BISEXUAL BRANCHING MODELS WITH IMMIGRATION*

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ABSTRACT

Modified bisexual Galton-Watson branching models allowing immigration of females and males, or mating units, are introduced. For the underlying Markov chains the classification of states is studied and relations between the probability generating functions are investigated. Estimators for some interesting parameter vectors are proposed and an illustrative example is given.

KEY WORDS

Bisexual Galton-Watson branching process, immigration models, moment method estimation.

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1. BISEXUAL GALTON-WATSON BRANCHING PROCESS

Introduced by Daley (1968) the Bisexual Galton-Watson Branching Process (BGWBP) is a two-type branching model with $F_n$ females and $M_n$ males in the $n$-th generation, $n = 1, 2, \ldots$, which form $Z_n = L(F_n, M_n)$ mating units. These mating units reproduce independently according to the same offspring distribution for each generation. The mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that $L(x, y) \leq xy$. Then, considering for simplicity $Z_0 = 1$,

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}), \quad n = 0, 1, \ldots \quad (1.1)$$

with the empty sum defined to be $(0, 0)$, where $f_{ni}$ (respectively $m_{ni}$) denotes the number of females (respectively males) produced by the $i$-th mating unit in the $n$-th generation, being $\{(f_{ni}, m_{ni})\}$ i.i.d. non-negative integer-valued bivariate random variables. We call $L$ superadditive if for all integer $n \geq 2$

$$L \left( \sum_{i=1}^{n} (x_i, y_i) \right) \geq \sum_{i=1}^{n} L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, \ i = 1, \ldots, n$$

The BGWBP has received some attention in the scientific literature. The extinction problem has been studied by Daley (1968), Hull (1982, 1984), Bruss (1984) and Daley et al. (1986). The main result, proved by Daley et al. (1986), is based on the concept of average reproduction rate per mating unit, namely $r_k := k^{-1}E[Z_{n+1} \mid Z_n = k], \ k = 1, 2, \ldots$, and establishes that
for a BGWBP with superadditive mating function the growth rate, defined as 
\[ r := \lim_{k \to \infty} r_k, \]
exists and if \( q_j := P[Z_n \to 0 \mid Z_0 = j], \ j = 1, 2, \ldots, \) then 
\[ q_j = 1 \] for all \( j \) if and only if \( r \leq 1. \) Thus, for \( r > 1 \) and \( Z_0 \) large enough,
there is a positive probability of survival. The limit behaviour has been in-
vestigated by Bagley (1986) and González and Molina (1996, 1997a, 1997b)
and some inferential problems have been considered by González Fragoso and
and Molina et al. (1998).

However, unlike asexual Galton-Watson branching processes theory, modified bisexual Galton-Watson models for description of
more realistic practical situations have not been, until now, sufficiently inves-
tigated.

In this paper two modified BGWBPs allowing immigration of females and
males (Section 2) or mating units (Section 3) are introduced. For the under-
lying Markov chains, the behaviour of the states is studied and some relations
between the probability generating functions are deduced. As a consequence
expressions for the main moments are derived. Section 4 is devoted to estimate,
for the first model, the mean vectors corresponding to offspring and immigra-
tion distributions. An illustrative simulated example is given. Throughout
the paper, we will assume a superadditive mating function, which is not a
serious restriction. As pointed out by Hull (1982), the vast majority of mating
functions used in the theory on two-sex population models are superadditive.

2. BGWBP WITH IMMIGRATION OF FEMALES AND MALES

From expression (1.1) we introduce the new process:
\[(F^*_{n+1}, M^*_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}) + (F^I_{n+1}, M^I_{n+1}) \quad (2.1)\]

\[Z_{n+1} = L(F^*_{n+1}, M^*_{n+1}), \quad n = 0, 1, \ldots\]

being \{((F^I_n, M^I_n))\} a sequence of i.i.d. non-negative integer-valued bivariate random variables independent of \{(f_{ni}, m_{ni})\}, where \(F^I_n\) (respectively \(M^I_n\)) represents the number of immigrant females (respectively males) in the \(n\)-th generation. It is easy to prove that \{\(Z_n\)\} and \{\((F^*, M^*)\)\} are Markov chains with state space in a subset \(S^*_Z\) of \(\mathbb{N}_0\) (the non-negative integers) and a subset \(S^*_n\times \mathbb{N}_0\), respectively, and non-stationary transition probabilities.

Let \(p_{hj} : = P[(f_{ni}, m_{ni}) = (h, j)]\), \(q_{kl} : = P[(F^I_n, M^I_n) = (k, l)]\), \(h, j, k, l \in \mathbb{N}_0\) be the offspring and immigration distributions, respectively (we assume both distributions non-degenerate). We denote by

\[B^*_z := \{x \in S^*_Z : P[Z_{m+n} = x \mid Z_n = z] > 0 \text{ for some } m\}\]

and

\[C^*_{(x,y)} = \{(r, s) \in S^* : P[(F^*_{n+m}, M^*_{n+m}) = (r, s) \mid (F^*_{n}, M^*_{n}) = (x, y)] > 0 \text{ for some } m\} \text{ where } z \in S^*_Z, (x, y) \in S^*.\]

Now we are going to prove two results (Lemma 2.1 and Theorem 2.1) establishing how the states of the underlying Markov chains communicate.
Lemma 2.1 If there exist \((h_0, j_0), (k_0, l_0) \in S^*\) such that \(p_{h_0j_0}, q_{k_0l_0},\ \ L(h_0, j_0), L(k_0, l_0) > 0\), then:

(i) For any \((x, y) \in S^*\) there exists \((r, s) \in S^*\) with \(L(r, s) > L(x, y)\) such that \((r, s) \in C_{(k_0, l_0)}^*\)

(ii) For any \(y \in S^*_Z\) there exists \(z \in S^*_Z\) with \(z > y\) such that \(z \in B_{L(k_0, l_0)}^*\)

Proof

(i) We define the sequence \((x_n, y_n)\) recursively in the form:

\[(x_0, y_0) = (k_0, l_0)\]

\[(x_{n+1}, y_{n+1}) = (h_0, j_0) \cdot L(x_n, y_n) + (k_0, l_0) , n = 0, 1, ...\]

Using the assumption that \(L\) is superadditive, we obtain

\[L(x_{n+1}, y_{n+1}) = L((h_0, j_0) \cdot L(x_n, y_n) + (k_0, l_0))\]

\[\geq L(h_0, j_0) \cdot L(x_n, y_n) + L(k_0, l_0) > L(x_n, y_n)\]

Thus \(L(x_n, y_n) \uparrow \infty\) as \(n \uparrow \infty\) and given \((x, y) \in S^*\) there exists \((x_m, y_m)\) such that \(L(x_m, y_m) > L(x, y)\). Consequently it is sufficient to prove that \((x_m, y_m) \in C_{(k_0, l_0)}^*\).

In fact
\[ P[(F_{n+m}^*, M_{n+m}^*) = (x_m, y_m) \mid (F_n^*, M_n^*) = (k_0, l_0)] \]

\[ \geq \prod_{i=0}^{m-1} P[(F_{n+1}^*, M_{n+1}^*) = (x_{i+1}, y_{i+1}) \mid (F_n^*, M_n^*) = (x_i, y_i)] \]

\[ \geq (p_{h_0 j_0})^{\sum_{i=0}^{m-1} L(x_i, y_i)} (q_{k_0 l_0})^m > 0 \]

where we have used that

\[ P[(F_{n+1}^*, M_{n+1}^*) = (x_{i+1}, y_{i+1}) \mid (F_n^*, M_n^*) = (x_i, y_i)] \]

\[ = P \left[ \sum_{j=1}^{L(x_i, y_i)} (f_{n_j}, m_{n_j}) + (F_{n+1}^I, M_{n+1}^I) = (h_0, j_0) \cdot L(x_i, y_i) + (k_0, l_0) \right] \]

\[ \geq (p_{h_0 j_0})^{L(x_i, y_i)} q_{k_0 l_0} \]

(ii) Given \( y \in \mathbb{N}_0 \) there exists an integer \( m > 0 \) such that \( L(x_m, y_m) > y \), then it is sufficient to prove that \( L(x_m, y_m) \in B_{L(k_0, l_0)}^* \). This is clear taking into account that:
\[ P[Z_{n+m} = L(x_m, y_m) \mid Z_n = L(k_0, l_0)] \]

\[ \geq \prod_{i=0}^{m-1} P[Z_{n+1} = L(x_{i+1}, y_{i+1}) \mid Z_n = L(x_i, y_i)] \]

\[ \geq (p_{h_0 j_0})^{m-1} L(x_i, y_i) (q_{k_0 l_0})^m > 0 \]

since

\[ P[Z_{n+1} = L(x_{i+1}, y_{i+1}) \mid Z_n = L(x_i, y_i)] \]

\[ = P \left[ L \left( \sum_{j=1}^{L(x_i, y_i)} \left( f_{ij}, m_{ij} \right) + \left( F^{i+1}_{n+1}, M^{i+1}_{n+1} \right) \right) = L(x_{i+1}, y_{i+1}) \right] \]

\[ \geq (p_{h_0 j_0})^{L(x_i, y_i)} q_{k_0 l_0} \]

\[ \square \]

**Theorem 2.1** Under the conditions of Lemma 2.1 and provided that \( p_{00} > 0 \), we have:

(i) \((k_0, l_0) \in C^*_{(x, y)}\) for all \((x, y) \in S^*\)

(ii) Given \((x, y), (u, t) \in S^*\) such that \((u, t) \in C^*_{(x, y)}\) then \((u, t) \in C^*_{(k_0, l_0)}\)

(iii) \(L(k_0, l_0) \in B^*_x\) for all \(x \in S^*_Z\)
(iv) Given \( x, y \in S^*_z \) such that \( y \in B^*_x \) then \( y \in B^*_{L(k_0,l_0)} \)

**Proof**

(i) It is immediate taking into account that

\[
P[(F^*_{n+1}, M^*_{n+1}) = (k_0, l_0) \mid (F^*_n, M^*_n) = (x, y)]
\]

\[
= P \left[ \sum_{i=1}^{L(x,y)} (f_{ni}, m_{ni}) + (F^I_{n+1}, M^I_{n+1}) = (k_0, l_0) \right] \geq (p_{00})^{L(x,y)} q_{k_0l_0} > 0
\]

(ii) For some integer \( m > 0 \) it is verified that

\[
P[(F^*_{n+m}, M^*_{n+m}) = (u, t) \mid (F^*_n, M^*_n) = (x, y)] > 0
\]

Applying Lemma 2.1 (i) there exists \((r, s) \in S^*\) with \( L(r, s) > L(x, y) \) such that \((r, s) \in C^*_{(k_0,l_0)}\).

Therefore it is sufficient to prove that \((u, t) \in C^*_{(r,s)}\).

If \( m = 1 \)

\[
P[(F^*_{n+1}, M^*_{n+1}) = (u, t) \mid (F^*_n, M^*_n) = (r, s)]
\]

\[
\geq (p_{00})^{L(r,s)-L(x,y)} P[(F^*_{n+1}, M^*_{n+1}) = (u, t) \mid (F^*_n, M^*_n) = (x, y)] > 0
\]
If $m > 1$

$$
P[(F_{n+m}^*, M_{n+m}^*) = (u, t) \mid (F_n^*, M_n^*) = (r, s)]
$$

$$
= \sum_{(a, b) \in S^*} P[(F_{n+m}^*, M_{n+m}^*) = (u, t) \mid (F_{n+1}^*, M_{n+1}^*) = (a, b)] 
\cdot P[(F_{n+1}^*, M_{n+1}^*) = (a, b) \mid (F_n^*, M_n^*) = (r, s)]
$$

(2.2)

Now

$$
P[(F_{n+1}^*, M_{n+1}^*) = (a, b) \mid (F_n^*, M_n^*) = (r, s)]
$$

$$
= P \left[ \sum_{i=1}^{L(r,s)} (f_{ni}, m_{ni}) + (F_{n+1}^I, M_{n+1}^I) = (a, b) \right]
$$

$$
\geq P \left[ \sum_{i=1}^{L(x,y)} (f_{ni}, m_{ni}) + (F_{n+1}^I, M_{n+1}^I) = (a, b) \right] \cdot (p_{00})^{L(r,s)-L(x,y)} > 0
$$

Consequently from (2.2) we can write

$$
P[(F_{n+m}^*, M_{n+m}^*) = (u, t) \mid (F_n^*, M_n^*) = (r, s)]
$$

$$
\geq P[(F_{n+m}^*, M_{n+m}^*) = (u, t) \mid (F_n^*, M_n^*) = (x, y)] \cdot (p_{00})^{L(r,s)-L(x,y)} > 0
$$
(iii) It is derived considering that
\[
P[Z_{n+1} = L(k_0, l_0) \mid Z_n = x)]
\]
\[
= P \left[ L \left( \sum_{i=1}^{x} (f_{ni}, m_{ni}) + (F_{n+1}^I, M_{n+1}^I) \right) = L(k_0, l_0) \right] \geq (p_{00})^x q_{k_0 l_0} > 0
\]

(iv) It is proved in a similar way that (ii) using Lemma 2.1 (ii).

\[\square\]

**Remark 2.1:** Using terminology of Chung (1967), from Lemma 2.1 and Theorem 2.1, it is concluded that in the Markov chain \{\(F_n^*, M_n^*\}\}, the states that communicate with some other also communicate with the state \((k_0, l_0)\) and no state leads to those that don’t communicate with any other. Consequently the state space will be the union of an essential class (the class of \((k_0, l_0)\)) and non-essential states that lead to \((k_0, l_0)\) in one step. Similar conclusions are obtained for the Markov chain \{\(Z_n\)\} being its state space the union of an essential class (the class of \(L(k_0, l_0)\)) and non-essential states that lead to \(L(k_0, l_0)\) in one step.

**Theorem 2.2** For a model (2.1) it is verified that:
\[
\psi_{n+1}(s, t) = h(s, t) \cdot g_n(\phi(s, t)), \quad n = 0, 1, \ldots \quad s, t \in [0, 1]
\]  
(2.3)

where \(\phi\), \(h\), \(g_n\) and \(\psi_{n+1}\) are the probability generating functions of \((f_{01}, m_{01})\), \((F_1^I, M_1^I)\), \(Z_n\) and \((F_{n+1}^*, M_{n+1}^*)\) respectively.
Proof

For $s,t \in [0,1]$ we have, taking into account (2.1)

$$
\Psi_{n+1}(s,t) = E[s^{F^*_{n+1}} t^{M^*_{n+1}}] = E[s^{F^*_{n+1}} t^{M^*_{n+1}}] E\left[ E\left[ s^{\sum_{i=1}^{Z_n} f_i n t I_i} t^{\sum_{i=1}^{Z_n} m_{ni} I_i} \Bigg| Z_n \right] \right]
$$

$$
= h(s,t) \sum_{k \in S^*_Z} E\left[ s^{\sum_{i=1}^{k} f_i n t I_i} t^{\sum_{i=1}^{k} m_{ni} I_i} \right] P[Z_n = k]
$$

$$
= h(s,t) \sum_{k \in S^*_Z} (E[s^{f_i n t m_{ni}}])^k P[Z_n = k] = h(s,t) g_n(\varphi(s,t))
$$

From Theorem 2.2, it is matter of straightforward computations to derive the following result:

**Corollary 2.1** Let $\mu^*_n$ and $\Sigma^*_n$ be the mean vector and covariance matrix of $(F^*_n, M^*_n)$, respectively. Then

(i) $\mu^*_n = E[Z_{n-1}] \mu + \mu^I \quad , n = 1, 2, ...$

(ii) $\Sigma^*_n = E[Z_{n-1}] \Sigma + \text{var}[Z_{n-1}] \mu^t \cdot \mu + \Sigma^I \quad , n = 1, 2, ...$

$(\mu^t$ denotes the transpose of the row vector $\mu$).

where $\mu$ and $\Sigma$ (respectively $\mu^I$ and $\Sigma^I$) denote the mean vector and covariance matrix of the offspring (immigration) distribution.
3. BGWBP WITH IMMIGRATION OF MATING UNITS

From expression (1.1), we now define the process:

\[(F'_{n+1}, M'_{n+1}) = \sum_{i=1}^{Y_n}(f_{ni}, m_{ni})\]

(3.1)

\[Y_{n+1} = L(F'_{n+1}, M'_{n+1}) + I_{n+1} \quad n = 0, 1, \ldots\]

where, without loss of generality we may assume \(Y_0 = 1\), and \(\{I_n\}\) is a sequence of i.i.d. non-negative integer-valued random variables independent of \(\{(f_{ni}, m_{ni})\}\). In this model \(I_n\) represents the number of immigrant mating units in the \(n\)-th generation. It can be easily shown that \(\{Y_n\}\) and \(\{(F'_n, M'_n)\}\) are Markov chains with state space in a subset \(S'_Y\) of \(\mathbb{N}_0\) and a subset \(S'\) of \(\mathbb{N}_0 \times \mathbb{N}_0\) respectively, and non-stationary transition probabilities. Let \(q_j := P[I_n = j], j \in \mathbb{N}_0\) be the non-degenerate immigration distribution.

For \(z \in S'_Y\) and \((x, y) \in S'\) we will denote by \(B'_z\) and \(C'_{(x,y)}\) the sets:

\[B'_z = \{x \in S'_Y : P[Y_{n+m} = x \mid Y_n = z] > 0 \text{ for some } m\}\]

\[C'_{(x,y)} = \{(r, s) \in S' : P[(F'_{n+m}, M'_{n+m}) = (r, s) \mid (F'_n, M'_n) = (x, y)] > 0 \text{ for some } m\}\]

We shall study (Lemma 3.1 and Theorem 3.1) how the states of these Markov chains communicate.
Lemma 3.1 Let $k_0 := \inf \{ j > 0 : q_j > 0 \}$ be. If there exists $(h_0, j_0) \in S'$ such that $L(h_0, j_0), p_{h_0 j_0} > 0$ then:

(i) For any $(x, y) \in S'$ there exists $(r, s) \in S'$ with $L(r, s) > L(x, y)$ such that $(r, s) \in C'_{(0,0)}$

(ii) For any $y \in S'_Y$ there exists $z \in S'_Y$ with $z > y$ such that $z \in B'_{k_0}$

Proof

It is analogous to that of Lemma 2.1 using the sequence $\{k_n\}$ defined recursively in the form:

$$k_{n+1} = L(k_n \cdot (h_0, j_0)) + k_0 \quad n = 0, 1, ...$$

and taking into account that $k_{n+1} \geq k_n \cdot L(h_0, j_0) + k_0 > k_n$ which implies that $k_n \uparrow \infty$ as $n \uparrow \infty.$

Theorem 3.1 Under the conditions of Lemma 3.1 and provided that $p_{00} > 0$, we have:

(i) $(0, 0) \in C'_{(x,y)}$ for all $(x, y) \in S'$

(ii) Given $(x, y), (u, t) \in S'$ such that $(u, t) \in C'_{(x,y)}$ then $C'_{(x,y)} = C'_{(0,0)}$

(iii) $k_0 \in B'_{x}$ for all $x \in S'_Y$
(iv) Given $x, y \in S'_Y$ such that $y \in B_x$ then $y \in B'_{k_0}$

**Proof**

(i)

$$P[(F'_{n+1}, M'_{n+1}) = (0, 0) \mid (F'_n, M'_n) = (x, y)] = P \left[ \sum_{i=1}^{L(x,y)+1} (f_{ni}, m_{ni}) = (0, 0) \right]$$

$$\geq P \left[ \sum_{i=1}^{L(x,y)+k_0} (f_{ni}, m_{ni}) = (0, 0) \right] \cdot P[I_n = k_0] \geq (p_{00})^{L(x,y)+k_0} q_{k_0} > 0$$

hence the result follows.

(ii) For some integer $m > 0$ we know that

$$P[(F'_{n+m}, M'_{n+m}) = (u, t) \mid (F'_n, M'_n) = (x, y)] > 0$$

From Lemma 3.1 (i) there exists $(r, s) \in S'$ with $L(r, s) > L(x, y)$ such that $(r, s) \in C'_{(0,0)}$.

Therefore, it will be sufficient to prove that $(u, t) \in C'_{(r,s)}$.

If $m = 1$
\[
P[(F'_{n+1}, M'_{n+1}) = (u, t) \mid (F'_n, M'_n) = (r, s)] = P \left[ \sum_{i=1}^{L(r,s)+I_n} (f_{ni}, m_{ni}) = (u, t) \right]
\]
\[
\geq (p_{00})^{L(r,s)-L(x,y)} P[(F'_{n+1}, M'_{n+1}) = (u, t) \mid (F'_n, M'_n) = (x, y)] > 0
\]

If \( m > 1 \), according to the Chapman-Kolmogorov relation, and using the fact that for all \( (a, b) \in S' \)

\[
P[(F'_{n+1}, M'_{n+1}) = (a, b) \mid (F'_n, M'_n) = (r, s)] = P \left[ \sum_{i=1}^{L(r,s)+I_n} (f_{ni}, m_{ni}) = (a, b) \right]
\]
\[
\geq \left( \sum_{l \in \mathbb{N}_0} P \left[ \sum_{i=1}^{L(x,y)+l} (f_{ni}, m_{ni}) = (a, b) \right] \cdot P[I_n = l] \right) \cdot (p_{00})^{L(r,s)-L(x,y)}
\]

we obtain that

\[
P[(F'_{n+m}, M'_{n+m}) = (u, t) \mid (F'_n, M'_n) = (r, s)]
\]
\[
\geq P[(F'_{n+1}, M'_{n+1}) = (u, t) \mid (F'_n, M'_n) = (x, y)] \cdot (p_{00})^{L(r,s)-L(x,y)} > 0
\]

and the result follows.
(iii) 

\[ P[Y_{n+1} = k_0 \mid Y_n = x] = P \left[ L \left( \sum_{i=1}^{x} (f_{ni}, m_{ni}) \right) + I_{n+1} = k_0 \right] \geq (p_{00})^x q_{k_0} > 0 \]

and the result holds.

(iv) It is derived in a similar way that (ii) applying Lemma 3.1 (ii) \qed

**Remark 3.1:** From Lemma 3.1 and Theorem 3.1, it follows that in the Markov chain \( \{(F'_n, M'_n)\} \) (respectively \( \{Y_n\} \)), the states that communicate with some other also communicate with the state \( (0,0) \) (respectively \( k_0 \)) and no state leads to those that don’t communicate with any other, being its state space the union of an essential class, the class of \( (0,0) \) (respectively \( k_0 \)), and non-essential states that lead to \( (0,0) \) (respectively \( k_0 \)) in one step.

Consequently, for the introduced models, we obtain a behaviour of states very similar to this one obtained for the non-supercritical asexual Galton-Watson model with immigration (see Wei and Winnicki (1989)).

**Remark 3.2:** The superadditivity in Lemmas 2.1 and 3.1 can be relaxed by assuming that \( L \) is strictly increasing in some argument. We have used the first one because it is a traditional condition in BGWBP theory.

**Theorem 3.2** Let \( h_{n+1} \) and \( \chi \) be the probability generating functions of \( Y_{n+1} \) and \( I_1 \), respectively. Then
\[ h_{n+1}(s) \leq \chi(s)h_n(g_1(s)) \quad , n = 0, 1, \ldots \quad s \in [0, 1] \quad (3.2) \]

**Proof**

For \( s \in [0, 1] \) we have

\[
h_{n+1}(s, t) = E[s^{Y_{n+1}}] = E[s^{I_{n+1}}] E \left[ E \left[ s^{L(\sum_{i=1}^{Y_n} f_{ni,m_{ni}})} \mid Y_n \right] \right]
\]

\[
\leq \chi(s) \sum_{k \in S'_Y} E \left[ s^{\sum_{i=1}^{k} L(f_{ni,m_{ni}})} \right] P[Y_n = k]
\]

\[
= \chi(s) \sum_{k \in S'_Y} \left( E \left[ s^{L(f_{ni,m_{ni}})} \right] \right)^k P[Y_n = k] = \chi(s)h_n(g_1(s))
\]

where the superadditivity has been used.

\[\Box\]

**Corollary 3.1** Let \( a = E[I_1] \) and \( \nu = E[Z_1] \) be, then it is verified that

\[ E[Y_{n+1}] \geq C_n \quad , n = 0, 1, \ldots \]

where \( C_n = a(1 + n) + 1 \) if \( \nu = 1 \) or \( (a + (1 - \nu - a)\nu^{n+1})(1 - \nu)^{-1} \) if \( \nu \neq 1 \).

**Proof**

\( h_{n+1} \) and \( \chi h_n \circ g_1 \) are convex and increasing functions on \([0, 1]\), moreover
\( h_{n+1}(1) = \chi(1)h_{n+1}(g_1(1)) = 1 \). So, from inequality (3.2) there exists \( \xi \) in \((0,1)\) such that, for \( s \in (\xi, 1] \)

\[
h'_{n+1}(s) \geq \chi'(s)h_n(g_1(s)) + \chi(s)h'_n(s)(g_1(s))g'_1(s)
\]

Thus, for \( s = 1 \) we obtain

\[
E[Y_{n+1}] \geq a + \nu E[Y_n]
\]

and it follows that

\[
E[Y_{n+1}] \geq a \sum_{i=0}^{n} \nu^i + \nu^{n+1}
\]

which completes the proof. \( \square \)

4. ESTIMATION OF \( \mu \) AND \( \mu^I \)

For a model of the form (2.1) we now consider the estimation problem of the mean vectors \( \mu = (\mu_1, \mu_2) \) and \( \mu^I = (\mu^I_1, \mu^I_2) \) (supposed finites), being

\[
\mu_1 = E[f_{01}], \quad \mu_2 = E[m_{01}], \quad \mu^I_1 = E[F^I_1], \quad \mu^I_2 = E[M^I_1].
\]

From (2.1) it follows that
\[ E[(F_n^*, M_n^*) \mid Z_{n-1}] = Z_{n-1} \mu + \mu^I \text{ a.s.} \] (4.1)

If for some integer \( k > 1 \), we know \( Z_{k-1} \) (assumed positive) \((F_k^*, M_k^*)\) and \((F_k^I, M_k^I)\) then, taking into account 4.1, the moment method suggests the following estimator for \( \mu \):

\[ \bar{\mu}_k = (\bar{\mu}_{1k}, \bar{\mu}_{2k}) = Z_{k-1}^{-1}(F_k^* - F_k^I, M_k^* - M_k^I) \] (4.2)

Theorem 4.1

(i) \( E[\bar{\mu}_k \mid Z_{k-1} > 0] = \mu \)

(ii) If \( \lim_{j \to \infty} j^{-1} L(j \mu_1, j \mu_2) > 1 \) then \( \lim_{k \to \infty} \bar{\mu}_k = \mu \) a.s. on \( \{Z_n \to \infty\} \)

Proof

(i)

\[ E[\bar{\mu}_k \mid Z_{k-1} > 0] = \]

\[ (P[Z_{k-1} > 0])^{-1} \sum_{z > 0} z^{-1} E[(F_k^* - F_k^I, M_k^* - M_k^I) \mid Z_{k-1} = z] \cdot P[Z_{k-1} = z] \]

\[ = (P[Z_{k-1} > 0])^{-1} \left( \sum_{z > 0} P[Z_{k-1} = z] \right) \mu = \mu \]
(ii) From \( \lim_{j \to \infty} j^{-1} L(j\mu_1, j\mu_2) > 1 \) it can be proved that

\[
\lim_{n \to \infty} Z_n^{-1} Z_{n+1} > 1 \quad \text{a.s. on} \quad \{Z_n \to \infty\} \quad (4.3)
\]

Using (4.3), the conditional Borel-Cantelli Lemma and some well known probabilistical arguments, the result is obtained.

If the sample \( Z_{k-1}, (F_k^*, M_k^*), (F_k^I, M_k^I), k = 1, ..., n \) is available, then a reasonable estimator for \( \mu \) could be obtained through an appropriate weighted average of the previous estimators \( \bar{\mu}_k, k = 1, ..., n \), i.e. an estimator of the form \( \sum_{k=1}^n \beta_k \bar{\mu}_k \).

Considering \( \beta_k \propto Z_{k-1}, k = 1, ..., n \) and \( \sum_{k=1}^n \beta_k = 1 \) we obtain the estimator

\[
\tilde{\mu}_n = (\tilde{\mu}_{1n}, \tilde{\mu}_{2n}) = U_n^{-1} \sum_{k=1}^n (F_k^* - F_k^I, M_k^* - M_k^I) \quad (4.4)
\]

where \( U_n = \sum_{k=1}^n Z_{k-1} \).

Obviously, for this sample, the proposed estimator for \( \mu^I \) will be

\[
\tilde{\mu}_n = (\tilde{\mu}_{1n}, \tilde{\mu}_{2n}) = n^{-1} \sum_{k=1}^n (F_k^I, M_k^I) \quad (4.5)
\]
Theorem 4.2

(i) \( E[\tilde{\mu}_n] = E \left[ U_n^{-1} \sum_{k=1}^{n-1} Z_{k-1} \tilde{\mu}_k \right] + E[U_n^{-1} Z_{n-1}] \mu \)

(ii) If \( \lim_{j \to \infty} j^{-1} L(j\mu_1, j\mu_2) > 1 \) then \( \lim_{k \to \infty} \tilde{\mu}_k = \mu \) a.s. on \( \{ Z_n \to \infty \} \)

Proof

(i) Let \( \mathcal{F}^*_n = \sigma((F^*_1, M^*_1), ..., (F^*_n, M^*_n)) \) and \( \mathcal{F}^I_n = \sigma((F^I_1, M^I_1), ..., (F^I_n, M^I_n)) \), \( n = 1, 2, ..., \) then

\[
E[\tilde{\mu}_n] = E \left[ E \left[ U_n^{-1} \sum_{k=1}^{n} (F^*_k - F^I_k, M^*_k - M^I_k) \right| \mathcal{F}^*_n \vee \mathcal{F}^I_n \right] \right]
\]

\[
= E \left[ U_n^{-1} \left( \sum_{k=1}^{n-1} (F^*_k - F^I_k, M^*_k - M^I_k) + E[(F^*_n - F^I_n, M^*_n - M^I_n) \mid \mathcal{F}^*_n \vee \mathcal{F}^I_n] \right) \right]
\]

\[
= E \left[ U_n^{-1} \sum_{k=1}^{n-1} Z_{k-1} \tilde{\mu}_k \right] + E[U_n^{-1} Z_{n-1}] \mu
\]

(ii) Applying Toeplitz Lemma we have

\[
\lim_{n \to \infty} U_n^{-1} \sum_{k=1}^{n} (F^*_k - F^I_k, M^*_k - M^I_k) = \lim_{n \to \infty} U_n^{-1} \sum_{k=1}^{n} Z_{k-1} \tilde{\mu}_k = \lim_{n \to \infty} \tilde{\mu}_k
\]

So, from Theorem 4.1 (ii) the proof is completed. \( \square \)
Remark 4.1: Using the fact that \( \{(F_n^I, M_n^I)\} \) is a sequence of i.i.d. bivariate random variables, the properties of \( \bar{\mu}_n^I \) can be easily derived. In particular, it is an unbiased and strongly consistent estimator for \( \mu^I \).

Illustrative example

Suppose a model (2.1) with offspring and immigration probability distributions given respectively by:

\[
P[f_{01} = i, m_{01} = j] = 6((3 - i - j)!i!j!)^{-1}(0, 5)^i(0, 35)^j(0, 15)^{3-i-j}
\]

\[i, j = 0, 1, 2, 3 \text{ such that } i + j \leq 3\]

and

\[
P[F_1^I = i, M_1^I = j] = e^{-2}(i!j!)^{-1} \quad i, j = 0, 1, ...
\]

and assume the historical mating function \( L(x, y) = \min\{x, y\} \).

Under these conditions, it is obtained that

\[\mu = (1.5, 1.05) \quad \text{and} \quad \mu^I = (1.1)\]
Moreover, it is clear that \( \lim_{j \to \infty} j^{-1} L(j\mu_1, j\mu_2) = 1.05 \).

Considering \( Z_0 = 1 \), we simulated a total of 200 generations and from (4.1), (4.4) and (4.5), from which the corresponding estimates for \( \mu \) and \( \mu^I \) were calculated.

Table 1 exhibits some values obtained and Figures 1, 2 and 3 show the evolution of estimates.

**TABLE 1**

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</table>
FIGURE 1

Estimates for $\mu$ obtained from $\bar{\mu}_n$

![Graph](image1.png)

FIGURE 2

Estimates for $\mu$ obtained from $\hat{\mu}_n$

![Graph](image2.png)
FIGURE 3

Estimates for \( \mu^I \) obtained from \( \tilde{\mu}_n^I \)

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REFERENCES


