

Indeed, from these statements and (2) we have for all $n \geq 1$ and $t \geq 0$, that

$$v_{\alpha_1}(t) = H_{\alpha_1}(v_{\alpha_1})(t) \leq H_{\alpha_2}(v_{\alpha_1})(t) \leq H_{\alpha_2}^n(v_{\alpha_1})(t) \leq u(t) = v_{\alpha_2}(t).$$

It remains to prove the statements S1-S5.

S1. Since $v_{\alpha_1}(\cdot)$ is a distribution function and $\alpha_1 < \alpha_2$, then

$$\alpha_1 + (1 - \alpha_1) v_{\alpha_1}(t - s) \leq \alpha_2 + (1 - \alpha_2) v_{\alpha_1}(t - s)$$

for all $0 \leq s \leq t$. Therefore

$$f_{\alpha_1}(v_{\alpha_1}(t - s)) = f(\alpha_1 + (1 - \alpha_1) v_{\alpha_1}(t - s)) \leq f(\alpha_2 + (1 - \alpha_2) v_{\alpha_1}(t - s)) = f_{\alpha_2}(v_{\alpha_1}(t - s)),$$

and the statement is shown.

S2. Reasoning as in (5), we obtain, for all $t \geq 0$, that

$$H_{\alpha_2}(v_{\alpha_1})(t) = f_{\alpha_2}(0) G(t) + (1 - f_{\alpha_2}(0))(F_{\alpha_1, \alpha_2} * G)(t),$$

where $F_{\alpha_1, \alpha_2}(t) = (1 - f_{\alpha_2}(0))^{-1}(f_{\alpha_2}(v_{\alpha_1}(t)) - f_{\alpha_2}(0))$ is a d.f. So $H_{\alpha_2}(v_{\alpha_1})(\cdot)$ will be also a d.f. as a convex linear combination of two d.f.

S3. Using S1 and (2), we have that $v_{\alpha_1}(t) \leq H_{\alpha_2}(v_{\alpha_1})(t)$, for all $t \geq 0$. Since $f_{\alpha_2}(\cdot)$ is an increasing function, applying an iterative method we obtain the statement.

S4. By S2 we obtain that $H_{\alpha_2}^n(v_{\alpha_1})(\cdot)$ is a distribution function, for all $n \geq 1$. Moreover, by S3, we have that $(H_{\alpha_2}^n(v_{\alpha_1})(t))_{n \geq 1}$ is a non-decreasing and bounded sequence, for each $t \geq 0$, and therefore the statement is shown.

S5. Applying S4, the continuity of $f_{\alpha_2}(\cdot)$ and the dominated convergence theorem, it follows for each fixed $t \geq 0$, that

$$u(t) = \lim_{n \to \infty} H_{\alpha_2}^{n+1}(v_{\alpha_1})(t) \leq \lim_{n \to \infty} \int_0^t f_{\alpha_2}(H_{\alpha_2}^n(v_{\alpha_1})(t - s)) dG(s)$$

$$= \int_0^t f_{\alpha_2}(\lim_{n \to \infty} H_{\alpha_2}^n(v_{\alpha_1})(t - s)) dG(s)$$

$$= \int_0^t f_{\alpha_2}(u(t - s)) dG(s) = H_{\alpha_2}(u)(t).$$
Moreover, since \( v_{\alpha_2}(\cdot) = H_{\alpha_2}(v_{\alpha_2})(\cdot) \), then \( u(\cdot) = v_{\alpha_2}(\cdot) \), because only a bounded function is solution of (2) (see [2], p. 139).

**Remark 2** We notice that with minor changes in the previous proof, we prove that \( v_{\alpha_2}(t) \leq v_{\alpha_1}(t) \) for \( t \geq 0 \) and \( 0 \leq \alpha_1 < \alpha_2 \leq \alpha_f \).

**Proof of Theorem 2**

Let \( \varepsilon > 0 \) and let \( \alpha \) be such that \( m_\alpha < m_{\alpha_{\alpha_f}} = \min\{1, m\} \). Since \( f(\cdot) \) is an uniformly continuous function on \( \{ s : 0 \leq s \leq 1 \} \), then there exists \( \eta = \eta(\varepsilon, \alpha) > 0 \) such that \( |f(s) - f(s^*)| \leq \varepsilon(1 - m_\alpha) \), for all \( s, s^* \) with \( 0 \leq s, s^* \leq 1 \) and \( |s - s^*| \leq \eta \). Let \( \alpha^* \) be such that \( m_{\alpha^*} \leq 1 \) and \( |\alpha - \alpha^*| \leq \eta \). Since \( |\alpha + (1 - \alpha)s - (\alpha^* + (1 - \alpha^*)s)| \leq |\alpha - \alpha^*| \), for all \( 0 \leq s \leq 1 \), therefore from (4), it follows that

\[
\sup_{0 \leq s \leq 1} |f_\alpha(s) - f_{\alpha^*}(s)| \leq \varepsilon(1 - m_\alpha). \tag{7}
\]

Taking into account this fact, next we show by induction on \( n \), for each \( n \geq 1 \), that

\[
|H^n_\alpha(G)(t) - H^n_{\alpha^*}(G)(t)| \leq \varepsilon(1 - m_\alpha), \quad t \geq 0. \tag{8}
\]

Fixed \( t \geq 0 \), for \( n = 1 \) we deduce from (7), that

\[
|H_\alpha(G)(t) - H_{\alpha^*}(G)(t)| \leq \int_0^t |f_\alpha(G(t-s)) - f_{\alpha^*}(G(t-s))|dG(s) \leq \varepsilon(1 - m_\alpha).
\]

By induction hypothesis, (8) holds for \( n \). Then for \( n + 1 \) we have that

\[
|H^{n+1}_\alpha(G)(t) - H^{n+1}_{\alpha^*}(G)(t)| \leq |H_\alpha(H^n_\alpha(G))(t) - H_\alpha(H^n_{\alpha^*}(G))(t)|
+ |H_\alpha(H^n_{\alpha^*}(G))(t) - H_{\alpha^*}(H^n_{\alpha^*}(G))(t)|.
\]

Moreover

\[
|H_\alpha(H^n_\alpha(G))(t) - H_\alpha(H^n_{\alpha^*}(G))(t)| \leq
\leq \int_0^t |f_\alpha(H^n_\alpha(G)(t-s)) - f_{\alpha^*}(H^n_{\alpha^*}(G)(t-s))|dG(s)
\leq m_\alpha \sup_{0 \leq s^* < \infty} |H^n_\alpha(G)(s^*) - H^n_{\alpha^*}(G)(s^*)|
\leq \varepsilon(1 - m_\alpha)m_\alpha = \varepsilon(1 - m_\alpha)\sum_{k=1}^n m_\alpha^k.
\]

and, from (7),

\[
|H_\alpha(H^n_{\alpha^*}(G))(t) - H_{\alpha^*}(H^n_{\alpha^*}(G))(t)| \leq
\leq \int_0^t |f_{\alpha^*}(H^n_{\alpha^*}(G)(t-s)) - f_{\alpha^*}(H^n_{\alpha^*}(G)(t-s))|dG(s)
\leq \varepsilon(1 - m_\alpha).
\]
Therefore, we conclude that

$$|H_{\alpha}^{n+1}(G)(t) - H_{\alpha}^{n+1}(G)(t)| \leq \varepsilon(1 - m_\alpha) \sum_{k=1}^{n} m_\alpha^k + \varepsilon(1 - m_\alpha) = \varepsilon(1 - m_\alpha^{n+1}).$$

Following the similar steps as in S3-S5 of the proof of Theorem 1, we obtain that $v_\alpha(t) = \lim_{n \to \infty} H_{\alpha}^{n}(G)(t)$, for $\tilde{\alpha}$ such that $m_\alpha \leq 1$. Finally, taking this fact into account and that $m_\alpha < 1$, from (8), we obtain that

$$\sup_{0 \leq t < \infty} |v_\alpha(t) - v_\alpha^*(t)| \leq \varepsilon,$$

and therefore the proof is completed.

**Proof of Theorem 3**

Let $p$ be such that $0 < p < 1$.

1. Let $\alpha_1, \alpha_2$ be such that $\alpha_{inf} \leq \alpha_1 < \alpha_2 \leq 1$. Since $\alpha_1 < \alpha_2$, then from Theorem 1, we have that

$$v_{\alpha_2}(t_{\alpha_1}^p) \geq v_{\alpha_1}(t_{\alpha_1}^p) \geq p,$$

and therefore, by definition of $t_{\alpha_1}^p$, we deduce that $t_{\alpha_1}^{p_2} \leq t_{\alpha_1}^{p_1}$.

2. Let $\alpha$ be such that $0 < m_\alpha < m_{\alpha_{inf}}$. From 1, we guarantee the existence of $\lim_{\tilde{\alpha} \to \alpha} t_{\tilde{\alpha}}^{\alpha}$, which is equal to $T = \sup\{t_{\tilde{\alpha}}^{\alpha} : \tilde{\alpha} > \alpha\}$. Therefore $T \leq t_{\alpha}^{\alpha}$. On the other hand, from Theorem 2, we deduce that for each $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, \alpha) > 0$ such that

$$p - \varepsilon \leq v_{\alpha}(t_{\alpha}^{\alpha}) - \varepsilon \leq v_{\alpha}(t_{\alpha}^{T}) \leq v_{\alpha}(T),$$

for all $\tilde{\alpha}$ with $m_{\tilde{\alpha}} \leq 1$ and $0 < \tilde{\alpha} - \alpha \leq \eta$. Then $p \leq v_{\alpha}(T)$ and so $t_{\alpha}^{\alpha} \leq T$.

(a) Applying 1, we deduce that $\lim_{\tilde{\alpha} \to \alpha} t_{\tilde{\alpha}}^{\alpha}$ exists, it is equal to $\underline{T} = \inf\{t_{\tilde{\alpha}}^{\alpha} : \tilde{\alpha} \leq \alpha\}$ and that $t_{\alpha}^{\alpha} \leq \underline{T}$. Next, we prove that $\underline{T} \leq T^\ast$. We split the proof in two cases, $v_{\alpha}(T^\ast) > p$ and $v_{\alpha}(T^\ast) = p$. First we consider the case $v_{\alpha}(T^\ast) > p$. Let $\varepsilon = v_{\alpha}(T^\ast) - p$. Applying Theorem 2, we deduce that there exists $\eta = \eta(\varepsilon, \alpha) > 0$ such that

$$v_{\alpha}(T^\ast) - v_{\alpha}(t_{\alpha}^{\alpha}) \leq \varepsilon = v_{\alpha}(T^\ast) - p,$$

for all $\tilde{\alpha}$, with $m_{\tilde{\alpha}} \leq 1$ and $0 < \alpha - \tilde{\alpha} \leq \eta$. Then $p \leq v_{\alpha}(T^\ast)$ and therefore we have that $t_{\alpha}^{\alpha} \leq T^\ast$ and consequently $\underline{T} \leq T^\ast$.

Finally, we consider the case $v_{\alpha}(T^\ast) = p$. By definition of $T^\ast$, we have that $p < v_{\alpha}(t)$ for all $t > T^\ast$. For each $t > T^\ast$, let $\varepsilon = v_{\alpha}(t) - p$. Applying Theorem 2, we deduce that there exists $\eta = \eta(\varepsilon, \alpha) > 0$ such that

$$v_{\alpha}(t) - v_{\alpha}(t_{\alpha}^{\alpha}) \leq \varepsilon = v_{\alpha}(t) - p,$$

for all $\tilde{\alpha}$, with $m_{\tilde{\alpha}} \leq 1$ and $0 < \alpha - \tilde{\alpha} \leq \eta$. Then $p \leq v_{\alpha}(t)$, $t_{\alpha}^{\alpha} \leq t$ and $\underline{T} \leq t$, and consequently $\underline{T} \leq T^\ast$.

(b) It is proved as the previous case when $v_{\alpha}(T^\ast) > p$, replacing $T^\ast$ by $t_{\alpha}^{\alpha}$.

(c) From (a) we obtain that $\lim_{\alpha \to \alpha} t_{\alpha}^{\alpha} = t_{\alpha}^{\alpha}$, and the proof is completed.
Proof of Theorem 4

1. Let $\alpha_1$, $\alpha_2$ be such that $\alpha_{inf} \leq \alpha_1 < \alpha_2 \leq 1$. From Theorem 1, we have that $v_{\alpha_1}(t) \leq v_{\alpha_2}(t)$, $t \geq 0$, and taking into account (6), it follows that $\mu_{\alpha_2} \leq \mu_{\alpha_1}$.

2. Let $\overline{\alpha}$ be such that $0 < m_{\overline{\alpha}} < m_{\alpha^2}$ and $\sup\{\mu_{\alpha} : \overline{\alpha} < \alpha \leq 1\} < \infty$. We denote by $M = \sup\{\mu_{\alpha} : \overline{\alpha} < \alpha \leq 1\}$. First we show that $\mu_{\alpha}$ is finite. It is enough to prove that for every $\varepsilon > 0$ and $N > 0$,

$$\int_0^N (1 - v_{\overline{\alpha}}(t))dt \leq \varepsilon + M. \quad (9)$$

Indeed,

$$\mu_{\overline{\alpha}} = \lim_{N \to \infty} \int_0^N (1 - v_{\overline{\alpha}}(t))dt \leq \varepsilon + M,$$

for all $\varepsilon > 0$, and therefore $\mu_{\overline{\alpha}} \leq M < \infty$.

Let us prove (9) for fixed $\varepsilon > 0$ and $N > 0$. Applying Theorem 2, there exists $\eta = \eta(\overline{\alpha}, \varepsilon, N)$ such that for all $\alpha > \overline{\alpha}$, with $\alpha - \overline{\alpha} \leq \eta$, if follows that

$$v_{\alpha}(t) - v_{\overline{\alpha}}(t) \leq N^{-1}\varepsilon, \ t \geq 0.$$

Therefore,

$$\int_0^N (1 - v_{\overline{\alpha}}(t))dt \leq \int_0^N (N^{-1}\varepsilon + 1 - v_{\alpha}(t))dt$$

$$= \varepsilon + \int_0^N (1 - v_{\alpha}(t))dt$$

$$\leq \varepsilon + M.$$

Finally, we prove the continuity properties. Let $\varepsilon > 0$. Since $\mu_{\overline{\alpha}}$ is finite, then there exists a non-negative number $n_0 = n_0(\varepsilon, \overline{\alpha}) > 0$ such that

$$\int_{n_0}^{\infty} (1 - v_{\overline{\alpha}}(t))dt \leq 2^{-1}\varepsilon. \quad (10)$$

Let $\alpha$ be such that $\alpha \geq \overline{\alpha}$. Then, after applying Theorem 2, we guarantee that there exists $\eta = \eta(\alpha, \varepsilon, n_0) > 0$ such that if $|\overline{\alpha} - \alpha| \leq \eta$, then $|v_{\overline{\alpha}}(t) - v_{\alpha}(t)| \leq (2n_0)^{-1}\varepsilon$ for all $t \geq 0$, and therefore

$$\int_0^{n_0} |v_{\overline{\alpha}}(t) - v_{\alpha}(t)|dt \leq 2^{-1}\varepsilon.$$

Moreover, since (10) holds, from Theorem 1, we have, for $\overline{\alpha} \geq \overline{\alpha}$, that

$$\int_{n_0}^{\infty} |v_{\overline{\alpha}}(t) - v_{\alpha}(t)|dt \leq 2^{-1}\varepsilon,$$

and the proof is completed.
Appendix A

We consider an age-dependent branching process initiated with one individual, with reproduction law \( \{p_k, \alpha\}_{k \geq 0} \), where \( 0 \leq \alpha \leq 1 \), with d.f. of the initial progenitor’s life-length \( G^*(\cdot) \) and with distribution function of the life-length \( G(\cdot) \) for other individuals. We suppose that \( G^*(t) \geq G(t) \) for all \( t \geq 0 \). In epidemiological context, this condition reflects the fact that the life time distribution \( G^*(\cdot) \) of the initial individual after vaccination, is always less or equal to its total life time, given by \( G(\cdot) \).

We denote by \( \hat{T}_\alpha \) the extinction time of such an age-dependent branching process. Also, we denote by \( \hat{v}_\alpha(\cdot) \) the d.f. of the extinction time \( \hat{T}_\alpha \), that is, \( \hat{v}_\alpha(t) = P(\hat{T}_\alpha \leq t) \), for all \( t \in \mathbb{R} \). Following a heuristic derivation as in [2] (see p. 138) we obtain the integral equation

\[
\hat{v}_\alpha(t) = \int_0^t f_\alpha(v_\alpha(t - s))dG^*(s), \quad t \geq 0. \tag{11}
\]

From (5) and (11) for all \( t \geq 0 \) one obtains

\[
v_\alpha(t) = f_\alpha(0)G(t) + (1 - f_\alpha(0))(F_\alpha * G)(t)
\]

and

\[
\hat{v}_\alpha(t) = f_\alpha(0)G^*(t) + (1 - f_\alpha(0))(F_\alpha * G^*)(t),
\]

where recall \( F_\alpha * G(\cdot) \) means the convolution of \( F_\alpha(\cdot) \) and \( G(\cdot) \), with \( F_\alpha(y) = (1 - f_\alpha(0))^{-1}(f_\alpha(y) - f_\alpha(0)) \), for all \( y \geq 0 \). Since \( G^*(t) \geq G(t) \) for all \( t \geq 0 \), then \( (F_\alpha * G^*)(t) \geq (F_\alpha * G)(t) \) for all \( t \geq 0 \) and therefore \( \hat{v}_\alpha(t) \geq v_\alpha(t) \), for all \( t \geq 0 \).

References