ON THE LIMIT BEHAVIOUR OF A SUPERADDITIVE BISEXUAL GALTON–WATSON BRANCHING PROCESS

M. GONZALEZ,* AND M. MOLINA,* Universidad de Extremadura

Abstract

The asymptotic behaviour of a superadditive bisexual Galton–Watson branching process is studied. Sufficient conditions for the almost sure and \( L^1 \) convergence of the suitably normed process are given. Finally, a first approach to the study of the \( L^1 \) convergence for a superadditive bisexual Galton–Watson branching process under the \( Z \log^4 Z \) condition is considered.

BISEXUAL GALTON–WATSON BRANCHING PROCESS; MARTINGALES; ALMOST SURE CONVERGENCE; \( L^1 \) CONVERGENCE

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1. Introduction

In this paper we study different aspects of the asymptotic behaviour of a bisexual Galton–Watson branching process (BGWBp). Introduced by Daley (1968), this process is a modification of the standard Galton–Watson process, which we can describe as follows.

Definition. Let \( \{(F_n, M_n) : n = 0, 1, \ldots; i = 1, 2, \ldots\} \) be a family of integer-valued, independent and identically distributed bivariate random variables. Let \( L: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) be a function which is monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that \( L(x, y) \leq xy \). We define processes \( \{Z_n\}_n \) and \( \{(F_n, M_n)\}_n \) by the iterative relation

\[
Z_0 = N \geq 1, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (F_i, M_i), \quad Z_{n+1} = L(F_{n+1}, M_{n+1})
\]

for \( n = 0, 1, \ldots \), with the empty sum defined to be \((0, 0)\).

In this model \( F_n \) and \( M_n \) represent, respectively, the number of females and males in the \( n \)th generation, \( n = 1, 2, \ldots \), which form \( Z_n = L(F_n, M_n) \) mating units. These reproduce independently with the same offspring distribution.

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* Postal address: Departamento de Matemáticas, Fac. de Ciencias, Universidad de Extremadura, 06071-Badajoz, Spain. E-mail: mvelasco@ba.unex.es
Using the independence assumption for the random variables \((F_{ni}, M_{ni})\) and (1.1), it is easy to verify that \(\{Z_n\}_n\) is a Markov chain with the non-negative integers as state space, the state 0 being absorbing and with stationary one-step transition probabilities.

We assume that the offspring distribution and the mating function \(L(x, y)\) are such that

\[
P(Z_n \to 0) + P(Z_n \to \infty) = 1
\]

holds. Daley (1968) introduces an important situation in which (1.2) holds.

**Daley’s model.** Let \(T_{ni} = F_{ni} + M_{ni}\) for \(i = 1, \ldots, Z_n\). The term \(T_{ni}\) denotes the total number of offspring produced by the \(i\)th mating unit in the \(n\)th generation. He supposes that \(\{T_{ni}\}_n\) are integer-valued, independent and identically distributed random variables and that each offspring is female with probability \(\alpha\) \((0 < \alpha < 1)\) or male with probability \(1 - \alpha\).

Daley et al. (1986) give a more general condition which guarantees that (1.2) holds. They consider that \(P(Z_{n+1} = j \mid Z_n = j) < 1, j = 1, 2, \ldots,\) is satisfied.

We say that a BGWBP is superadditive if the mating function, \(L(x, y)\), satisfies, for all \(x_1, x_2, y_1, y_2\) in \(\mathbb{R}^4\),

\[
L(x_1 + x_2, y_1 + y_2) \geq L(x_1, y_1) + L(x_2, y_2).
\]

Daley (1968), Hull (1982, 1984), Bruss (1984) and Daley et al. (1986) have studied the problem of finding necessary and sufficient conditions for the process to become extinct (i.e. \(Z_n = 0\) for some \(n\)) with probability one. The main result is based on the concept of mean growth rate, defined by \(r_k = k^{-1}E[Z_{n+1} \mid Z_n = k]\), for \(k = 1, 2, \ldots\).

**Theorem** (Daley et al. 1986). In a superadditive BGWBP the mean growth rates satisfy \(r = \lim_{j \to \infty} r_j = \sup_{j > 0} r_j\), and

\[
q_j = \lim_{n \to \infty} P(Z_n = 0 \mid Z_0 = j) = 1 \quad \text{for all } j \text{ if and only if } r \leq 1.
\]

Bagley (1986) has done some research on the limit behaviour of the process for a specific superadditive mating function, \(L(x, y) = \min \{x, y\}\), and when Daley’s model holds.

For \(r > 1\), we know that if \(Z_0\) is large enough, there is a positive probability of survival. Then, in this case, it is very important to study the asymptotic behaviour of \(\{r^{-n}Z_n\}_n\) — and consequently that of \(\{r^{-n}F_n\}_n\) and \(\{r^{-n}M_n\}_n\) — and to find conditions that can guarantee that the limits of these sequences are non-degenerate in 0. In this paper we study these questions for BGWBP with any superadditive mating function and not assuming, at least not explicitly, that Daley’s model holds. In Section 2 we give sufficient conditions for the almost sure and \(L^1\) convergence of \(\{r^{-n}Z_n\}_n\) to a non-degenerate limit. This section is an adaptation of results of Klebaner (1984, 1985) to bisexual branching processes. Section 3 is devoted to a study of the strong relation between the almost sure and \(L^1\) convergence of \(\{r^{-n}Z_n\}_n\) and the almost sure and \(L^1\) convergence of \(\{r^{-n}F_n\}_n\) and \(\{r^{-n}M_n\}_n\). Finally, in Section 4, we study for a superadditive BGWBP the \(L^1\) convergence
to a non-degenerate limit of $\{r^{-n}Z_n\}_n$, $\{r^{-n}F_n\}_n$ and $\{r^{-n}M_n\}_n$ under a $Z \log^+ Z$ condition. Also, as an example, we study the asymptotic behaviour of an important superadditive BGWBP under that classical condition.

2. Almost sure and $L^1$ convergence of $\{r^{-n}Z_n\}_n$

In this section we consider a superadditive BGWBP such that (1.2) is satisfied and we do not need to suppose necessarily that Daley's model holds.

**Theorem 2.1.** $r^{-n}Z_n \to W$ almost surely as $n \to \infty$, $W$ being a non-negative and finite random variable.

**Proof.** Taking into account that $\{Z_n\}_n$ is a Markov chain with stationary one-step transition probabilities and since $r = \sup_{k > 0} r_k$, we have that $\{r^{-n}Z_n\}_n$ is a non-negative supermartingale relative to the family of $\sigma$-algebras $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$, $n \geq 0$:

$$E[r^{-n-1}Z_{n+1} \mid \mathcal{F}_n] = r^{-n-1}E[Z_{n+1} \mid Z_n] = r^{-n-1}Z_n r_{Z_n} \leq r^{-n}Z_n.$$ 

Then, by the supermartingale convergence theorem, the proof is complete.

Let $\varepsilon_k = r - r_k$, $k = 1, 2, \ldots$. It is clear that $\varepsilon_k \geq 0$ for all $k$ and $\varepsilon_k \to 0$ as $k \to \infty$.
If we denote $W_n = r^{-n}Z_n$ and $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$, $n \geq 0$, it is easy to obtain that

$$E[W_{n+1} \mid \mathcal{F}_n] = W_n - r^{-1}W_n \varepsilon_{Z_n}, \quad n \geq 0,$$

which implies that

$$E[W_{n+1}] = Z_0 - r^{-1} \sum_{k=0}^{n} E[W_k \varepsilon_{Z_k}], \quad n \geq 0. \tag{2.1}$$

**Theorem 2.2.** Let $\varepsilon(x)$ be a positive function, monotonic non-increasing such that $\varepsilon_n \leq \varepsilon(n)$, $n \geq 1$, and $\varepsilon(x)$ is concave in $\mathbb{R}^+$. If $\Sigma_{n=1}^{\infty} \varepsilon(n)/n < \infty$, then $\lim_{n \to \infty} E[W_n \mid Z_0 = i] > 0$ for all $i$ such that $q_i < 1$.

**Proof.** First, we shall prove that $\{E[W_n]\}_n$ converges to a limit larger than 0 if the process starts with a large enough number of mating units.

Since $\varepsilon(x)$ is positive and monotonic non-increasing, $\Sigma_{n=1}^{\infty} \varepsilon(n)/n < \infty$ implies that $\varepsilon(x) \to 0$ as $x \to \infty$. Hence, there is a $\delta > 0$ such that for $x \geq \delta$, $\varepsilon(x) < 1$ is satisfied. We define $x_0 = \Pi_{n=0}^{\infty} (1 - r^{-1}\varepsilon(r^n)^{-1})$. It is clear, taking into account that $\Sigma_{n=1}^{\infty} \varepsilon(n)/n < \infty$ if and only if $\Sigma_{n=1}^{\infty} \varepsilon(r^n) < \infty$, that $1 < x_0 < \infty$.

Since $\{W_n\}_n$ is a supermartingale relative to $\{\mathcal{F}_n\}_n$, $\{E[W_n]\}_n$ is a monotonic non-increasing sequence and consequently it is convergent. Moreover, if there is an $n_0$ such that $E[W_{n_0}] \geq \delta x_0$, then $E[W_k] \geq \delta$ for all $k \geq n_0$ (see Klebaner (1984), p. 46).

Therefore, we have proved that if $i \geq [\delta x_0]+1$ ([x] denotes the largest integer $\leq x$), then $\lim_{n \to \infty} E[W_n \mid Z_0 = i] = a_i > 0$. Finally, using an argument of Markov chains, we prove that $\lim_{n \to \infty} E[W_n] \geq \delta > 0$ for all $i$ such that $q_i < 1$ (see Klebaner (1984), p. 46), and this completes the proof of the theorem.
Corollary 2.3. If \( \{e_n\}_n \) is a monotonic non-increasing sequence and \( \Sigma_{n=1}^{\infty} e_n/n < \infty \), then \( \lim_{n \to \infty} E[W_n | Z_0 = i] > 0 \) for all \( i \) such that \( q_i < 1 \).


Let \( R_k = E[|Z_{n+1} - k r_k| | Z_n = k] \), \( k = 1, 2, \cdots \).

Theorem 2.4. Suppose that \( \{e_n\}_n \) and \( \{R_n/n\}_n \) are monotonic non-increasing. If \( \Sigma_{n=1}^{\infty} e_n/n < \infty \) and \( \Sigma_{n=1}^{\infty} R_n/n^2 < \infty \), then \( \{W_n\}_n \) converges in \( L^1 \) to a finite and non-degenerate limit.

Proof. We shall suppose that the process starts with a large enough number of mating units to guarantee that the process does not become extinct. To prove \( L^1 \) convergence of \( \{W_n\}_n \), it is sufficient to obtain that \( \{W_n\}_n \) satisfies the \( L^1 \)-Cauchy convergence criterion. Once this has been obtained, from Corollary 2.3, \( \Sigma_{n=1}^{\infty} e_n/n < \infty \) guarantees that the limit is non-degenerate in 0:

\[
E[|W_{n+1} - W_n|] \leq r^{-n-1}(E[|Z_{n+1} - r Z_n|] + E[|r Z_n - r Z_n Z_n|])
= r^{-n-1}(E[R Z_n] + E[Z_n e Z_n]).
\]

If we define \( g(x) = (R_1 + e_1)I_{[0 \leq x < 1]} + x(R_{[x]} + e_{[x]})I_{[x \geq 1]} \), then

\[
E[|W_{n+1} - W_n|] \leq r^{-n-1}E[g(Z_n)].
\]

From (2.2), we prove that, for some \( \delta > 0 \),

\[
E[|W_{n+1} - W_n|] \leq r^{-n-1}Z_0 \frac{\hat{g}(\delta r^n)}{\delta r^n},
\]

where \( \hat{g}(x) \) is a function such that \( \Sigma_{n=0}^{\infty} \hat{g}(\delta r^n)/\delta r^n < \infty \) (see Klebaner (1985), pp. 53–54). This completes the proof of the theorem.

Remark 1. Obviously, the almost sure limit obtained in Theorem 2.1 is equal a.s. to the \( L^1 \)-limit obtained in the last theorem.

Remark 2. If \( R_n = O(\log n) \) or \( R_n = O((\log n)^{-p}) \), \( p > 0 \), then \( R_n \) satisfies the conditions of the theorem.

3. Almost sure and \( L^1 \) convergence of \( \{r^{-n}F_n\}_n \) and \( \{r^{-n}M_n\}_n \)

There is a close relation between almost sure and \( L^1 \) convergence of \( \{r^{-n}Z_n\}_n \) and almost sure and \( L^1 \) convergence of \( \{r^{-n}F_n\}_n \) and \( \{r^{-n}M_n\}_n \). This can be stated by the following results obtained under the same conditions considered in Section 2.

Theorem 3.1. \( r^{-n}F_n \to r^{-1}E[F_0]W \) almost surely as \( n \to \infty \), where \( W \) is the almost sure limit of \( r^{-n}Z_n \).

Proof. Let \( S_n = r^{-n}F_n \), \( W_n = r^{-n}Z_n \) and \( \sigma = \sigma(Z_0, \cdots, Z_n) \), for \( n \geq 0 \). Let

\[
S_{n+1} = r^{-n-1} \sum_{i=1}^{Z_n} F_{i | F_n = x},
\]
We verify

\[ E[\hat{S}_{n+1} | \mathcal{F}_n] = r^{-1} W_n E[F_{01} I_{(F_{01} \leq r^n)}]. \]

Since \( r > 1 \), we obtain that \( F_{01} I_{(F_{01} \leq r^n)} \to F_{01} \) almost surely as \( n \to \infty \). Hence, by the dominated convergence theorem, \( E[F_{01} I_{(F_{01} \leq r^n)}] \to E[F_{01}] \). And, taking the limit \( n \to \infty \) in (3.1),

\[ E[\hat{S}_{n+1} | \mathcal{F}_n] \to r^{-1} E[F_{01}] W \quad \text{almost surely as } n \to \infty. \]

On the other hand

\[ \text{Var}[\hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n]] \leq Z_0 r^{-2} r^{-n} \text{Var}[F_{01} I_{(F_{01} \leq r^n)}] \leq Z_0 r^{-2} r^{-n} \int_0^{\infty} x^2 I_{(x \leq r^n)} \, dF(x) \]

where \( F(x) = P(F_{01} \leq x) \). Hence

\[ \sum_{n=0}^{\infty} \text{Var}[\hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n]] \leq Z_0 r^{-2} \int_0^{\infty} x^2 \sum_{n=0}^{\infty} \frac{1}{r^n} I_{(x \leq r^n)} \, dF(x) \]

\[ \leq Z_0 r^{-2} \int_0^{\infty} x^2 O(x^{-1}) \, dF(x) < \infty. \]

Therefore, by the convergence theorem for martingales, \( \Sigma_{n=0}^{\infty} (\hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n]) \) converges almost surely and in \( L^1 \). Then \( \hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n] \to 0 \) almost surely as \( n \to \infty \), and taking into account (3.2) \( \hat{S}_{n+1} \to E[F_{01}] r^{-1} W \) almost surely as \( n \to \infty \).

To complete the proof of the theorem we show that \( S_n \) and \( \hat{S}_n \) are equivalent sequences:

\[ \sum_{n=0}^{\infty} P(S_{n+1} \neq \hat{S}_{n+1}) = \sum_{n=0}^{\infty} E[P(F_{n+i} > r^n \text{ for some } i = 1, \ldots, Z_n | \mathcal{F}_n)] \]

\[ \leq Z_0 \sum_{n=0}^{\infty} \int_0^{\infty} r^n I_{(x > r^n)} \, dF(x) \]

\[ = Z_0 \int_0^{\infty} \sum_{n=0}^{\infty} r^n I_{(x > r^n)} \, dF(x) \]

\[ = Z_0 \int_0^{\infty} O(x) \, dF(x) < \infty. \]

Therefore, \( S_n \) converges almost surely to \( E[F_{01}] r^{-1} W \).

In a similar way, we obtain that \( r^{-n} M_n \to r^{-1} E[M_{01}] W \) almost surely as \( n \to \infty \), where \( W \) is the almost sure limit of \( r^{-n} Z_n \). Taking into account these results it is easy to determine the equivalence in \( L^1 \) convergence between \( r^{-n} Z_n \), \( r^{-n} F_n \) and \( r^{-n} M_n \).

**Remark.** From these results it is clear that any condition that guarantees the convergence of \( r^{-n} Z_n \) to a non-degenerate in 0 random variable (see Section 2), is also sufficient for the convergence of \( r^{-n} F_n \) and \( r^{-n} M_n \) to a non-degenerate limit.
4. Convergence of \( \{r^{-n}Z_n\}_n \) under the Z log\(^+\)Z condition: a first approach

In general, it is very difficult to check whether the condition \( \Sigma_{n=1}^{\infty} R_n/h^2 < \infty \) holds. For this reason we consider whether it is possible to find a Z log\(^+\)Z condition which, with \( \Sigma_{n=1}^{\infty} \epsilon_n/h < \infty \) (we saw in Section 2 that this condition guarantees that \( \lim_{n \to \infty} E[r^{-n}Z_n] > 0 \)), can be sufficient to prove the convergence of \( r^{-n}Z_n \) to a non-degenerate in 0 random variable. In this sense, we have obtained a general result for some superadditive BGWBP and, as a corollary of this, we have studied one of the mating functions considered by Daley (1968): \( L(x, y) = x \min\{1, y\} \).

We consider a superadditive BGWBP, as in the other sections, and assume that \( r > 1 \).

Let \( S_n = r^{-n}F_n \), \( W_n = r^{-n}Z_n \) and \( D_n = S_n - r^{-1}W_n E[F_0 I_{\{F_0 \leq r^n\}}] \), \( n = 0, 1, \ldots \).

**Theorem 4.1.** Suppose that \( \{\epsilon_n\}_n \) is monotonic non-increasing such that \( \Sigma_{n=1}^{\infty} \epsilon_n/h < \infty \) and \( D_n \geq 0 \) for all \( n \). If \( E[F_0 I_{\{F_0 \leq r^n\}}] < \infty \), then \( \{r^{-n}F_n\}_n \) converges in \( L^1 \) to a non-degenerate in 0 random variable, \( W \).

**Proof.** As in the other theorems, we shall suppose that the process starts with a large enough number of mating units to guarantee that the process does not become extinct. First, taking into account that \( \Sigma_{n=1}^{\infty} \epsilon_n/h < \infty \) and Corollary 2.3, we have that \( \lim_{n \to \infty} E[r^{-n}Z_n] > 0 \).

Next, we shall prove that if \( E[F_0 I_{\{F_0 \leq r^n\}}] < \infty \), then \( \{r^{-n}F_n\}_n \) converges in \( L^1 \) to a non-negative and finite random variable, \( W \). Let

\[
\hat{S}_{n+1} = r^{-n-1} \sum_{i=1}^{Z_n} F_{ni} I_{\{F_n \leq r^n\}}.
\]

We consider the martingale increments \( \hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n] \), where \( \mathcal{F}_n = \sigma(Z_0, \ldots, Z_n) \):

\[
\sum_{n=0}^{\infty} (\hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n]) = \sum_{n=0}^{\infty} (\hat{S}_{n+1} - S_n + D_n).
\]

In Theorem 3.1, we proved that \( \Sigma_{n=0}^{\infty} (\hat{S}_{n+1} - E[\hat{S}_{n+1} | \mathcal{F}_n]) \) converges in \( L^1 \). Taking into account this result, we will show that \( E[F_0 I_{\{F_0 \leq r^n\}}] < \infty \) implies that \( \Sigma_{n=0}^{\infty} (\hat{S}_{n+1} - S_n) \) converges in \( L^1 \). We only need to prove that \( \Sigma_{n=0}^{\infty} D_n \) converges in \( L^1 \), and since \( D_n \geq 0 \) for all \( n \) it is sufficient to show that \( \Sigma_{n=0}^{\infty} E[D_n] < \infty \):

\[
E[D_n] = E[S_n - r^{-1}W_n E[F_0 I_{\{F_0 \leq r^n\}}]]
= E[E[S_n - r^{-1}W_n E[F_0 I_{\{F_0 \leq r^n\}}] | \mathcal{F}_{n-1}]]
= r^{-1} E[W_{n-1} E[F_0 I_{\{F_0 \leq r^n\}}] - r^{-1} E[F_0 I_{\{F_0 \leq r^n\}}] r^{-n} E[Z_{n-1} r Z_{n-1}]]
= r^{-1} E[W_{n-1} E[F_0 I_{\{F_0 > r^n\}}] + r^{-2} E[F_0 I_{\{F_0 \leq r^n\}}] E[W_{n-1} I_{\{F_0 \leq r^n\}}] E[Z_{n-1} + E(Z_{n-1})], \quad n \geq 1.
\]

If \( F(x) = P(F_0 \leq x) \), then
\[
\sum_{n=0}^{\infty} E[F_{01} I_{\{F_{01}>r^*\}}] = \sum_{n=0}^{\infty} \int_{r^n}^{\infty} x dF(x) = \int_{0}^{\infty} x \sum_{n=0}^{\infty} I_{\{x>r^n\}} dF(x) = \int_{0}^{\infty} xO(\log^+ x) dF(x),
\]

which is finite by \( E[F_{01} \log^+ F_{01}] < \infty \). Also, by (2.1) and taking into account that \( 0 \leq E[W_n] \leq Z_0 \) for all \( n \geq 0 \), we have that \( \Sigma_{n=0}^{\infty} E[W_n E_{Z_n}] < \infty \).

Hence, if \( E[F_{01} \log^+ F_{01}] < \infty \), then \( \Sigma_{n=0}^{\infty} E[D_n] < \infty \) and therefore \( \Sigma_{n=0}^{\infty} (\tilde{S}_{n+1} - S_n) \) converges in \( L^1 \).

By Theorem 3.1, there is a finite and non-negative random variable \( W \), such that \( r^{-n} F_n \to W \) almost surely as \( n \to \infty \). Hence, since \( \tilde{S}_n \leq S_n \) for all \( n \), we obtain

\[
E[W] \geq E[S_n] + E\left[ \sum_{k=n}^{\infty} (\tilde{S}_{k+1} - S_k) \right], \quad n \geq 0.
\]

(4.1)

From (4.1), taking into account that \( \Sigma_{n=0}^{\infty} (\tilde{S}_{n+1} - S_n) \) converges in \( L^1 \), we have that \( E[W] \geq \limsup_{n \to \infty} E[S_n] \). On the other hand, by Fatou's lemma, \( E[W] \leq \liminf_{n \to \infty} E[S_n] \). Hence \( E[W] = \lim_{n \to \infty} E[S_n] \), and therefore \( r^{-n} F_n \to W \) in \( L^1 \) as \( n \to \infty \). Finally, the \( L^1 \) convergence of \( r^{-n} F_n \) implies that \( r^{-n} Z_n \) converges in \( L^1 \) (see Section 3) to a random variable equal a.s. to \( rE[F_{01}]^{-1} W \). Hence, since we have proved that \( \lim_{n \to \infty} E[r^{-n} Z_n] > 0 \), we get that \( E[W] > 0 \) and \( W \) is non-degenerate in \( 0 \). This completes the proof of the theorem.

Remark. It is easy to prove that a sufficient condition for \( D_n \geq 0 \) for all \( n \) is that \( L(x, y) \leq x/y E[F_{01}] \) or, by symmetry, \( L(x, y) \leq y/r E[M_{01}] \). The two mating functions introduced by Daley (1968) satisfy one of these properties.

Corollary 4.2. Suppose a BGWBP with superadditive mating function \( L(x, y) = x \min \{1, y\} \) and such that Daley's model holds. If \( E[F_{01} \log^+ F_{01}] < \infty \), then \( \{r^{-n} F_n\}_n \) converges in \( L^1 \) to a non-degenerate in \( 0 \) random variable.

Proof. If Daley's model holds, then the probability generating function (p.g.f.) of \( (F_n, M_n) \) will be \( f(s_1, s_2) = g(\alpha s_1 + (1-\alpha)s_2), 0 \leq s_1, s_2 \leq 1 \), where \( g(s) \) is the p.g.f. of \( T_n = F_n + M_n \) and \( \alpha (0 < \alpha < 1) \) is the probability for an offspring to be a female. In this case, and for \( L(x, y) = x \min \{1, y\} \), it is known (Daley 1968) that

\[
E[s^{Z_{n+1}} | Z_n = j] = g(\alpha s + 1 - \alpha) s^j - g(\alpha s)^j + g(\alpha)^j, \quad 0 \leq s \leq 1.
\]

From (4.2), we have that \( \varepsilon_n = r - r_n = xg'(\alpha)g(\alpha^{n-1}) \) (monotonic non-increasing) and therefore, \( \Sigma_{n=1}^{\infty} \varepsilon_n/n < \infty \).

Finally, taking into account that in this case \( r = E[F_{01}] \),

\[
D_n = S_n - r^{-1} W_n E[F_{01} I_{\{F_{01} \leq r^*\}}] \geq r^{-n}(F_n - F_n \min \{1, M_n\}) \geq 0.
\]

Considering Theorem 4.1, the proof is completed.

Remark. In the case considered by Bagley (1986), the corresponding \( D_n \) are also \( \geq 0 \). Hence, using the same method as in the theorem we can prove that \( E[F_{01} \log^+ F_{01}] < \infty \) implies the convergence in \( L^1 \) of \( r^{-n} F_n \) to a non-negative and finite random variable.
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References


