Limit behaviour for a subcritical bisexual Galton–Watson branching process with immigration

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Received March 1999; received in revised form September 1999

Abstract
A bisexual Galton–Watson branching process allowing immigration of females and males is considered and for the subcritical case, i.e. growth rate less than one, the limit behaviour of related sequences is investigated. © 2000 Elsevier Science B.V. All rights reserved

Keywords: Bisexual Galton–Watson process; Bisexual models with immigration; Asymptotic behaviour

1. Introduction

Daley (1968) defined the bisexual Galton–Watson branching process (BGWP) \( \{(F_n, M_n), n = 1, 2, \ldots \} \) in the form:

\[
Z_0 = N \geq 1, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}), \quad Z_{n+1} = L(F_{n+1}, M_{n+1}), \quad n = 0, 1, \ldots ,
\]

where \( N \) is a positive integer and the empty sum is considered to be \((0, 0)\). Intuitively \( f_{ni} (m_{ni}) \) represents the number of females (males) originated by the \( i \)th mating unit in the \( n \)th generation, being \( \{(f_{ni}, m_{ni}), i = 1, 2, \ldots ; n = 0, 1, \ldots \} \) a sequence of i.i.d., non-negative, integer-valued random variables and the mating function \( L: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is assumed to be monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that \( L(x, y) \leq xy \). Thus, \( F_n (M_n) \) will be the number of females (males) in the \( n \)th generation, which form \( Z_n = L(F_n, M_n) \) mating units. These mating units reproduce independently through the same offspring probability distribution for each generation.

* Research supported by the Consejería de Educación y Juventud de la Junta de Extremadura and the Fondo Social Europeo, Grant IPR98A023.
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A BGWP is said to be superadditive if the mating function is superadditive, i.e. satisfies, whatever \( k \), that
\[
L(\sum_{i=1}^{k} (x_i, y_i)) \geq \sum_{i=1}^{k} L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+,
\]
for all \( k \).

For a superadditive BGWP, Daley et al. (1986) proved that the growth rate, namely \( r := \lim_{k \to \infty} k^{-1} E[\sum_{i=1}^{k} (f_{ni}, m_i)] \), exists and it can be calculated as \( r = \lim_{k \to \infty} k^{-1} L(k \mu) \) where \( \mu = (E[f_{01}], E[m_{01}]) \).

Moreover,
\[
P[Z_n \to 0| Z_0 = N] = 1, \quad N = 1, 2, \ldots \text{ if and only if } r \leq 1.
\]

The BGWP has been considered in the scientific literature about branching models (Hull, 1982, 1984; Bruss, 1984; Bagley, 1986; González Fragoso, 1995; González and Molina, 1996, 1997; Molina et al., 1998) however, unlike asexual Galton–Watson branching processes theory, modified bisexual Galton–Watson models for description of more realistic practical situations have not been developed.

In an attempt to widen the knowledge about this subject, in this paper we consider a modified bisexual Galton–Watson process allowing immigration of females and males, which will be interesting for description of bisexual populations that receive immigrants (females and males) from other ones. In Section 2, the probability model is introduced, notation and some basic concepts are given, and the limit behaviour of the sequence corresponding to the mean growth rates per mating unit is studied. Section 3 is devoted to look at, for the subcritical case (i.e. growth rate less than one), the distribution convergence of the sequence describing the number of mating units per generation.

2. The probability model

From model (1), we derive the bisexual Galton–Watson branching process with immigration \( \{(F^*_n, M^*_n), \ n = 1, 2, \ldots\} \) defined:
\[
Z^*_0 = N, \quad (F^*_n, M^*_n) = \sum_{i=1}^{Z^*_n} (f_{ni}, m_i),
\]
\[
Z^*_n+1 = L(F_{n+1}, M_{n+1}), \quad n = 0, 1, \ldots,
\]

where \( \{(F^l_n, M^l_n), \ n = 1, 2, \ldots\} \) is a sequence of i.i.d. non-negative, integer valued random variables, independent of \( \{(f_{ni}, m_i)\} \). Intuitively, \( F^*_n \) (\( M^*_n \)) may be viewed as the number of immigrant females (males) in the \( n \)th generation. It can readily be shown that \( \{Z^*_n\} \) and \( \{(F^*_n, M^*_n)\} \) are Markov chains with stationary transition probabilities.

Denote by \( p_{kl} = P[F_{01} = k, M_{01} = l] \) and \( q_{kl} = P[F^l_{1} = k, M^l_{1} = l] \), \( k, l = 0, 1, \ldots \) the reproduction and immigration probability distributions, respectively, and assume that the corresponding mean vectors \( \mu := (E[f_{01}], E[m_{01}]) \) and \( \mu^l := (E[F^l_{1}], E[M^l_{1}]) \) are finite.

Definition 2.1. For every positive integer \( k \), we define the mean growth rate per mating unit as \( r^*_k := k^{-1} E[Z^*_n| Z^*_0 = k] \).

Obviously, \( r^*_k \) is independent of \( n \) and can be rewritten as
\[
r^*_k = k^{-1} E \left[ L \left( \sum_{i=1}^{k} (f_{ni}, m_i) + (F^l_{n+1}, M^l_{n+1}) \right) \right], \quad k = 1, 2, \ldots
\]

Remark 2.1. In general, it is not true that \( \{r^*_k\} \) has limit \( r \) as \( k \to \infty \). In fact, suppose model (3) such that \( p_{00} + q_{01} = 1, \) \( q_{01} = 1 \), and let \( L(x, y) = x \min\{1, y\} \) be the promiscuous mating function considered in
Daley et al. (1986). Under these conditions, it is easy to verify that \( \mu = (p_{10}, 0) \) and consequently \( r = 0 \). Nevertheless,

\[
\lim_{k \to \infty} f_k^* = \lim_{k \to \infty} k^{-1} \sum_{j=0}^{k} \binom{k}{j} p_{10}^{k-j} p_{00} = p_{10} > 0.
\]

**Proposition 2.1.** Suppose model (3) with \( \mu \) positive and superadditive mating function verifying \( L(x, y) \leq x + y \), then \( \{r_k^*\} \) converges to \( r \) as \( k \to \infty \).

**Proof.** From strong law of large numbers \( \{k^{-1} \sum_{i=1}^{k} (f_{mi} - m_i, m_i) + (F_{n+1}^I, M_{n+1}^I)\} \) converges a.s to \( \mu \) as \( k \to \infty \).

Taking into account that \( L \) is a non-decreasing function, and considering a similar argument to this one used in Lemma 2.3 (Daley et al., 1986), it is derived that the sequence \( \{k^{-1} L(\sum_{i=1}^{k} (f_{mi}, m_i) + (F_{n+1}^I, M_{n+1}^I))\} \) converges a.s. to \( r \) as \( k \to \infty \).

Now,

\[
L \left( \sum_{i=1}^{k} (f_{ni}, m_i) + (F_{n+1}^I, M_{n+1}^I) \right) \leq \sum_{i=1}^{k} (f_{ni} - m_i) + F_{n+1}^I + M_{n+1}^I
\]

and therefore applying again the strong law of large numbers it follows that

\[
\left\{k^{-1} L \left( \sum_{i=1}^{k} (f_{ni}, m_i) + (F_{n+1}^I, M_{n+1}^I) \right) \right\}
\]

is uniformly integrable, which implies the result. \( \square \)

**Remark 2.2.** The assumption of superadditivity and the condition \( L(x, y) \leq x + y \) are not serious restrictions. Both are verified by the vast majority of mating functions used in the literature on two-sex population models (for example, the mating functions considered in Daley et al., 1986).

**Definition 2.2.** Model (3) is said to be subcritical, critical or supercritical if \( r < 1 \), \( =1 \) or \( > 1 \), respectively.

3. **Limit behaviour for the subcritical case**

We now consider subcritical model (3) with superadditive mating function and shall investigate the limit behaviour of \( \{Z_n^*\} \) as \( n \to \infty \). We will prove that imposing some conditions on the mean growth rates per mating unit, analogous results to these ones obtained for asexual subcritical branching processes with immigration (see Jagers, 1975) are deduced. Firstly, it will be necessary to verify that the Markov chain \( \{Z_n^*\} \) is irreducible and aperiodic. Let \( S^* \) denote the state space of \( \{Z_n^*\} \) and \( k_0 := \inf \{k : P[L(F_1^I, M_1^I) = k] > 0\} \).

**Lemma 3.1.** If \( E[L(f_{01}, m_{01})] > 0 \) and \( E[L(F_1^I, M_1^I)] > 0 \) then for every \( t \in S^* \) there exists \( s \in S^* \), \( s > t \) such that \( k_0 \) leads to \( s \).

**Proof.** Since \( E[L(f_{01}, m_{01})] > 0 \) and \( E[L(F_1^I, M_1^I)] > 0 \), there exist positive integers \( i, j, \xi, \eta \in S^* \) such that \( p_{ij}, q_{ij}, L(i, j), L(\xi, \eta) > 0 \) (without loss of generality, it can be supposed that \( k_0 = L(\xi, \eta) \)). We define the sequences \( \{(x_n, y_n)\} \) and \( \{k_n\} \) recursively in the form

\[
(x_0, y_0) = (\xi, \eta), \quad (x_{n+1}, y_{n+1}) = (i, j)k_n + (\xi, \eta), \quad k_{n+1} = L(x_{n+1}, y_{n+1}).
\]

Taking into account that \( L \) is superadditive, \( \{k_n\} \) is an (strictly) increasing sequence of positive integers, so \( \lim_{n \to \infty} k_n = \infty \). Therefore will be sufficient to prove that for every \( n \), \( k_0 \) leads to \( k_n \).
In fact, whatever $m$, we obtain
\[
P(Z_{m+n}^* = k_n | Z_m^* = k_0) \geq \prod_{i=1}^{n} P(Z_{m+i}^* = k_i | Z_{m+i-1}^* = k_{i-1}) \geq (p_{ij})^{\sum_{i=1}^{n} k_{i-1}} (q_{k_0})^n > 0. \]

**Theorem 3.1.** If $E[L(f_{01}, m_{01})] > 0$, $E[L(F_{1i}^t, M_1^t)] > 0$ and $p_{00} > 0$ then
\begin{itemize}
  \item[(i)] Every state $t \in S^*$ leads to $k_0$ in only one step.
  \item[(ii)] If one state $t$ leads to another state $u$ then $k_0$ also leads to $u$.
\end{itemize}

**Proof.** (i) There exist positive integers $\xi, \eta$ such that $q_{\xi \eta}, L(\xi, \eta) > 0$. Considering again $k_0 = L(\xi, \eta)$ we have
\[
P(Z_1^* = k_0 | Z_0^* = t) = P \left[L \left( \sum_{i=1}^{t} (f_{1i}, m_{1i}) + (F_{1i}^t, M_1^t) \right) = k_0 \right] \geq (p_{00})^{t} q_{\xi \eta} > 0.
\]

(ii) Suppose that $t$ leads to $u$, then for some positive integer $n$ $P(Z_n^* = u | Z_0^* = t) > 0$. From Lemma 3.1 there exists $s > t$ such that $k_0$ leads to $s$. Now,
\[
P(Z_n^* = u | Z_0^* = s) = \sum_{v \in S^*} P(Z_n^* = u | Z_1^* = v) P(Z_1^* = v | Z_0^* = s)
\]
and taking into account that $s > t$
\[
P(Z_1^* = v | Z_0^* = s) \geq P(Z_1^* = v | Z_0^* = t) (p_{00})^{s-t}.
\]
From (4) and (5)
\[
P(Z_n^* = u | Z_0^* = s) \geq (p_{00})^{s-t} \sum_{v \in S^*} P(Z_n^* = u | Z_1^* = v) P(Z_1^* = v | Z_0^* = t)
\]
\[
= (p_{00})^{s-t} P(Z_n^* = u | Z_0^* = t) > 0. \]

**Remark 3.1.** From Theorem 3.1, it is deduced that $S^*$ is formed by only one essential aperiodic class (the class of $k_0$) and some inessential states leading to $k_0$ in one step. Therefore $\{Z_n^*\}$ is an irreducible and aperiodic Markov chain.

**Remark 3.2.** In the subcritical case, from (2) it is deduced that $\{Z_n\}$, i.e. the sequence representing the number of mating units per generation in model (1) with superadditive mating function, converges a.s. to 0 as $n \to \infty$. We will prove (Theorem 3.2) that, under certain conditions, $\{Z_n^*\}$ converges in distribution to a non-degenerate variable in 0 as $n \to \infty$.

**Lemma 3.2.** Let $\{X_n\}$ denote a stationary irreducible Markov chain where all states are non-negative integers. Assume that $E[X_n | X_0 = k_0] \leq M$, for all $n$ where $M > 0$ and $k_0$ is the smallest essential state. Then $\{X_n\}$ is positive recurrent.

**Proof.** Assume $k_0$ is not positive recurrent and let $p_{k_0}^{(n)}$ denote the $n$-step transition probability from $k_0$ to $k_0$. Then $\lim_{n \to \infty} p_{k_0}^{(n)} = 0$ and since the chain is irreducible also $\sum_{i=0}^{M-1} p_{k_0}^{(n)} = 0$. Thus, we have
\[
\lim_{n \to \infty} E[X_n | X_0 = k_0] \geq \lim_{n \to \infty} \sum_{i=0}^{M-1} i p_{k_0}^{(n)} \geq (M + 1) \lim_{n \to \infty} \left(1 - \sum_{i=0}^{M-1} p_{k_0}^{(n)}\right) = M + 1
\]
which gives a contradiction. \(\square\)
Theorem 3.2. If it is verified that \( E[L(f_{01}, m_{01})] > 0 \), \( E[L(F_{1}^{t}, M_{1}^{t})] > 0 \), \( p_{00} > 0 \) and \( \limsup_{n \to \infty} r_{k}^{*} < 1 \), then \( \{Z_{n}^{*}\} \) converges in distribution to a positive, finite and non-degenerate random variable \( Z^{*} \) as \( n \to \infty \).

Proof. Under the considered assumptions, from Theorem 3.1, it is deduced that \( \{Z_{n}^{*}\} \) is an irreducible Markov chain. If \( k_{0} \) is defined as in Lemma 3.1 then, using that \( L \) is non-decreasing in each argument, it is derived that \( P[Z_{n}^{*} > k_{0}] = 1 \), \( n = 1, 2, \ldots \) and therefore if \( k^{*} \) is an essential state it is obtained that \( k^{*} \geq k_{0} \) (i.e. \( k_{0} \) is the smallest essential state). Taking into account Lemma 3.2, to prove positive recurrence of \( \{Z_{n}^{*}\} \) it is enough to verify that \( \limsup_{n \to \infty} E[Z_{n+1}^{*} | Z_{0}^{*} = k_{0}] \) is finite.

Let \( k_{1} > 0 \) and \( r' < 1 \) such that \( r_{k}^{*} < r' \) for all \( k > k_{1} \) and set

\[
C = \max_{0 < k < k_{1}} E[Z_{n+1}^{*} | Z_{0}^{*} = k].
\]

Then \( E[Z_{n+1}^{*} | Z_{0}^{*} = k_{0}] \leq r' Z_{n}^{*} + C \).

Hence,

\[
\limsup_{n \to \infty} E[Z_{n+1}^{*} | Z_{0}^{*} = k_{0}] = \limsup_{n \to \infty} E[E[Z_{n+1}^{*} | Z_{0}^{*} = k_{0}] | Z_{n}^{*} = k_{0}]
\]

\[
\leq r' \limsup_{n \to \infty} E[Z_{n}^{*} | Z_{0}^{*} = k_{0}] + C
\]

and, consequently,

\[
\limsup_{n \to \infty} E[Z_{n+1}^{*} | Z_{0}^{*} = k_{0}] \leq (1 - r')^{-1} C < \infty.
\]

From Markov chains theory, it follows that \( \{Z_{n}^{*}\} \) converges in distribution to a positive, finite and non-degenerate random variable \( Z^{*} \) whose probability distribution will be the corresponding stationary distribution. \( \square \)

Corollary 3.1. Under the conditions of Theorem 3.2 it is verified that the sequence \( \{E[(F_{n}^{*}, M_{n}^{*}) | Z_{n-1}^{*}]\} \), \( n = 1, 2, \ldots \) converges in distribution to \( Z^{*} \mu + \mu^{t} \) as \( n \to \infty \).

Proof. It is immediate taking into account that

\[
E[(F_{n}^{*}, M_{n}^{*}) | Z_{n-1}^{*}] = Z_{n-1}^{*} \mu + \mu^{t}
\]

a.s. \( n = 1, 2, \ldots \)

and using Theorem 3.2. \( \square \)

Remark 3.3. As a direct consequence of Proposition 2.1, if \( E[L(f_{01}, m_{01})] > 0 \), \( E[L(F_{1}^{t}, M_{1}^{t})] > 0 \), \( p_{00} > 0 \) and the mating function verifies the condition \( L(x, y) \leq x + y \), Theorem 3.2 holds.

Acknowledgements

The authors would like to thank the referee for his comments and helpful suggestions which have improved this paper, and very specially for the shorter proof of Theorem 3.2 proportionated.

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