A Note on Bisexual Galton-Watson Branching Processes with Immigration

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1. INTRODUCTION

Recently, from the branching model introduced in [1], new bisexual Galton-Watson branching processes allowing immigration have been developed in [2] and some probabilistical analysis about them has been obtained. In particular, for the bisexual Galton-Watson process allowing the immigration of females and males, it has been proved (see [3]) that, under certain conditions, the sequence representing the number of mating units per generation converges in distribution to a positive, finite and non-degenerate random variable. The aim of this paper is to provide, through a different methodology, an alternative proof of this limit result. In this new, and more technical proof, we make use of the underlying probability generating functions. In Section 2, a brief description of the probability model is considered and some basic definitions and results are given. Section 3 is devoted to prove the asymptotic result previously indicated.

2. THE PROBABILITY MODEL

The bisexual Galton-Watson process with immigration of females and males (BGWPI) denoted by \( \{(F_n^*, M_n^*), n = 1, 2, \ldots \} \) is defined, see [2], in the form:

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\[ Z_0^* = N, \quad (F_{n+1}^*, M_{n+1}^*) = \sum_{i=1}^{Z_n^*} (f_{ni}, m_{ni}) + (F_{n+1}^I, M_{n+1}^I), \]

\[ Z_{n+1}^* = L(F_{n+1}^*, M_{n+1}^*), \quad n = 0, 1, \ldots \]

where \( N \) is a positive integer and the empty sum is considered to be \((0, 0)\). \={(f_{ni}, m_{ni})\} and \{\{F_n^I, M_n^I\}\} are independent sequences of i.i.d. non-negative integer-valued random variables with mean vectors \( \mu = (\mu_1, \mu_2) \) and \( \mu^I = (\mu_1^I, \mu_2^I) \), respectively. Intuitively \( f_{ni} \) \( (m_{ni}) \) represents the number of females (males) produced by the \( i \)-th mating unit in the \( n \)-th generation and \( F_{n}^I \) \( (M_{n}^I) \) may be viewed as the number of immigrating females (males) in the \( n \)-th generation. The mating function \( L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is non-decreasing in each argument, integer-valued for integer-valued arguments and such that \( L(x, y) \leq xy \). Consequently, from an intuitive outlook, \( F_{n}^* \) \( (M_{n}) \) will be the number of females (males) in the \( n \)-th generation, which form \( Z_{n}^* = L(F_{n}^*, M_{n}^*) \) mating units. These mating units reproduce independently through the same offspring distribution for each generation.

It can be shown that \{\{Z_{n}^*\}\} and \{\{(F_{n}^*, M_{n}^*)\}\} are Markov chains with stationary transition probabilities. We denote by \( p_{kl} = P[(f_{01}, m_{01}) = (k, l)] \), \( k, l = 0, 1, \ldots \) and assume that \( \mu \) and \( \mu^I \) are finite.

**Definition 2.1.** A BGWPI is said to be superadditive if the mating function \( L \) is superadditive, i.e. satisfies, for every positive integer \( n \), that

\[ L \left( \sum_{i=1}^{n} (x_i, y_i) \right) \geq \sum_{i=1}^{n} L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, \quad i = 1, \ldots, n. \]

**Definition 2.2.** For a BGWPI and each positive integer \( k \), we define the average reproduction rate per mating unit, denoted by \( r_k^* \), as:

\[ r_k^* = k^{-1} E[Z_{n+1}^* \mid Z_n^* = k], \quad k = 1, 2, \ldots. \]

For a superadditive BGWPI with finite mean vector \( \mu \) and mating function verifying that \( L(x, y) \leq x + y \) it is derived (see [3]) that \( \lim_{k \to \infty} r_k^* = r \) being \( r \) the named asymptotic growth rate (or growth rate).

**Definition 2.3.** A superadditive BGWPI is said to be subcritical, critical or supercritical if \( r \) is \(<, = \) or \( >, \) respectively.
3. A LIMIT RESULT FOR THE SEQUENCE \{Z_{n}^{*}\}

In this section we consider a subcritical and superadditive BGWPI defined by (1) and, considering as a tool the underlying probability generating functions, we provide an alternative proof to theorem 3.2 in [3]. Previously it will be necessary to introduce the following lemma:

**Lemma 3.1.** Let \( \psi \) a positive, non decreasing and continuous function on \([0, 1]\) such that \( \psi(1) = 1 \) and \( \psi'(1^{-}) \in (0, \infty) \). Then for \( \delta \in (0, 1) \) it is verified that

\[
\sum_{k=1}^{\infty} (1 - \psi(1 - \delta^k)) < \infty.
\]

**Proof.** Consider \( h(x) = 1 - \psi(1 - \delta^x), x \in \mathbb{R}^1 \). It is clear that \( h \) is a positive, non increasing and continuous function. Moreover, it follows that \( \lim_{x \to \infty} h(x) = 0 \). Then, from the integral criteria for convergence of series, it will be sufficient to prove that \( \int_{1}^{\infty} h(x)dx < \infty \). Making use of the transformation \( s = 1 - \delta^x \) we get that \( \int_{1}^{\infty} h(x)dx \) is proportional to \( \int_{0}^{1} (1 - s)^{-1}(1 - \psi(s))ds \) which is convergent taking into account that \( \psi'(1^{-}) < \infty \) and \( (1 - s)^{-1}(1 - \psi(s)) \) is bounded on \([0, 1]\). \( \blacksquare \)

**Theorem 3.1.** If \( E[L(F_{01}, m_{01})] > 0, E[L(F_{11}, M_{11})] > 0, p_{00} > 0 \) and there exists \( \alpha > 0 \) and \( N_0 \geq 1 \) such that for \( k > n_0, r^*_k \leq r + k^{-1}\alpha \) then \( \{Z_{n}^{*}\} \) converges in distribution to a positive and finite random variable \( Z^{*} \) as \( n \to \infty \).

**Proof.** Under the considered assumptions, it can be proved in [2] that \( \{Z_{n}^{*}\} \) is an irreducible Markov chain. If \( k_0 = \inf\{k : P[L(F_{11}, M_{11}) = k] > 0\} \) then, using that \( L \) is non decreasing in each argument, it is derived that \( P[Z_{n}^{*} \geq k_0] = 1, n = 1, 2, \ldots \), and therefore if \( k^{*} \) is an essential state it is obtained that \( k^{*} \geq k_0 \).

Thus, if \( f_{n}^{*} \) and \( h_{k}^{*} \) denote the probability generating functions associated with \( Z_{n}^{*} \) and with the kth row of the transition matrix of \( \{Z_{n}^{*}\} \), respectively, i.e. \( f_{n}^{*}(s) = E[s^{Z_{n}^{*}}] \) and \( h_{k}^{*}(s) = E[s^{Z_{n+1}^{*}} \mid Z_{n}^{*} = k], s \in [0, 1] \), then it is followed that:

\[
f_{n}^{*}(s) = \sum_{j=k_0}^{\infty} s^{j} P[Z_{n}^{*} = j] \quad \text{and} \quad h_{k}^{*}(s) = \sum_{j=k_0}^{\infty} s^{j} P[Z_{n+1}^{*} = j \mid Z_{n}^{*} = k].
\]
From Jensen's inequality we obtain:

\[(h_k^*(s))^{1/k} \geq \varphi_k(s), \quad s \in [0, 1],\]

where

\[\varphi_k(s) = E \left[ s^{k-1} r_k \left( \sum_{i=1}^{k} (f_{m_i}, m_{ni}) + (P^{I}_{n+1}, M^{I}_{n+1}) \right) \right].\]

Since, for some \(\xi \in (s, 1)\):

\[\varphi_k(s) = 1 - r_k^*(1 - s) + \frac{\varphi_k''(\xi)}{2} (1 - s)^2\]

we have for \(k > N_0\), that

\[\varphi_k(s) \geq a(s) \left( 1 - \frac{(1 - s)\alpha}{ka(s)} \right)\]

being \(a(s) = 1 - r(1 - s)\). Now

\[0 \leq \frac{(1 - s)\alpha}{ka(s)} \leq (1 - r)^{-1} \alpha, \quad s \in [0, 1].\]

Therefore for \(k > N_1 > \max\{N_0, (1 - r)^{-1} \alpha\}\), taking into account (2) and (3), it is deduced that:

\[h_k^*(s) \leq (a(s))^k \left( 1 - \frac{(1 - s)\alpha}{ka(s)} \right)^k \leq (a(s))^k A(s), \quad s \in [0, 1]\]

where \(A(s) = \left( 1 - \frac{(1 - s)\alpha}{N_1a(s)} \right)^{N_1}\).

It is clear that \(A\) is a positive, non decreasing and continuous function on \(\mathbb{R}^+\) verifying that \(A(1) = 1\) and \(A'(1) = \alpha\). Let \(u(s)\) be an arbitrary probability generating function such that \(u'(1) < \alpha\) (for example the probability generating function of a Poisson distribution with mean \(\lambda < \alpha\)) and for \(s \in [0, 1]\) we define the function:

\[\hat{h}_k(s) = \begin{cases} (a(s))^k u(s) & \text{if } k = 1, \ldots, N_1, \\ h_k^*(s) & \text{if } k > N_1 + 1. \end{cases}\]

If \(\psi(s) = \min\{u(s), A(s)\}\), it follows that

\[\hat{h}_k(s) \geq (a(s))^k \psi(s), \quad s \in [0, 1], \quad k = 1, 2, \ldots\]
and from the comparison theorem for Markov chains (see [4], p.45) it will be sufficient to prove that \( k_0 \) is a positive recurrent state for the Markov chain with transition matrix rows associated to \( \hat{h}_k(s) \). If we denote this Markov chain by \( \{ \hat{Z}_n \} \) then, without loss of generality, it may be assumed that \( k_0 = 0 \).

Let
\[
\hat{f}_m(s) = E[s^{\hat{Z}_{n+m}} \mid \hat{Z}_n = 0], \quad m = 0, 1, \ldots
\]

It is not difficult to verify that:
\[
\hat{f}_m(s) \geq \prod_{j=0}^{m-1} \psi(a_j(s)), \quad s \in [0, 1],
\]
where \( a_j \) denotes the \( j \) times composition of the function \( a \) and \( a_0(s) = s \). Consequently, if \( p^{(m)}_{00} \) represents the \( m \) step transition probability from 0 to 0, taking into account (4) we deduced that
\[
\lim_{m \to \infty} p^{(m)}_{00} = \lim_{m \to \infty} \hat{f}_m(0) \geq \prod_{j=0}^{\infty} \psi(1 - r^j)
\]
and therefore 0 will be a positive recurrent state if the limit above is positive or, equivalently, if \( \sum_{j=0}^{\infty} (1 - \psi(1 - r^j)) < \infty \) which holds as a consequence of Lemma 1. From Markov chains theory we deduce that \( \{ \hat{Z}_n \} \) converges in distribution to a positive and finite random variable \( \hat{Z}^* \) whose probability distribution will be the corresponding stationary distribution. 

References


